Spetses

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Abstract. We report on the properties of unipotent degrees of complex reflection groups.

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1. Introduction

About two hours by boat south of Athens in the Aegean sea lies the small island Spetses. On a conference there in July 1993 Michel Broué first asked whether maybe every finite complex reflection group occurs as Weyl group of some object which is an analogue of a finite group of Lie type. This question has instigated some fruitful research and led to the discovery of fascinating structures associated to finite reflection groups, for example the so-called unipotent degrees (see [14]). It therefore seems appropriate to call these yet unknown objects *spetses*.

It was first noted by Springer [20] that non-real reflection groups naturally appear inside Weyl groups as what is now called relative Weyl groups, that is, normalizers modulo centralizers of subspaces. The importance of this construction was revealed in the work [3] of Broué, Michel and the author on the ℓ-blocks of characters of finite groups of Lie type. There it was shown that the unipotent characters inside an ℓ-block are parametrized by the irreducible characters of a relative Weyl group, which in general is a non-real reflection group. A possible interpretation of this result was subsequently proposed by Broué and the author [2] in terms of the so-called cyclotomic Hecke algebra attached to a finite complex reflection group (see also [6]).

The results of Lusztig show that the set of unipotent characters of a finite group of Lie type only depends on the Weyl group of the associated algebraic group, together with the action of the Frobenius endomorphism on it. In the course of this classification Lusztig observed that similar sets can formally also be attached to those finite real reflection groups which are not Weyl groups [11,12].

It is the purpose of this article to give a brief introduction into a similar construction, for complex reflection groups, of 'unipotent characters', Fourier transform matrices, and eigenvalues of Frobenius, which satisfy combinatorial properties like unipotent characters of actual finite groups of Lie type, just as if there existed an algebraic group whose Weyl group is non-real. It is tempting to speculate about an underlying algebraic structure giving rise to the unipotent characters attached to complex reflection groups. Until now, such an object has not been found, but a lot of intriguing evidence for its existence has been collected.

We would like to conclude this introduction by stating that many of the properties of unipotent degrees were found by computer experiments, and some of the results for large exceptional spetses could only be verified using computer algebra systems. We think that this might serve as a good illustration of the power of experimental/computational algebra.

2. Reflection data

2.1. Complex reflection groups. Let V be a finite dimensional vector space over a subfield k of the field of complex numbers $\mathbb{C}$. An element $1 \neq \sigma \in \mathrm{GL}(V)$ is called a *(complex) reflection* if σ pointwise fixes some hyperplane in V. A finite subgroup $W \leq \mathrm{GL}(V)$ generated by complex reflections will be called a *complex reflection group*. Thus the finite Coxeter groups, being real reflection groups, are examples of complex reflection groups, and in particular all finite Weyl groups fall into this class.

A *parabolic subgroup* of a complex reflection group W is the centralizer (pointwise stabilizer) in W of some subspace of V. It is a remarkable result of Steinberg that all parabolic subgroups of a complex reflection group are again generated by reflections.

Let $S(V)$ denote the symmetric algebra of V. By the theorem of Shephard-Todd and Chevalley, the ring of invariants $S(V)^W$ of W in $S(V)$ is a polynomial ring. The quotient $S(V)_W$ of $S(V)$ by the ideal generated by the invariants of strictly positive degree is called the *coinvariant algebra*. Chevalley has shown that the W-module $S(V)_W$ affords a graded version of the regular representation. For an irreducible character $\chi \in \mathrm{Irr}(W)$ the *fake degree* is defined as the graded multiplicity of χ in $S(V)_W$ (the generating function for the embedding degrees),

$$R_\chi := \langle S(V)_W, \chi\rangle_W = (x-1)^{\dim(V)} P_W \frac{1}{|W|} \sum_{w\in W} \frac{\det_V(w)\chi(w)}{\det_V(x-w)} \in \mathbb{Z}[x]\,,$$

where P_W denotes the Poincaré polynomial of the coinvariant algebra $S(V)_W$. The fake degrees can be considered as a first elementary approximation of the unipotent degrees of W from which they differ in a subtle way (see Section 5.1).

2.2. Families of groups of Lie type. Our aim is to introduce for a complex reflection group W an object which behaves like a family of finite groups of Lie type with Weyl group W. To motivate this, let first $\mathbf{G}$ be a connected reductive algebraic group over the algebraic closure of a finite field of positive characteristic. We assume that $\mathbf{G}$ is already defined over the finite field $\mathbb{F}_q$ and let $F : \mathbf{G} \to \mathbf{G}$ be the corresponding Frobenius morphism. The group of fixed points $\mathbb{G}(q) := \mathbf{G}^F$ is then a finite group of Lie type. Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$ contained in an F-stable Borel subgroup of $\mathbf{G}$, and Y the cocharacter group of $\mathbf{T}$. Via its action on Y, W can be considered as a subgroup of the automorphism group of the real vector space $Y_\mathbb{R} := Y \otimes_\mathbb{Z} \mathbb{R}$. The Frobenius endomorphism F also acts on $Y_\mathbb{R}$ as a product $q\phi$ where ϕ is an automorphism of finite order normalizing W. Replacing the Borel subgroup by another one containing $\mathbf{T}$ changes ϕ by an element of the Weyl group W of $\mathbf{G}$ with respect to $\mathbf{T}$. Hence ϕ is determined as automorphism of $Y_\mathbb{R}$ up to elements of W.

Conversely, the real vector space $Y_{\mathbb{R}}$ together with the actions of W and ϕ determines F and $\mathbf{G}$ up to isogeny. In this way, a whole series $\{\mathbb{G}(q) \mid q \text{ a prime power}\}$ of finite groups of Lie type can be encoded by the data $(Y_{\mathbb{R}}, W\phi)$.

2.3. REFLECTION DATA. This leads us to consider complex reflection groups together with certain automorphisms. We define a *reflection datum with Weyl group* W to be a pair $\mathbb{G} = (V, W\phi)$ where W is a complex reflection group on V and $\phi \in \mathrm{GL}(V)$ normalizes W. A *Levi subdatum of* $\mathbb{G}$ is a reflection datum $\mathbb{L} = (V, W_{\mathbb{L}} w\phi)$ where $w \in W$ and $W_{\mathbb{L}}$ is a $w\phi$-stable parabolic subgroup of W. A reflection datum with trivial Weyl group $W = 1$ is called a *torus*. For a torus $\mathbb{S} = (V', (w\phi)|_{V'})$ of $\mathbb{G}$ we define its *centralizer* to be the Levi subdatum

$$C_{\mathbb{G}}(\mathbb{S}) := (V, C_W(V')w\phi)$$

(note that $C_W(V')$ is a reflection subgroup of W by Steinberg's theorem).

2.4. SYLOW THEORY. Let $\mathbb{G}$ be a reflection datum. The *(polynomial) order of* $\mathbb{G}$ is defined as

$$|\mathbb{G}| := \frac{x^N}{\frac{1}{|W|}\sum_{w \in W} \frac{\det_V(w)}{\det_V(x - w\phi)}},$$

where N is the number of reflecting hyperplanes of W in V. For example, if $\phi = 1$ then $|\mathbb{G}| = (x-1)^{\dim(V)} P_W$, and if $\mathbb{G} = (V, \phi)$ is a torus then $|\mathbb{G}| = \det_V(x - \phi)$ is the characteristic polynomial of ϕ on V. Steinberg has shown that in the case of groups of Lie type, $|\mathbb{G}|(q)$ gives the order of $\mathbb{G}(q)$. By an extension of Molien's formula for the ring of invariants $S(V)^W$ it follows that $|\mathbb{G}|$ is a product of cyclotomic polynomials over k, that is, of k-irreducible polynomials whose roots are roots of unity.

Let Φ be a cyclotomic polynomial over k. A torus $\mathbb{S}$ is called a Φ*-torus* if its order $|\mathbb{S}|$ is a power of Φ. Thus $\mathbb{S} = (V, \phi)$ is a Φ-torus if and only if all eigenvalues of ϕ on V are roots of Φ. The centralizers of generic Φ-tori of $\mathbb{G}$ are called Φ*-split* Levi subdata. Note that, in particular, $\mathbb{G}$ itself is Φ-split. The results of Springer [20, 3.4 and 6.2] on eigenspaces of elements in complex reflection groups imply that the Φ-tori of reflection data satisfy an analogue of Sylow theory:

THEOREM 2.5. *Let $\mathbb{G} = (V, W\phi)$ be a reflection datum and $\Phi \neq x$ a prime divisor of $|\mathbb{G}|$ over k.*

(a) There exist non-trivial Φ-tori of $\mathbb{G}$.

(b) For any maximal Φ-torus $\mathbb{S}$ of $\mathbb{G}$ we have $|\mathbb{S}| = \Phi^{a(\Phi)}$, where $a(\Phi)$ is the precise power of Φ dividing $|\mathbb{G}|$.

(c) Any two maximal Φ-tori of $\mathbb{G}$ are W-conjugate.

The concept of reflection data allows to capture much of the structural properties of groups of Lie type. The unipotent degrees play a similar role for the description of the irreducible representations.

2.6. CLASSIFICATION. The finite irreducible complex reflection groups have been classified by Shephard and Todd [19]. The irreducible groups fall into an infinite series of monomial groups $G(m,p,n)$ (with $n \geq 1$, $m \geq 2$, $p|m$), the symmetric groups, and 34 exceptional groups which all occur in dimension at most eight. All the results to be given in the sequel depend on this classification in the sense that their proofs are case-by-case.

3. Cyclotomic Hecke algebras

We now introduce a deformation of the group algebra of a complex reflection group which generalizes the usual Iwahori-Hecke algebra of a Coxeter group.

3.1. Cyclotomic algebras. Starting from the classification mentioned in 2.6 one can find for any complex reflection group a so-called *good presentation*

$$W = \langle s \in S \mid s^{d_s} = 1, \text{ certain homogeneous relations}\rangle$$

similar to the Coxeter presentation of real reflection groups with various good properties. For example, the elements of S map to reflections, and S has minimal size subject to this (equal to n or $n+1$ if $W \le \mathrm{GL}_n(k)$ is irreducible). Moreover, any subset S' of S together with those relations involving only elements of S' gives a presentation of a parabolic subgroup of W. However, in contrast to the Coxeter case, no conceptual definition of good presentations is available at present.

Let $\mathbf{u} = (u_{s,j} \mid s \in S, 0 \le j \le d_s - 1)$ be transcendentals over $\mathbb{Z}$, such that $u_{s,j} = u_{t,j}$ whenever s and t are conjugate in W. Let $A := \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$. The *generic cyclotomic Hecke algebra* $\mathcal{H}(W, \mathbf{u})$ *of* W *with parameter set* $\mathbf{u}$ is defined to be the A-algebra on generators $\{T_s \mid s \in S\}$ subject to the homogeneous relations from the good presentation of W, and the deformed order relations

$$\prod_{j=0}^{d_s-1} (T_s - u_{s,j}) = 0 \qquad (s \in S).$$

For a ring homomorphism $f : A \to R$ we write $\mathcal{H}_R = \mathcal{H} \otimes_A R$ for the corresponding specialization of $\mathcal{H} := \mathcal{H}(W, \mathbf{u})$. Clearly, such a homomorphism is uniquely determined by the images $f(u_{s,j})$. Under the specialization defined by

$$u_{s,j} \mapsto \exp(2\pi i j/d_s) \qquad \text{for } s \in S,\ 0 \le j \le d_s - 1, \tag{3.2}$$

$\mathcal{H}$ maps to the group algebra of the complex reflection group W. Any specialization through which (3.2) factors will be called *admissible*. One particularly important example is the *1-parameter specialization* $\mathcal{H}(W, x)$ of $\mathcal{H}(W, \mathbf{u})$ induced by the map

$$u_{s,j} \mapsto \begin{cases} x & j = 0, \\ \exp(2\pi i j/d_s) & j > 0, \end{cases} \tag{3.3}$$

where x is an indeterminate. This is the analogue of the classical 1-parameter Iwahori-Hecke algebra for real W.

3.4. Braid groups. Let us sketch a more conceptual construction of cyclotomic algebras (see [5]). For an irreducible complex reflection group $W \le \mathrm{GL}(V)$ we let $\mathcal{A}$ be the set of the reflecting hyperplanes of W and denote by

$$M := V \setminus \bigcup_{H \in \mathcal{A}} H$$

the complement. For a fixed base point $x_0 \in M$ we define the *pure braid group of* W as the fundamental group $P(W) := \pi_1(M, x_0)$. Let $\bar{M}$ be the quotient of M

by W. By the theorem of Steinberg on parabolic subgroups M is an unramified Galois cover of $\bar{M}$, with group W. Thus we have the canonical exact sequence

$$1 \longrightarrow P(W) \longrightarrow B(W) \longrightarrow W \longrightarrow 1$$

with the *braid group* $B(W) := \pi_1(\bar{M}, \bar{x}_0)$, where $\bar{x}_0$ is the image of x_0. To each hyperplane $H \in \mathcal{A}$ is attached a class of elements in $B(W)$, the generators of monodromy around H, which in W map to reflections along H. It is shown in [5] (for all but six irreducible types) that $\mathcal{H}(W, \mathbf{u})$ is isomorphic to the quotient

$$\mathcal{H}(W, \mathbf{u}) := \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]B(W) \Big/ \big(\prod_{j=0}^{d_s-1} (\mathbf{s} - u_{s,j}) \mid \mathbf{s} \text{ generator of monodromy}\big),$$

of the group algebra of $B(W)$ by the ideal generated by the deformed order relations. This approach gives a definition of cyclotomic algebras independent of the choice of a good presentation. It also allows to study $\mathcal{H}(W, \mathbf{u})$ via monodromy representations of the braid group, see [5,18]. For example, Opdam uses this to derive symmetry properties of the fake degrees of W.

3.5. STRUCTURE OF CYCLOTOMIC ALGEBRAS. The following important structure result for $\mathcal{H}(W, \mathbf{u})$ has been proved for all but finitely many irreducible complex reflection groups (see [1,2]); it is conjectured to hold in all cases:

THEOREM 3.6. *The cyclotomic Hecke algebra $\mathcal{H}(W, \mathbf{u})$ is free over $A = \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ of rank $|W|$.*

Assume that W is defined over the number field k. Let $\mu(k)$ denote the group of roots of unity in k and let $\mathbf{v} = (v_{s,j} \mid s \in S,\, 0 \leq j \leq d_s - 1)$ where $v_{s,j}^{|\mu(k)|} = \exp(-2\pi i j/d_s) u_{s,j}$. In terms of this, one has ([15, Theorem 5.2]):

THEOREM 3.7. *The field $K_W := k(\mathbf{v})$ is a splitting field for $\mathcal{H}(W, \mathbf{u})$.*

In the rational case, when W is a Weyl group and $k = \mathbb{Q}$, we have $|\mu(k)| = 2$ and thus recover the classical result of Benson/Curtis and Lusztig.

It follows from Theorem 3.6, the fact that k is a splitting field for W, and Tits' deformation theorem that $\mathcal{H}_{K_W}$ is isomorphic to the group algebra $K_W W$. Thus any extension to $\mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}]$ of the specialization (3.2) defines a bijection $\mathrm{Irr}(W) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{H}_{K_W})$, $\chi \mapsto \chi_{\mathbf{v}}$, between $\mathrm{Irr}(W)$ and $\mathrm{Irr}(\mathcal{H}_{K_W})$.

Since the group algebra $K_W W$ is symmetric, the same is true for $\mathcal{H}_{K_W}$. But in fact, it is known for all but finitely many irreducible W that this statement already holds over A (see [17]):

THEOREM 3.8. *The cyclotomic algebra $\mathcal{H}(W, \mathbf{u})$ is a symmetric algebra over A.*

Let us choose a symmetric form $\langle\ ,\ \rangle : \mathcal{H} \otimes \mathcal{H} \to A$ on $\mathcal{H}$ with Gram matrix invertible over A such that the associated trace form $t_{\mathbf{u}} : \mathcal{H}(W, \mathbf{u}) \to A$ defined by $t_{\mathbf{u}}(h) := \langle 1, h \rangle$ under (3.2) specializes to the canonical trace form on the group algebra of W. Over the splitting field K_W of $\mathcal{H}(W, \mathbf{u})$, we may write $t_{\mathbf{u}}$ as a sum over the irreducible characters of $\mathcal{H}_{K_W}$ with non-vanishing coefficients, so

$$t_{\mathbf{u}} = \sum_{\chi \in \mathrm{Irr}(W)} \frac{1}{c_\chi} \chi_{\mathbf{v}}\,,$$

where c_χ is integral over A. The c_χ are called *Schur elements* of $\mathcal{H}(W, \mathbf{u})$ (with respect to $t_{\mathbf{u}}$).

3.9. SPETSIAL REFLECTION GROUPS. We now come to an important property of some complex reflection groups which seems to lie at the heart of the existence of unipotent degrees. Let W be a finite complex reflection group defined over k, $\mathbb{Z}_k$ the ring of integers of k, $\mathcal{H}(W, x)$ the 1-parameter cyclotomic algebra (3.3) over $\mathbb{Z}[x, x^{-1}]$, $k(y)$ with $y^{|\mu(k)|} = x$ a splitting field for $\mathcal{H}(W, x)$ (see Theorem 3.7). Let $\chi \in \mathrm{Irr}(W)$. The *generic degree* $\delta_\chi := P(W)/c_\chi$ *of* χ is a Laurent polynomial in y. We write $a(\chi)$ for the order of zero of δ_χ at $y = 0$ divided by $|\mu(k)|$, and $b(\chi)$ for the order of zero of the fake degree R_χ at $x = 0$. If $a(\chi) = b(\chi)$ then χ is called *special*. In all cases where the generic degrees are explicitly known, the following can be checked (see [16]):

PROPOSITION 3.10. *The following are equivalent:*

(i) for all $\chi \in \mathrm{Irr}(W)$ *there exists a special* $\psi \in \mathrm{Irr}(W)$ *with* $a(\chi) = a(\psi)$*;*
(ii) (rationality) $\delta_\chi \in k(x)$ *for all* $\chi \in \mathrm{Irr}(W)$*;*
(iii) (integrality) $\delta_\chi \in k[y]$ *for all* $\chi \in \mathrm{Irr}(W)$*;*
(iv) $\delta_1 = 1$*;*
(v) (representability) the k*-subspaces* $\langle \delta_\chi \mid \chi \rangle$ *and* $\langle R_\chi \mid \chi \rangle$ *of* $k(y)$ *coincide.*

Except for the obvious implications, no a priori proof is known for any of these statements.

A reflection group satisfying the above (very special) equivalent conditions will be called *spetsial*. It can be checked that all parabolic subgroups of spetsial reflection groups are again spetsial. A reflection datum with spetsial reflection group is called a *spets*. Thus, Levi subdata of spetses are again spetses.

The spetsial reflection groups include all the real ones, as well as all those irreducible n-dimensional ones which are generated by n reflections of order two. The complete list can be found in [16].

4. UNIPOTENT DEGREES

4.1. UNIPOTENT CHARACTERS. Let $\mathbb{G} = (V, W\phi)$ be a spets with rational Weyl group W and $\{\mathbb{G}(q) \mid q \text{ prime power}\}$ an associated family of finite groups of Lie type as in Section 2.3. By the results of Lusztig the unipotent characters $\mathcal{E}(\mathbb{G}(q))$ of the groups $\mathbb{G}(q)$ can be parametrized by a set $\mathcal{E}(\mathbb{G})$ independent of q, and there is a function $\deg : \mathcal{E}(\mathbb{G}) \to k[x]$, $\gamma \mapsto \deg(\gamma)$, such that for any choice of q there is a bijection $\psi_q^{\mathbb{G}} : \mathcal{E}(\mathbb{G}) \to \mathcal{E}(\mathbb{G}(q))$ such that $\psi_q^{\mathbb{G}}(\gamma)$ has degree $\psi_q^{\mathbb{G}}(\gamma)(1) = \deg(\gamma)(q)$ (see [11,3]). Then $\mathcal{E}(\mathbb{G})$ is called the set of *(generic) unipotent characters of* $\mathbb{G}$. This set has many interesting combinatorial properties, some of which we will now describe. We formulate these for the case that W is a Weyl group, $k = \mathbb{Q}$, but the reader should already have in mind the case of a more general reflection group.

4.2. GENERALIZED HARISH-CHANDRA THEORY. The functors of Lusztig induction and restriction give, for any Levi subspets $\mathbb{L}$ of $\mathbb{G}$ linear maps

$$R_{\mathbb{L}}^{\mathbb{G}} : \mathbb{Z}\mathcal{E}(\mathbb{L}) \to \mathbb{Z}\mathcal{E}(\mathbb{G}), \quad {}^*R_{\mathbb{L}}^{\mathbb{G}} : \mathbb{Z}\mathcal{E}(\mathbb{G}) \to \mathbb{Z}\mathcal{E}(\mathbb{L}),$$

satisfying $\psi_q^{\mathbb{G}} \circ R_{\mathbb{L}}^{\mathbb{G}} = R_{\mathbb{L}(q)}^{\mathbb{G}(q)} \circ \psi_q^{\mathbb{L}}$ for all q (when extending $\psi_q^{\mathbb{G}}$ linearly to $\mathbb{Z}\mathcal{E}(\mathbb{G})$), where $R_{\mathbb{L}(q)}^{\mathbb{G}(q)}$ denotes Lusztig induction between finite reductive groups $\mathbb{L}(q) \leq \mathbb{G}(q)$ associated to $\mathbb{L}, \mathbb{G}$.

Let Φ be a cyclotomic polynomial over k dividing $|\mathbb{G}|$. A unipotent character $\gamma \in \mathcal{E}(\mathbb{G})$ is called Φ-*cuspidal* if ${}^*R_{\mathbb{L}}^{\mathbb{G}}(\gamma) = 0$ for any Φ-split proper Levi subspets $\mathbb{L}$ of $\mathbb{G}$. It can be shown that this is equivalent to γ being of central Φ-defect, that is, to $|\mathbb{G}_{\rm ss}|/\deg(\gamma)$ not being divisible by Φ. Here, $\mathbb{G}_{\rm ss}$ is the semisimple quotient $(V/V^W, W\phi)$ of $\mathbb{G}$.

A pair $(\mathbb{L}, \lambda)$ consisting of a Φ-split Levi subspets of $\mathbb{G}$ and a unipotent character $\lambda \in \mathcal{E}(\mathbb{L})$ is called a Φ-*split pair*. It is called Φ-*cuspidal* if moreover λ is Φ-cuspidal. Let $(\mathbb{M}_1, \mu_1)$ and $(\mathbb{M}_2, \mu_2)$ be Φ-split in $\mathbb{G}$. Then we say that $(\mathbb{M}_1, \mu_1) \leq_\Phi (\mathbb{M}_2, \mu_2)$ if $\mathbb{M}_1$ is a Φ-split Levi subspets of $\mathbb{M}_2$ and μ_2 occurs in $R_{\mathbb{M}_1}^{\mathbb{M}_2}(\mu_1)$.

For a Φ-cuspidal pair $(\mathbb{L}, \lambda)$ of $\mathbb{G}$ we write

$$\mathcal{E}(\mathbb{G}, (\mathbb{L}, \lambda)) := \{\gamma \in \mathcal{E}(\mathbb{G}) \mid (\mathbb{L}, \lambda) \leq_\Phi (\mathbb{G}, \gamma)\}$$

for the set of unipotent characters of $\mathbb{G}$ lying above $(\mathbb{L}, \lambda)$. We call $\mathcal{E}(\mathbb{G}, (\mathbb{L}, \lambda))$ the Φ-*Harish-Chandra series above* $(\mathbb{L}, \lambda)$ because of the following fundamental result (see [3]), which is a complete analogue of the usual Harish-Chandra theory (the case $\Phi = x - 1$):

THEOREM 4.3. *(a) (Disjointness) The sets* $\mathcal{E}(\mathbb{G}, (\mathbb{L}, \lambda))$ *(where* $(\mathbb{L}, \lambda)$ *runs over a system of representatives of the* $W_{\mathbb{G}}$*-conjugacy classes of* Φ*-cuspidal pairs) form a partition of* $\mathcal{E}(\mathbb{G})$*.*

(b) (Transitivity) Let $(\mathbb{L}, \lambda)$ *be* Φ*-cuspidal and* $(\mathbb{M}, \mu)$ *be* Φ*-split such that* $(\mathbb{L}, \lambda) \leq_\Phi (\mathbb{M}, \mu)$ *and* $(\mathbb{M}, \mu) \leq_\Phi (\mathbb{G}, \gamma)$*. Then* $(\mathbb{L}, \lambda) \leq_\Phi (\mathbb{G}, \gamma)$*.*

4.4. PERFECT ISOMETRIES. The only known proof of Theorem 4.3 in the case $\Phi \neq x - 1$ consists in the explicit determination of the Lusztig induced of Φ-cuspidal unipotent characters. To state this result from [3] we need to introduce an important invariant of a Φ-Harish-Chandra series. Let $(\mathbb{L}, \lambda)$ be a Φ-cuspidal pair in $\mathbb{G}$. By results of Lusztig it is possible to define an action of $N_W(W_{\mathbb{L}})/W_{\mathbb{L}}$ on $\mathcal{E}(\mathbb{L})$ which is the generic version of the corresponding actions in the series of finite groups of Lie type attached to $\mathbb{G}$ (see [3]). We then call $W_{\mathbb{G}}(\mathbb{L}, \lambda) := N_W(W_{\mathbb{L}}, \lambda)/W_{\mathbb{L}}$ the *relative Weyl group of* $(\mathbb{L}, \lambda)$ *in* $\mathbb{G}$.

THEOREM 4.5. *For each* Φ *there exists a collection of isometries*

$$I_{(\mathbb{L},\lambda)}^{\mathbb{M}} : \mathbb{Z}\mathrm{Irr}(W_{\mathbb{M}}(\mathbb{L}, \lambda)) \to \mathbb{Z}\mathcal{E}(\mathbb{M}, (\mathbb{L}, \lambda)),$$

such that for all $\mathbb{M}$ *and all* $(\mathbb{L}, \lambda)$ *we have*

$$R_{\mathbb{M}}^{\mathbb{G}} \circ I_{(\mathbb{L},\lambda)}^{\mathbb{M}} = I_{(\mathbb{L},\lambda)}^{\mathbb{G}} \circ \mathrm{Ind}_{W_{\mathbb{M}}(\mathbb{L},\lambda)}^{W_{\mathbb{G}}(\mathbb{L},\lambda)}.$$

Here $\mathbb{M}$ *runs over the* Φ*-split Levi subgroups of* $\mathbb{G}$ *and* $(\mathbb{L}, \lambda)$ *over the set of* Φ*-cuspidal pairs of* $\mathbb{M}$*.*

For $\Phi = x - 1$ and $\lambda = 1$, this gives an embedding $\mathrm{Irr}(W^\phi) \subseteq \mathcal{E}(\mathbb{G})$; the image consists of the so-called principal series unipotent characters.

4.6. Generalized Howlett/Lehrer-Lusztig theory. A deeper understanding of Theorem 4.5 can be gained starting from the following surprising fact (which is verified in a case-by-case analysis, see [2]; for a general argument in the case $\lambda = 1$ see [9]):

Proposition 4.7. *For any Φ-cuspidal pair $(\mathbb{L}, \lambda)$ of $\mathbb{G}$ the relative Weyl group $W_{\mathbb{G}}(\mathbb{L}, \lambda)$ is a complex reflection group. Moreover, if W acts irreducibly on V then $W_{\mathbb{G}}(\mathbb{L}, \lambda)$ is also irreducible in its natural reflection representation.*

In particular there is a cyclotomic algebra $\mathcal{H}(W_{\mathbb{G}}(\mathbb{L}, \lambda))$ attached to the relative Weyl group. In view of this, Theorem 4.5 can be considered as the shadow of an even more precise result on the decomposition of $R_{\mathbb{L}}^{\mathbb{G}}$ which was verified in [2, 14] for all the cases where the Schur elements are known:

Theorem 4.8. *For any Φ dividing $|\mathbb{G}|$ and any Φ-cuspidal pair $(\mathbb{L}, \lambda)$ of $\mathbb{G}$ there exists an admissible specialization $f_{(\mathbb{L},\lambda)} : \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}] \to \mathbb{C}[x]$ of the cyclotomic Hecke algebra $\mathcal{H}(W_{\mathbb{G}}(\mathbb{L}, \lambda), \mathbf{u})$, such that for all $\chi \in \mathrm{Irr}(W_{\mathbb{G}}(\mathbb{L}, \lambda))$*

$$\deg(I_{(\mathbb{L},\lambda)}^{\mathbb{G}}(\chi)) = \frac{|\mathbb{G}|_{x'}}{|\mathbb{L}|_{x'}} \frac{\deg(\lambda)}{f_{(\mathbb{L},\lambda)}(c_\chi)} \, .$$

The specialization $f_{(\mathbb{L},\lambda)}$ is determined locally, that is, by situations in which $W_{\mathbb{G}}(\mathbb{L}, \lambda)$ is a 1-dimensional reflection group. Furthermore, in the case $\phi = 1$, for $\Phi = x - 1$ and $\lambda = 1$, $f_{(\mathbb{L},\lambda)}$ is the 1-parameter specialization (3.3), which shows that the degrees of the principal series unipotent characters are the generic degrees of $\mathcal{H}(W, x)$.

This is precisely the formula one would obtain if the specialization of the cyclotomic algebra $\mathcal{H}(W_{\mathbb{G}}(\mathbb{L}, \lambda), \mathbf{u})$ of the relative Weyl group were the endomorphism algebra of $R_{\mathbb{L}(q)}^{\mathbb{G}(q)}(\psi_q^{\mathbb{L}}(\lambda))$ (see [2] and also [6] for a conjectural explanation).

4.9. Unipotent degrees. Let now $\mathbb{G} = (V, W\phi)$ be an arbitrary spets, with spetsial reflection group W over k. Assume given for any Levi subspets $\mathbb{L} = (V, W_{\mathbb{L}} w\phi)$ of $\mathbb{G}$ a set $\mathcal{E}(\mathbb{L})$ with an action of $N_W(W_{\mathbb{L}})$, a degree map $\deg : \mathcal{E}(\mathbb{L}) \to k[x]$, and for any Φ and any pair $\mathbb{L} \leq \mathbb{M}$ of Φ-split Levi subspetses two homomorphisms

$$R_{\mathbb{L}}^{\mathbb{M}} : \mathbb{Z}\mathcal{E}(\mathbb{L}) \to \mathbb{Z}\mathcal{E}(\mathbb{M}), \quad {}^*R_{\mathbb{L}}^{\mathbb{M}} : \mathbb{Z}\mathcal{E}(\mathbb{M}) \to \mathbb{Z}\mathcal{E}(\mathbb{L}),$$

adjoint to each other with respect to the scalar products for which $\mathcal{E}(\mathbb{L})$, $\mathcal{E}(\mathbb{M})$ are orthonormal. If these data satisfy the analogues of Theorems 4.5 and 4.8 (and hence also of Theorem 4.3) then we say that $\mathbb{G}$ *has unipotent degrees* attached to it. Thus, by the above, spetses for Weyl groups have unipotent degrees. Amazingly, this property is shared by all spetsial reflection groups, even the non-real ones:

Theorem 4.10. *Spetses have unipotent degrees.*

For the explicit construction of the sets $\mathcal{E}(\mathbb{G})$ see [11] for real reflection groups, [14] for the infinite series (where also the properties in Theorem 4.5 and 4.8 are verified for the infinite series of Weyl groups) and [4] for the exceptional spetses. Note that non spetsial reflection data cannot have unipotent degrees by Proposition 3.10.

5. Fusion rules

5.1. Fourier transforms. Let $\mathbb{G} = (V, W\phi)$ be a spets and $\mathcal{E}(\mathbb{G})$ be the associated set of unipotent degrees. By Proposition 3.10 the generic degrees lie in the space spanned by the fake degrees $\{R_\chi\}$. But more is true: all unipotent degrees lie in this space. This gives rise to further fascinating properties of unipotent degrees. To explain these, let us return to the case of rational spetses $\mathbb{G}$ originating from families of groups of Lie type, as in Section 2.3. For simplicity of exposition, let us also assume that $\phi = 1$.

By the fundamental results of Lusztig, the subspace of the space of class functions spanned by the unipotent characters coincides with the space spanned by the so-called unipotent almost characters, and both sets form orthonormal bases of this subspace. Let S denote the corresponding base change matrix, the *Fourier transform matrix*. Then S is unitary and of order 2. Moreover, S transforms the vector of unipotent degrees of $\mathbb{G}$ into the vector of degrees of almost characters, that is, a vector consisting of the fake degrees of W, extended by a suitable number of zeros. Furthermore, Lusztig associates to each unipotent character a root of unity, the so-called eigenvalue of Frobenius. Let F denote the diagonal matrix formed by these Frobenius eigenvalues. Then $(FS)^3 = 1$, hence F and S give rise to a representation of the modular group $\mathrm{PSL}_2(\mathbb{Z})$.

A similar situation occurs in the case of arbitrary spetses: There exists a matrix S over k which is symmetric, unitary, and which transforms the unipotent degrees into the fake degrees of W. Moreover, to each $\gamma \in \mathcal{E}(\mathbb{G})$ can be attached an eigenvalue of Frobenius (a root of unity), such that S together with the diagonal matrix F formed by the Frobenius eigenvalues satisfy:

$$S^4 = 1, \quad (F\,S)^3 = 1, \quad [F, S^2] = 1,$$

hence S, F give rise to a representation of $\mathrm{SL}_2(\mathbb{Z})$.

5.2. Families. In the case of spetses coming from groups of Lie type, the Fourier matrix has a block diagonal shape, with blocks given by the cells, or families, of the Weyl group. In the general case, a similar statement holds. There exists a partition of $\mathcal{E}(\mathbb{G})$ into *families*, such that the a-function is constant on families, each family contains a unique special character, and S becomes block diagonal with respect to this partition. Via the embedding in Theorem 4.5 this induces a subdivision of $\mathrm{Irr}(W)$ into families, which is not yet well understood, since the concept of cells does not seem to generalize easily to complex reflection groups.

In the case of Coxeter groups W, two characters $\chi, \chi' \in \mathrm{Irr}(W)$ lie in the same family if and only if the corresponding characters of the 1-parameter Hecke algebra $\mathcal{H}(W, x)_\mathcal{O}$ lie in the same block, with $\mathcal{O} := A[(1 + x\mathbb{Z}[x])^{-1}]$ (as was pointed out by Raphaël Rouquier; see also [8]). We expect the same statement to be true in the case of arbitrary spetsial reflection groups. It seems interesting to study the projective characters of these blocks.

Let $\mathcal{F} \subseteq \mathcal{E}(\mathbb{G})$ be a family with Fourier matrix $S_\mathcal{F} = (s_{jk})_{j,k\in\mathcal{F}}$. We define structure constants

$$n_{jk}^l := \sum_{m\in\mathcal{F}} \frac{s_{jm} s_{km} \overline{s_{lm}}}{s_{j_0 m}} \qquad \text{for } j, k, l \in \mathcal{F}, \tag{5.3}$$

where $j_0 \in \mathcal{F}$ is the special character (by the above, $s_{j_0 m} \neq 0$ for all $m \in \mathcal{F}$). In the case of finite Coxeter groups, all structure constants are non-negative integers (see [10, 12, 13]). In the complex case, the following slightly weaker result holds:

Theorem 5.4. *The structure constants of families of spetses are rational integers.*

The proof is case by case and will be published elsewhere. This result shows that (apart from the missing positivity) the data $S_{\mathcal{F}}, F_{\mathcal{F}}$ of a family satisfy the axioms of a fusion rule (see e.g. [7]). In particular, for any family $\mathcal{F}$ of a spets $\mathbb{G}$ we obtain an associative, commutative ring $B(\mathcal{F})$ with 1, free and indecomposable over $\mathbb{Z}$ with basis indexed by $\mathcal{F}$ and structure constants given by (5.3).

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