On the Burnside Problem for Groups of Even Exponent

Sergei V. Ivanov[1]

Abstract. The Burnside problem about periodic groups asks whether any finitely generated group with the law $x^n \equiv 1$ is necessarily finite. This is proven only for $n \leq 4$ and $n = 6$. A negative solution to the Burnside problem for odd $n \gg 1$ was given by Novikov and Adian. The article presents a discussion of a recent solution of the Burnside problem for even exponents $n \gg 1$ and related results.

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Recall that the notorious Burnside problem about periodic groups (posed in 1902, see [B]) asks whether any finitely generated group that satisfies the law $x^n \equiv 1$ (n is a fixed positive integer called the *exponent* of G) is necessarily finite. A positive solution to this problem is obtained only for $n \leq 4$ and $n = 6$. Note the case $n \leq 2$ is obvious, the case $n = 3$ is due to Burnside [B], $n = 4$ is due to Sanov [S], and $n = 6$ to M. Hall [Hl] (see also [MKS]). A negative solution to the Burnside problem for odd exponents was given in 1968 by Novikov and Adian [NA] (see also [Ad]) who constructed infinite m-generator groups with $m \geq 2$ of any odd exponent $n \geq 4381$ (later Adian [Ad] improved on this estimate bringing it down to odd $n \geq 665$). A simpler geometric solution to this problem for odd $n > 10^{10}$ was later given by Ol'shanskii [Ol1] (see also [Ol2]). We remark that attempts to approach the Burnside problem via finite groups gave rise to a restricted version of the Burnside problem [M] that asks whether there exists a number $f(m,n)$ so that the order of any finite m-generator group of exponent n is less than $f(m,n)$. The existence of such a bound $f(m,n)$ was proven for prime n by Kostrikin [K1] (see also [K2]) and for $n = p^\ell$ with prime p by Zelmanov [Z1]-[Z2]. By a reduction theorem due to Ph. Hall and Higman [HH] it then follows from this Zelmanov result that, modulo the classification of finite simple groups, the function $f(m,n)$ does exist for all m, n.

However, the Burnside problem for even exponents n without odd divisor ≥ 665, being especially interesting for $n = 2^\ell \gg 1$, remained open. The principal difference between odd and even exponents in the Burnside problem can be illustrated by pointing out that, on the one hand, for every odd $n \gg 1$ there are infinite

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2-generator groups of exponent n all of whose proper subgroups are cyclic [IA] (see also [Ol2]) and, on the other hand, any 2-group the orders of whose finite (or abelian) subgroups are bounded is itself finite [Hd].

A negative solution to the Burnside problem for even exponents $n \gg 1$ is given in recent author's article [Iv] and based on the following inductive construction (which is analogous to Ol'shanskii's construction [Ol1] for odd $n > 10^{10}$).

Let F_m be a free group of rank m over an alphabet $\mathcal{A} = \{a_1^{\pm 1}, \dots, a_m^{\pm 1}\}$, $m > 1$, $n \geq 2^{48}$ and n be divisible by 2^9 provided n is even (from now on we impose these restrictions on m and n unless otherwise stated; note this estimate $n \geq 2^{48}$ has been improved on by Lysënok [L] to $n \geq 2^{13}$). By induction on i, let $B(m,n,0) = F_m$ and, assuming that the group $B(m,n,i-1)$ with $i \geq 1$ is already constructed as a quotient group of F_m, define A_i to be a shortest element of F_m (if any) the order of whose image (under the natural epimorphism $\psi_{i-1} : F_m \to B(m,n,i-1)$) is infinite. Then $B(m,n,i)$ is constructed as a quotient group of $B(m,n,i-1)$ by the normal closure of $\psi_{i-1}(A_i^n)$. Clearly, $B(m,n,i)$ has a presentation of the form

$$B(m,n,i) = \langle\, a_1, \dots, a_m \parallel A_1^n, \dots, A_{i-1}^n, A_i^n \,\rangle, \tag{1}$$

where $A_1^n, \dots, A_{i-1}^n, A_i^n$ are the defining relators of $B(m,n,i)$.

The quotient group F_m/F_m^n, where F_m^n is the subgroup of the free group F_m generated by all nth powers, is denoted by $B(m,n)$ and called the free m-generator Burnside group of exponent n. Now we give a summary of basic results of [Iv].

Theorem 1 ([Iv]). *Let $m > 1$, $n \geq 2^{48}$, and 2^9 divide n provided n is even. Then the following hold.*

(a) *The free m-generator Burnside group $B(m,n)$ of exponent n is infinite.*
(b) *The word A_i does exist for each $i \geq 1$.*
(c) *The direct limit $B(m,n,\infty)$ of the groups $B(m,n,i)$, $i = 1,2,\dots,$ is the free m-generator Burnside group $B(m,n)$ of exponent n, that is, $B(m,n)$ has the presentation*

$$B(m,n) = \langle\, a_1, \dots, a_m \parallel A_1^n, \dots, A_i^n, A_{i+1}^n, \dots \,\rangle. \tag{2}$$

(d) *There are algorithms that solve the word and conjugacy problems for the group $B(m,n)$ given by presentation (2).*
(e) *Let $n = n_1 n_2$, where n_1 is odd and n_2 is a power of 2. If n is odd, then every finite subgroup of $B(m,n)$ is cyclic. If n is even, then every finite subgroup of $B(m,n)$ is isomorphic to a subgroup of the direct product $D(2n_1) \times D(2n_2)^\ell$ for some ℓ, where $D(2k)$ is a dihedral group of order $2k$.*
(f) *For every $i \geq 0$ the group $B(m,n,i)$ given by presentation (1) is hyperbolic (in the sense of Gromov [G]).*

Note that the part (a) of Theorem 1 is immediate from part (b) because if $B(m,n)$ were finite it could be given by finitely many defining relators and so A_i would fail to exist for sufficiently large i. To prove part (d) the word and conjugacy problems for the group $B(m,n)$ are effectively reduced to the word problem for some $B(m,n,i)$ and it is shown that every $B(m,n,i)$ satisfies a linear isoperimetric inequality and so is hyperbolic. It should be pointed out that for odd exponents $n \gg 1$ all parts of Theorem 1 had been known due to Novikov and Adian (parts (a), (e), (f); see [Ad]) and Ol'shanskii (parts (b), (c), (f); see [Ol1], [Ol2]).

It is worth noting that the structure of finite subgroups of the free Burnside group $B(m,n)$ and $B(m,n,i)$ is very complex when the exponent n is even and, in fact, finite subgroups of groups $B(m,n,i)$, $B(m,n)$ turn out to be so important in proofs of [Iv] that at least a third of article [Iv] is an investigation of their various properties and another third is a preparation of necessary techniques to conduct this investigation. Part (e) of Theorem 1 may be regarded as a central result on finite subgroups of $B(m,n)$. To state more results on finite subgroups of groups $B(m,n,i)$, $B(m,n)$, denote by $\mathcal{F}(A_i)$ a maximal finite subgroup of $B(m,n,i-1)$ relative to the property that A_i (that is, the image of A_i in $B(m,n,i-1)$) normalizes $\mathcal{F}(A_i)$. A word U is called an $\mathcal{F}(A_i)$-*involution* if $U^2 \in \mathcal{F}(A_i)$, U normalizes the subgroup $\mathcal{F}(A_i)$ of $B(m,n,i-1)$, and

$$UA_iU^{-1} = A_i^{-1}F$$

in $B(m,n,i-1)$, where $F \in \mathcal{F}(A_i)$.

For example, if A_i is a letter then $\mathcal{F}(A_i) = \{1\}$ and there are no $\mathcal{F}(A_i)$-involutions. If $n \gg 1$ is odd then for every i one has $\mathcal{F}(A_i) = \{1\}$ and there are no $\mathcal{F}(A_i)$-involutions. If now $A_i = a_1^{n/2}a_2^{n/2}$ then $\mathcal{F}(A_i) = \{1\}$ and $a_1^{n/2}$, $a_2^{n/2}$ are $\mathcal{F}(A_i)$-involutions. An example when $\mathcal{F}(A_i) \neq \{1\}$ is provided by

$$A_i = (a_1^{n/2}(a_1a_2)^{n/2})^{n/2}(a_2^{n/2}(a_1a_2)^{n/2})^{n/2}$$

for $(a_1a_2)^{n/2} \in \mathcal{F}(A_i)$.

Next, define

$$\mathcal{G}(A_i) = \langle\, U_i, A_i, \mathcal{F}(A_i)\,\rangle$$

to be a subgroup in $B(m,n,i-1)$ generated by A_i, by the subgroup $\mathcal{F}(A_i)$, and by a word U_i, where U_i is an $\mathcal{F}(A_i)$-involution provided there are $\mathcal{F}(A_i)$-involutions and $U_i = 1$ otherwise. It follows from definitions that $\mathcal{G}(A_i)$ is either an extension of $\mathcal{F}(A_i)$ by an infinite dihedral group generated by elements U_i, A_i of order $2, \infty$, respectively, modulo $\mathcal{F}(A_i)$ provided there are $\mathcal{F}(A_i)$-involutions or $\mathcal{G}(A_i)$ is an extension of $\mathcal{F}(A_i)$ by an infinite cyclic group generated by A_i provided there are no $\mathcal{F}(A_i)$-involutions. Basic properties of finite subgroups of groups $B(m,n,i)$, $B(m,n)$ are collected in the following.

Theorem 2 ([Iv]). *Let $B(m,n)$ be a free m-generator Burnside group of even exponent $n \geq 2^{48}$ given by presentation (2), where n is divisible by 2^9. Then the following are true.*

(a) *The subgroup $\mathcal{F}(A_i)$ is defined uniquely and is a 2-group.*

(b) *The word $A_i^{n/2}$ centralizes in $B(m,n,i-1)$ the subgroup $\mathcal{F}(A_i)$ and hence the quotient $\mathcal{G}(A_i)/\langle A_i^n\rangle$, denoted by $\mathcal{K}(A_i)$ is either an extension of $\mathcal{F}(A_i)$ by a dihedral group of order $2n$ generated by elements U_i, A_i, or $\mathcal{K}(A_i)$ is an extension of $\mathcal{F}(A_i)$ by a cyclic group of order n generated by A_i. In addition, the group $\mathcal{K}(A_i)$ naturally embeds in $B(m,n,i)$ and $B(m,n)$.*

(c) *Every word W of finite order in $B(m,n,i-1)$ is conjugate in $B(m,n,i-1)$ to a word of the form $A_j^k T$ with some integers k, $j < i$ and $T \in \mathcal{F}(A_i)$. Moreover, the conjugacy in $B(m,n,i-1)$ of nontrivial in $B(m,n,i-1)$ words $A_{j_1}^{k_1} T_1$ and $A_{j_2}^{k_2} T_2$, where $T_1 \in \mathcal{F}(A_{j_1})$ and $T_2 \in \mathcal{F}(A_{j_2})$, $j_1, j_2 < i$, yields $j_1 = j_2$ and $k_1 \equiv \pm k_2 \pmod n$. (Therefore, given a nontrivial in $B(m,n,i-1)$ word W of finite order such number j is defined uniquely in $B(m,n,i-1)$ as well as in $B(m,n)$ and called the height of the word W.)*

(d) *Every finite subgroup of $B(m,n)$, consisting of words of heights $\le i$ and containing a word of height i, is conjugate to a subgroup of $\mathcal{K}(A_i) = \langle\, U_i, A_i, \mathcal{F}(A_i) \,\rangle \subseteq B(m,n)$.*

(e) *The words A_i and U_i act on the subgroup $\mathcal{F}(A_i)$ of $B(m,n,i-1)$ by conjugations in the same way as some words V_{A_i} and V_{U_i} act respectively, where V_{A_i} and V_{U_i} are such that the subgroup $\langle\, V_{A_i}, V_{U_i}, \mathcal{F}(A_i) \,\rangle$ of $B(m,n,i-1)$ is finite and the equality $U_i^2 = V_{U_i}^2$ (as well as $(U_i A_i{}^2 = (V_{U_i} V_{A_i})^2$ provided $U_i \ne 1$) holds in $B(m,n,i-1)$.*

Let us see how the algebraic description of finite subgroups of $B(m,n)$ of Theorem 1 (e) can be derived from Theorem 2 by induction on the height $h(G)$ of a finite subgroup G of $B(m,n)$ ($h(G)$ is the maximum of heights of elements of G). Let G be a finite subgroup of $B(m,n)$ with $h(G) = i$. By Theorem 2 (d), one may assume that G is a subgroup of $\mathcal{K}(A_i)$. It follows from definitions and Theorem 2 (e) that there are homomorphisms

$$\kappa_1 : \mathcal{K}(A_i) \to D(2n), \quad \kappa_2 : \mathcal{K}(A_i) \to G_0$$

such that

$$\operatorname{Ker} \kappa_1 = \mathcal{F}(A_i), \quad G_0 = \langle\, \mathcal{F}(A_i), V_{A_i}, V_{U_i} \,\rangle \subseteq B(m,n,i-1)$$

and $h(G_0) < i$. Since $\operatorname{Ker} \kappa_1 \cap \operatorname{Ker} \kappa_2 = \{1\}$, the group G embeds in the direct product $D(2n) \times G_0$. By the induction hypothesis, G_0 embeds in $D(2n_1) \times D(2n_2)^\ell$ for some ℓ. By Theorem 2 (a), $\mathcal{F}(A_i)$ is a 2-group, therefore the subgroup of $D(2n_1)$ of index 2 has the trivial intersection with the image of $\mathcal{F}(A_i)$ in $D(2n) \times D(2n_1) \times D(2n_2)^\ell$ and hence $D(2n_1)$ can be replaced by $D(2n_2)$. Since $D(2n)$ embeds in $D(2n_1) \times D(2n_2)$, we have that G is embeddable in $D(2n_1) \times D(2n_2)^{\ell+1}$ as required. This algebraic description of finite subgroups is very important in many parts of article [Iv], especially, in making the inductive step from the group $B(m,n,i-1)$ to $B(m,n,i)$. For example, this description helps a great deal in proving one of the hardest and absolutely crucial for the whole work technical results: If the subgroup

$$\langle\, A_i^k T A_i^{-k} \mid k = 0, 1, \dots, 7 \,\rangle$$

of $B(m,n,i-1)$ is finite then A_i normalizes this subgroup. We will make an informal remark that if a similar claim (where instead of 7 one could put a number as large as $n^{1/2}$) were false then A_i^n would not have to centralize $\mathcal{F}(A_i)$ and imposing the relation $A_i^n = 1$ on $B(m,n,i-1)$ would result in extra relations of type $R = 1$ where R is a nontrivial element of $\mathcal{F}(A_i)$ (R has the form $R =$

$A_i^n F A_i^{-n} F^{-1} \neq 1$ with $F \in \mathcal{F}(A_i)$). This secondary factorization would make a complete mess implying that one could be far better off trying to solve the Burnside problem for $n = 2^\ell$ in the affirmative.

The proofs in [Iv] are based on geometric techniques of van Kampen diagrams (which are labelled planar 2-complexes representing consequences of defining relators of group presentations; see [Ol2], [LS], [IO1]) and may be regarded as a further development of Ol'shanskii's method [Ol1] for solving the Burnside problem for odd exponents $n \gg 1$. The main obstacle in carrying over Ol'shanskii's proof to even exponents is in making the inductive step from $B(m, n, i-1)$ to $B(m, n, i)$. Curiously, in odd case the inductive step from $B(m, n, i-1)$ to $B(m, n, i)$ is being made in [Ol1] by boiling everything down to an elementary fact that the fundamental group of an annulus is cyclic (however, the reduction itself is highly nontrivial). The same fact is ultimately responsible for the cyclicity of finite subgroups of $B(m, n)$ with odd $n \gg 1$. This reduction naturally fails in even case due to the existence of self-compatible cells in nonsimply connected diagrams over $B(m, n, i-1)$. (A self-compatible cell is a 2-cells that surrounds hole(s) of a diagram and has two long arcs of its boundary with a narrow strip squeezed between the arcs.) Informally, turning these self-compatible cells from the main obstacle into a source of new information is what the article [Iv] is all about. However, extracting gems from this mine is quite a challenge and that partially explains an extraordinary length (over 300 pages) of the article and its complex logical structure (over 110 lemmas are proved by simultaneous induction on the parameter i with quite a few back references; note a similar simultaneous induction is carried out in [NA], [Ol1]).

It is implied by results of [Iv] that finite (as well as locally finite) subgroups of $B(m, n)$ with even n are very interesting subject for investigation on their own. In particular, one might wonder if their description given in Theorem 1 (e) is complete, that is, every group $D(2n_1) \times D(2n_2)^\ell$ embeds in $B(m, n)$. Another natural question is to ask whether every locally finite subgroup of $B(m, n)$ is an FC-group. Recall that a group G is *locally finite* if every finitely generated subgroup of G is finite. A group G is termed an *FC-group* if every conjugacy class of G is finite. Also, let D_i, $i = 1, 2, \dots$, be groups isomorphic to $D(2n_2)$, $\mathcal{D}$ be the cartesian product of D_i, $i = 1, 2, \dots$, C_i be the normal cyclic subgroup of D_i of order n_2, and $b_i \in D_i$ be an element of order 2 that together with C_i generate $D_i = \langle C_i, b_i \rangle$. By $\mathcal{B}$ denote the subgroup of $\mathcal{D}$ that consists of all elements whose projection on every D_i is either b_i or 1. By $\mathcal{C}$ denote the direct product of groups C_i naturally embedded in $\mathcal{D}$. At last, let $\mathcal{E} = \langle \mathcal{B}, \mathcal{C} \rangle$. Clearly, $\mathcal{E} = \mathcal{B}\mathcal{C}$ is a semidirect product of $\mathcal{B}$ and $\mathcal{C}$.

The following Theorem 3 is a summary of joint with Ol'shanskii results on (locally) finite subgroups of $B(m, n)$.

Theorem 3 ([IO2]). *Let $B(m, n)$ be a free m-generator Burnside group of even exponent n, where $m > 1$ and $n \geq 2^{48}$, $n = n_1 n_2$, n_1 is odd, n_2 is a power of 2, $n_2 \geq 2^9$. Then the following hold:*

(a) *Suppose $\mathcal{G}$ is a finite 2-subgroup of $B(m, n)$. Then the centralizer $C_{B(m,n)}(\mathcal{G})$ of $\mathcal{G}$ in $B(m, n)$ contains a subgroup $\mathcal{M}$ isomorphic to a free*

Burnside group $B(\infty, n)$ of infinite countable rank such that $\mathcal{G} \cap \mathcal{M} = \{1\}$. In particular, $\langle \mathcal{G},\ \mathcal{M} \rangle = \mathcal{G} \times \mathcal{M}$.

(b) *The centralizer $C_{B(m,n)}(\mathcal{H})$ of a subgroup $\mathcal{H}$ of $B(m, n)$ is infinite if and only if $\mathcal{H}$ is a locally finite 2-subgroup. In particular, $C_{B(m,n)}(\mathcal{H})$ is finite provided $\mathcal{H}$ is not locally finite.*

(c) *An arbitrary infinite group G embeds in $B(m, n)$ as a locally finite subgroup if and only if G is isomorphic to a countable subgroup of $\mathcal{E}$.*

(d) *An arbitrary infinite group G embeds in $B(m, n)$ as a maximal locally finite subgroup if and only if G is isomorphic to a countable subgroup of $\mathcal{E}$.*

(e) *An infinite locally finite subgroup $\mathcal{L}$ of $B(m, n)$ is contained in a unique maximal locally finite subgroup. That is, the intersection of two distinct maximal locally finite subgroups of $B(m, n)$ is always finite.*

(f) *Given a finite 2-subgroup $\mathcal{G}$ of $B(m, n)$ there are continuously many pairwise nonisomorphic maximal locally finite subgroups that contain $\mathcal{G}$.*

(g) *If a finite subgroup $\mathcal{G}$ of $B(m, n)$ contains a nontrivial element of odd order, then $\mathcal{G}$ is contained in a unique maximal finite subgroup. In particular, the intersection of two distinct maximal finite subgroups of $B(m, n)$ is always a 2-group.*

Note the mutual disposition of infinite maximal locally finite subgroups stated in Theorem 3 (e), (f) is reminiscent of a known puzzle-type problem: Find, in a countably infinite set, continuously many subsets whose pairwise intersections are all finite (note this is impossible if the cardinalities of the intersections are bounded).

A couple of questions mentioned above can now be easily answered:

Corollary. *Let $B(m, n)$ be defined as in Theorem 2. Then the following are true.*

(a) *A finite group G embeds in $B(m, n)$ if and only if G is isomorphic to a subgroup of the direct product $D(2n_1) \times D(2n_2)^\ell$ for some $\ell > 0$, where $D(2k)$ is a dihedral group of order $2k$.*

(b) *The group $B(m, n)$ contains (maximal) locally finite subgroups that are not FC-groups.*

(c) *A subgroup $\mathcal{S}$ of $B(m, n)$ is locally finite if and only if every 2-generator subgroup of $\mathcal{S}$ is finite.*

The machinery developed in [Iv] for solving the Burnside problem for even exponents $n \gg 1$ has made it possible to prove a conjecture of Gromov on quotients of hyperbolic groups of bounded exponent.

To state the results we recall several definitions. Let G be a finitely generated group, $\mathcal{A}$ be a finite set of generators for G. By $|g| = |W|$ denote the length of a shortest word W in the alphabet $\mathcal{A}$ that represents an element $g \in G$. One of definitions of a hyperbolic group G is given by means of the Gromov product

$$(g \cdot h) = \tfrac{1}{2}(|g| + |h| - |g^{-1}h|)$$

as follows: A group G is called *hyperbolic* [G] if there exists a constant $\delta \geq 0$ such that for every triple $g, h, f \in G$

$$(g \cdot h) \geq \min((g \cdot f), (h \cdot f)) - \delta. \tag{3}$$

In turns out [G], [GH] that the property of being hyperbolic does not depend on a particular generating set $\mathcal{A}$ (but the constant δ does depend on $\mathcal{A}$). A trivial example of hyperbolic group is the free group $F = F(\mathcal{A})$ over $\mathcal{A}$ for which inequality (3) is satisfied with $\delta = 0$ because in this case $(g \cdot h)$ is the length of the maximal common beginning of reduced words g, h in $\mathcal{A}$. Perhaps, the most complicated (in terms of proving that) examples of hyperbolic groups are provided by the series of groups $B(m, n, i)$ of Theorems 1–2.

Similar to free groups, an arbitrary nonelementary hyperbolic group has many homomorphic images (recall a group G is termed in [G] *elementary* if G has a cyclic subgroup of finite index). Discussing an approach to construction of an infinite periodic quotient group $\bar{G}$ of a nonelementary hyperbolic group G, Gromov [G] (see also [GH], [Ol4]) points out that his approach does not let bound the orders of elements in $\bar{G}$ (and so $\bar{G}$ will not be of finite exponent). Nevertheless, Gromov conjectures (see 5.5E, 5.5F in [G]) that it is possible in principle to bound the orders of elements in $\bar{G}$ and obtain $\bar{G}$ of finite exponent n, that is, he conjectures that for every nonelementary hyperbolic group G there is an $n = n(G)$ such that the quotient G/G^n is infinite. Thus Gromov suggested a natural expansion of the Burnside problem to nonelementary hyperbolic groups. This Gromov conjecture was proven by Ol'shanskii [Ol3] for torsion free hyperbolic groups. However, in the general case of a hyperbolic group with torsion there are serious obstacles connected with nonelementary centralizers of elements of G and noncyclic finite subgroups in G/G^n that are essentially the same as those in solving the classical Burnside problem for even exponents n.

The solution [Iv] of the "even" Burnside problem discussed above combined with ideas of [Ol3] enabled Ol'shanskii and the author to prove the Gromov conjecture in full generality.

Theorem 4 ([IO3]). *For every nonelementary hyperbolic group G there exists a positive even integer $n = n(G)$ such that the following are true:*

(a) *The quotient group G/G^n is infinite.*
(b) *The word and conjugacy problems are solvable in G/G^n.*
(c) *Let $n = n_1 n_2$, where n_1 is odd and n_2 is a power of 2. Then every finite subgroup of G/G^n is isomorphic to an extension of a finite subgroup K of G by a subgroup of the direct product $D(2n_1) \times D(2n_2)^\ell$ for some ℓ, where $D(2k)$ is a dihedral group of order $2k$.*
(d) *The subgroup G^n is torsion free and $\bigcap_{k=1}^{\infty} G^{kn} = \{1\}$.*

When proving Theorem 4, we encounter several restrictions to be imposed on n and end up with that n must be divisible by $2^{k_0+5} n_0$ (to say nothing of that $n \gg \delta = \delta(G)$), where $\frac{n_0}{2}$ is the least common multiple of the exponents of the holomorphs $\mathrm{Hol}(K)$ over all finite subgroups K of G and k_0 is the minimal integer with $2^{k_0-3} > \max |K|$ over all finite subgroups K of G. We note that almost all lemmas of [Iv] are reproved in [IO3] with necessary modifications which are analogous to those made by Ol'shanskii [Ol3] to adjust his solution [Ol1] of "odd" Burnside problem for proving Gromov conjecture for torsion free hyperbolic groups. We also note that, just like in [Iv], information on finite subgroups of G/G^n is very important in proofs of [IO3] and their description (Theorem 4 (c)) is given as in [Iv] (Theorem 1 (e)) modulo finite subgroups of G. Naturally, proofs in [IO3] also make use of various facts of general theory of hyperbolic groups, see [G], [GH], [CDP].

REFERENCES

[Ad] S.I. Adian, *The Burnside problem and identities in groups*, Nauka, Moscow, 1975; English translation: Springer-Verlag, 1979.

[AI] V.S. Atabekian and S.V. Ivanov, Two remarks on groups of bounded exponent, # 2243-B87, VINITI, Moscow, 1987 (this is kept in the Depot of VINITI, Moscow, and is available upon request), 23 pp.

[B] W. Burnside, On unsettled question in the theory of discontinuous groups, *Quart. J. Pure and Appl. Math.* **33**(1902), 230-238.

[CDP] E. Coornaert, T. Delzant, and A. Papadopoulos (ed's), *Géométric et théorie des groupes: Les groupes, huperboliques de Gromov*, Lecture Notes in Math. **1441** (1991), Springer-Verlag.

[GH] E. Ghys and P. de la Harpe (ed's), *Sur les groupes huperboliques d'aprés Mikhael Gromov*, Birkhaüser, 1990.

[G] M. Gromov, Hyperbolic groups, in *Essays in Group Theory*, ed. S.M. Gersten, M.S.R.I. Pub. 8, Springer, 1987, 75–263.

[Hl] M. Hall, Jr., Solution of the Burnside problem for exponent 6, *Proceedings Nat. Acad. Sci. USA* **43**(1957), 751–753.

[Hd] D. Held, On abelian subgroups of an infinite 2-group, *Acta Sci. Math. (Szeged)* **27**(1966), 97–98.

[HH] Ph. Hall and G. Higman, On the p-length of p-soluble groups and reduction theorems for Burnside's problem, *Proc. London Math. Soc.* **6**(1956), 1-42.

[Iv] S.V. Ivanov, The free Burnside groups of sufficiently large exponents, *Intern. J. Algebra Comp.* **4** (1994), 1–308.

[IO1] S.V. Ivanov and A.Yu. Ol'shanskii, Some applications of graded diagrams in combinatorial group theory, *London Math. Soc. Lecture Notes Ser.*, vol.160(1991), Cambridge Univ. Press, Cambridge and New York, 1991, 258–308

[IO2] S.V. Ivanov and A.Yu. Ol'shanskii, On finite and locally finite subgroups of free Burnside groups of large even exponents, *J. Algebra* **195** (1997), 241-284.

[IO3] S.V. Ivanov and A.Yu. Ol'shanskii, Hyperbolic groups and their quotients of bounded exponents, *Trans. Amer. Math. Soc.* **348** (1996), 2091–2138.

[K1] A.I. Kostrikin, On the Burnside problem, *Math. USSR Izv.* **23** (1959), 3-34.

[K2] A.I. Kostrikin, *Around Burnside*, Nauka, Moscow, 1986.

[L] I.G. Lysënok, Infinite Burnside groups of even period, *Math. Ross. Izv.* **60** (1996), 3-224.

[LS] R.C. Lyndon and P.E. Schupp, *Combinatorial group theory*, Springer-Verlag, 1977.

[M] W. Magnus, A connection between the Baker-Hausdorff formula and a problem of Burnside, *Ann. Math.* **52** (1950), 11-26; **57** (1953), 606.

[MKS] W. Magnus, J. Karras, and D. Solitar, *Combinatorial group theory*, Interscience Pub., John Wiley and Sons, 1966.

[NA] P.S. Novikov and S.I. Adian, On infinite periodic groups, I, II, III, *Math. USSR Izv.* **32** (1968), 212–244, 251–524, 709–731.

[Ol1] A.Yu. Ol'shanskii, On the Novikov-Adian theorem, *Math. USSR Sbornik* **118** (1982), 203–235.

[Ol2] A.Yu. Ol'shanskii, *Geometry of defining relations in groups*, Nauka, Moscow, 1989; English translation in *Math. and Its Applications* (Soviet series), **70** (Kluwer Acad. Publishers, 1991).

[Ol3] A.Yu. Ol'shanskii, Periodic quotient groups of hyperbolic groups, *Math. USSR Sbornik* **72**(1992), 519–541.

[Ol4] A.Yu. Ol'shanskii, On residualizing homomorphisms and G-subgroups of hyperbolic groups, *Intern. J. Algebra Comp.* **3**(1993), 365–409.

[S] I. N. Sanov, Solution of the Burnside problem for exponent 4, *Uch. Zapiski Leningrad State Univ., Ser. Matem.* **10** (1940), 166–170.

[Zl1] E.I. Zelmanov, Solution of the restricted Burnside problem for groups of odd exponent, *Math. USSR Izv.* **36**(1991), 41-60.

[Zl2] E.I. Zelmanov, A solution of the restricted Burnside problem for 2-groups, *Math. USSR Sbornik* **72**(1992), 543-565.

Sergei Ivanov
Department of Mathematics
University of Illinois at Urbana-Champaign
1409 West Green Street,
Urbana, IL 61801, U.S.A.