The Intrinsic Hodge Theory of p-Adic Hyperbolic Curves

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§1. Introduction

(A.) The Fuchsian Uniformization

A *hyperbolic curve* is an algebraic curve obtained by removing r points from a smooth, proper curve of genus g, where g and r are nonnegative integers such that $2g - 2 + r > 0$. If X is a hyperbolic curve over the field of complex numbers C, then X gives rise in a natural way to a Riemann surface $\mathcal{X}$. As one knows from complex analysis, the most fundamental fact concerning such a Riemann surface (due to Köbe) is that it may be *uniformized by the upper half-plane*, i.e.,

$$\mathcal{X} \cong \mathfrak{H}/\Gamma$$

where $\mathfrak{H} \stackrel{\text{def}}{=} \{z \in C \mid \text{Im}(z) > 0\}$, and $\Gamma \cong \pi_1(\mathcal{X})$ (the topological fundamental group of $\mathcal{X}$) is a discontinuous group acting on $\mathfrak{H}$. Note that the action of Γ on $\mathfrak{H}$ defines a canonical representation

$$\rho_{\mathcal{X}} : \pi_1(\mathcal{X}) \to PSL_2(R) \stackrel{\text{def}}{=} SL_2(R)/\{\pm 1\} = \text{Aut}_{\text{Holomorphic}}(\mathfrak{H})$$

The goal of the present manuscript is to survey various work ([Mzk1-5]) devoted to generalizing Köbe's uniformization to the p-adic case.

First, we observe that it is not realistic to expect that hyperbolic curves over p-adic fields may be literally uniformized by some sort of p-adic upper half-plane in the fashion of the Köbe uniformization. Of course, one has the theory of Mumford ([Mumf]), but this theory furnishes a p-adic analogue not of Köbe's *Fuchsian uniformization* (i.e., uniformization by a Fuchsian group), but rather of what in the complex case is known as the *Schottky uniformization*. Even in the complex case, the Fuchsian and Schottky uniformizations are fundamentally different: For instance, as the moduli of the curve vary, its Schottky periods vary holomorphically, whereas its Fuchsian periods vary only *real analytically*. This fact already suggests that the Fuchsian uniformization is of a more *arithmetic nature* than the Schottky uniformization, i.e., it involves

real analytic structures $\Longleftrightarrow$ complex conjugation $\Longleftrightarrow$
Frobenius at the infinite prime

Thus, since one cannot expect a p-adic analogue in the form of a literal global uniformization of the curve, the first order of business is to reinterpret the Fuchsian uniformization in more abstract terms that generalize naturally to the p-adic setting.

(B.) The Physical Interpretation

The first and most obvious approach is to observe that the Fuchsian uniformization gives a new physical, geometric way to reconstruct the original *algebraic curve* X. Namely, one may think of the Fuchsian uniformization as defining a *canonical arithmetic structure* $\rho_{\mathcal{X}} : \pi_1(\mathcal{X}) \to PSL_2(R)$ on the purely *topological* invariant $\pi_1(\mathcal{X})$. Alternatively (and essentially equivalently), one may think of the Fuchsian uniformization as the datum of a *metric* (given by descending to $\mathcal{X} \cong \mathfrak{H}/\Gamma$ the Poincaré metric on $\mathfrak{H}$) – i.e., an *arithmetic* (in the sense of arithmetic at the infinite prime) structure – on the differential manifold underlying $\mathcal{X}$ (which is a purely *topological* invariant). Then the equivalence

$$X \Longleftrightarrow SO(2)\backslash PSL_2(R)/\Gamma$$

between the *algebraic* curve X and the physical/analytic object $SO(2)\backslash PSL_2(R)/\Gamma$ obtained from $\rho_{\mathcal{X}}$ is given by considering *modular forms* on $\mathfrak{H} = SO(2)\backslash PSL_2(R)$, which define a projective (hence, *algebraizing*) embedding of $\mathcal{X}$.

(C.) The Modular Interpretation

Note that $\rho_{\mathcal{X}}$ may also be regarded as a representation into $PGL_2(C) = GL_2(C)/C^{\times}$, hence as defining an action of $\pi_1(\mathcal{X})$ on P^1_C. Taking the quotient of $\mathfrak{H} \times P^1_C$ by the action of $\pi_1(\mathcal{X})$ on both factors then gives rise to a projective bundle with connection on $\mathcal{X}$. It is immediate that this projective bundle and connection may be algebraized, so we thus obtain a projective bundle and connection $(P \to X, \nabla_P)$ on X. This pair (P, ∇_P) has certain properties which make it an *indigenous bundle* (terminology due to Gunning). More generally, an indigenous bundle on $\mathcal{X}$ may be thought of as a *projective structure* on $\mathcal{X}$, i.e., a subsheaf of the sheaf of holomorphic functions on $\mathcal{X}$ such that locally any two sections of this subsheaf are related by a linear fractional transformation. Thus, the Fuchsian uniformization defines a special *canonical indigenous bundle* on X.

In fact, the notion of an indigenous bundle is entirely *algebraic*. Thus, one has a natural moduli stack $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$ of indigenous bundles, which forms a torsor (under the affine group given by the sheaf of differentials on $\mathcal{M}_{g,r}$) – called the *Schwarz torsor* – over the moduli stack $\mathcal{M}_{g,r}$ of hyperbolic curves of type (g, r). Moreover, $\mathcal{S}_{g,r}$ is not only algebraic, it is defined over $Z[\frac{1}{2}]$. Thus, the canonical indigenous bundle defines a *canonical real analytic section*

$$s : \mathcal{M}_{g,r}(C) \to \mathcal{S}_{g,r}(C)$$

of the Schwarz torsor at the infinite prime. Moreover, not only does s "contain" all the information that one needs to define the Fuchsian uniformization of an individual hyperbolic curve (indeed, this much is obvious from the definition of s!), it also essentially "is" (interpreted properly) the Bers uniformization of the universal covering space (i.e., "Teichmüller space") of $\mathcal{M}_{g,r}(C)$ (cf. the discussions in the Introductions of [Mzk1,4]). That is to say, from this point of view, one may regard the uniformization theory of hyperbolic curves and their moduli as the study of the canonical section s. Alternatively, from the point of view of Teichmüller theory, one may regard the uniformization theory of hyperbolic curves and their moduli as the theory of (so-called) *quasi-fuchsian deformations of the representation* $\rho_{\mathcal{X}}$.

(D.) The Notion of "Intrinsic Hodge Theory"

Note that both the physical and modular approaches to the Fuchsian uniformization assert that there is a certain equivalence

$$\text{algebraic geometry} \iff \text{topology endowed with an arithmetic structure}$$

That is, on the algebraic geometry side, we have the scheme (respectively, stack) given by the curve X itself in the physical approach (respectively, its moduli $\mathcal{M}_{g,r}$ in the modular approach), whereas on the "topology plus arithmetic structure" side, we have the theory of the canonical representation $\rho_{\mathcal{X}}$ of $\pi_1(\mathcal{X})$ (i.e., $SO(2)\backslash PSL_2(R)/\Gamma$ in the physical approach; quasi-fuchsian deformations of $\rho_{\mathcal{X}}$ in the modular approach). This sort of equivalence is reminiscent of that given by classical or p-adic *Hodge theory* between the de Rham or Hodge cohomology of an algebraic variety (on the algebraic geometry side), and the singular or étale cohomology (equipped with Galois action) on the topology plus arithmetic side. In our case, however, instead of dealing with the cohomology of the curve, we are dealing with "the curve itself" and its moduli. It is for this reason that we refer to this sort of theory as the *intrinsic Hodge theory* of the curve X.

Finally, we note that this formal analogy with classical/p-adic Hodge theory is by no means merely philosophical. Indeed, even in the classical theory reviewed in (B.) and (C.) above, the methods of classical Hodge theory play an important technical role in the proofs of the main theorems. Similarly, in the theory of [Mzk1-5] – which constitute our main examples of *intrinsic Hodge theory for hyperbolic curves* – the more recently developed techniques of p-adic Hodge theory play a crucial technical role in the proofs of the main results.

§2. The Physical Approach in the p-adic Case

(A.) The Arithmetic Fundamental Group

Let K be a field of characteristic zero. Let us denote by $\overline{K}$ an algebraic closure of K. Let $\Gamma_K \stackrel{\text{def}}{=} \text{Gal}(\overline{K}/K)$. Let X_K be a *hyperbolic curve over* K; write $X_{\overline{K}} \stackrel{\text{def}}{=} X \times_K \overline{K}$. Then one has an exact sequence

$$1 \to \pi_1(X_{\overline{K}}) \to \pi_1(X_K) \to \Gamma_K \to 1$$

of *algebraic fundamental groups*. (Here, we omit the base-points from the notation for the various fundamental groups.)

We shall refer to $\pi_1(X_{\overline{K}})$ as the *geometric fundamental group* of X_K. Note that the structure of $\pi_1(X_{\overline{K}})$ is determined entirely by (g, r) (i.e., the "type" of the hyperbolic curve X_K). In particular, $\pi_1(X_{\overline{K}})$ *does not depend on the moduli of* X_K. Of course, this results from the fact that K is of *characteristic zero*; in positive characteristic, on the other hand, preliminary evidence ([Tama2]) suggests that the fundamental group of a hyperbolic curve over an algebraically closed field (far from being independent of the moduli of the curve!) may in fact *completely determine* the moduli of the curve.

On the other hand, we shall refer to $\pi_1(X_K)$ (equipped with its augmentation to Γ_K) as the *arithmetic fundamental group* of X_K. Although it is made up of two "parts" – i.e., $\pi_1(X_{\overline{K}})$ and Γ_K – which do not depend on the moduli of X_K, it is not unreasonable to expect that the extension class defined by the above exact sequence, i.e., the structure of $\pi_1(X_K)$ as a group equipped with augmentation to Γ_K, may in fact depend quite strongly on the moduli of X_K. Indeed, according to the *anabelian philosophy* of Grothendieck (cf. [LS]), for "sufficiently arithmetic" K, one expects that *the structure of the arithmetic fundamental group* $\pi_1(X_K)$ *should be enough to determine the moduli of* X_K. Although many important versions of Grothendieck's anabelian conjectures remain unsolved (most notably the so-called *Section Conjecture* (cf., e.g., [LS], p. 289, 2)), in the remainder of this §, we shall discuss various versions that have been resolved in the affirmative. Finally, we note that this anabelian philosophy is a special case of the notion of "intrinsic Hodge theory" discussed above: indeed, on the algebraic geometry side, one has "the curve itself," whereas on the topology plus arithmetic side, one has the arithmetic fundamental group, i.e., the purely (étale) topological $\pi_1(X_{\overline{K}})$, equipped with the structure of extension given by the above exact sequence.

(B.) The Main Theorem

Building on earlier work of H. Nakamura and A. Tamagawa (see, especially, [Tama1]), the author applied the p-adic Hodge theory of [Falt2] and [BK] to prove the following result (cf. Theorem A of [Mzk5]):

Theorem 1. *Let p be a prime number. Let K be a subfield of a finitely generated field extension of Q_p. Let X_K be a hyperbolic curve over K. Then for any smooth variety S_K over K, the natural map*

$$X_K(S_K)^{\mathrm{dom}} \to \mathrm{Hom}^{\mathrm{open}}_{\Gamma_K}(\pi_1(S_K), \pi_1(X_K))$$

is bijective. Here, the superscripted "dom" denotes dominant ($\Longleftrightarrow$ nonconstant) K-morphisms, while $\mathrm{Hom}^{\mathrm{open}}_{\Gamma_K}$ denotes open, continuous homomorphisms compatible with the augmentations to Γ_K, and considered up to composition with an inner automorphism arising from $\pi_1(X_{\overline{K}})$.

Note that this result constitutes an analogue of the "physical aspect" of the Fuchsian uniformization, i.e., it exhibits the *scheme* X_K (in the sense of the functor defined by considering (nonconstant) K-morphisms from arbitrary smooth S_K to X_K) as equivalent to the "physical/analytic object"

$$\mathrm{Hom}^{\mathrm{open}}_{\Gamma_K}(-, \pi_1(X_K))$$

defined by the topological $\pi_1(X_{\overline{K}})$ together with some additional canonical arithmetic structure (i.e., $\pi_1(X_K)$).

In fact, the proof of Theorem 1 was also motivated by this point of view: That is to say, just as one may regard the algebraic structure of a hyperbolic curve over C as being defined by certain (a priori) *analytic* modular forms on $\mathfrak{H}$, the proof of Theorem 1 proceeds by considering certain p-adic analytic representations of differential forms on X_K. In the p-adic case, however, the domain of definition of these analytic forms (i.e., the analogue to the upper half-plane) is the spectrum of the p-adic completion of the maximal tame extension of the function field of X_K along various irreducible components of the special fiber of a stable model $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$ of X_K (where $\mathcal{O}_K$ is the ring of integers of a finite extension K of Q_p). It turns out that this object is, just like the upper half-plane, independent of the moduli of X_K.

In fact, various slightly stronger versions of Theorem 1 hold. For instance, instead of the whole geometric fundamental group $\pi_1(X_{\overline{K}})$, it suffices to consider its maximal pro-p quotient $\pi_1(X_{\overline{K}})^{(p)}$. Another strengthening allows one to prove the following result (cf. Theorem B of [Mzk5]), which generalizes a result of Pop ([Pop]):

Corollary 2. *Let p be a prime number. Let K be a subfield of a finitely generated field extension of Q_p. Let L and M be function fields of arbitrary dimension over K. Then the natural map*

$$\mathrm{Hom}_K(\mathrm{Spec}(L), \mathrm{Spec}(M)) \to \mathrm{Hom}^{\mathrm{open}}_{\Gamma_K}(\Gamma_L, \Gamma_M)$$

is bijective. Here, $\mathrm{Hom}^{\mathrm{open}}_{\Gamma_K}(\Gamma_L, \Gamma_M)$ is the set of open, continuous group homomorphisms $\Gamma_L \to \Gamma_M$ over Γ_K, considered up to composition with an inner homomorphism arising from $\mathrm{Ker}(\Gamma_M \to \Gamma_K)$.

(C.) Comparison with the Case of Abelian Varieties

Note that there is an obvious formal analogy between Theorem 1 above and Tate's conjecture on homomorphisms between abelian varieties (cf., e.g., [Falt1]). Indeed, in discussions of Grothendieck's anabelian philosophy, it was common to refer to statements such as that of Theorem 1 as the "anabelian Tate conjecture," or the "Tate conjecture for hyperbolic curves." In fact, however, there is an important difference between Theorem 1 and the "Tate conjecture" of, say, [Falt1]: Namely, *the Tate conjecture for abelian varieties is false over local fields (i.e., finite extensions of Q_p)*. Moreover, until the proof of Theorem 1, it was generally thought that, just like its abelian cousin, the "anabelian Tate conjecture" was essentially global in nature. That is to say, it appears that the point of view of the author, i.e., that Theorem 1 should be regarded as a p-adic version of the "physical aspect" of the Fuchsian uniformization of a hyperbolic curve, does not exist in the literature (prior to the work of the author).

§3. The Modular Approach in the p-adic Case

(A.) The Example of Shimura Curves

As discussed in §1, (C.), classical complex Teichmüller theory may be formulated as the study of the canonical real analytic section s of the Schwarz torsor $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$. Thus, it is natural suppose that the p-adic analogue of classical Teichmüller theory should revolve around some sort of *canonical p-adic section* of the Schwarz torsor. Then the question arises:

How does one define a canonical p-adic section of the Schwarz torsor?

Put another way, for each (or at least most) p-adic hyperbolic curves, we would like to associate a (or at least a finite, bounded number of) canonical indigenous bundles. Thus, we would like to know what sort of properties such a "canonical indigenous bundle" should have.

The model that provides the answer to this question is the theory of *Shimura curves*. In fact, the theory of canonical Schwarz structures, canonical differentials, and canonical coordinates on Shimura curves localized at finite primes has been extensively studied by Y. Ihara (see, e.g., [Ihara]). In some sense, *Ihara's theory provides the prototype for the "p-adic Teichmüller theory" of arbitrary hyperbolic curves* ([Mzk1-4]) to be discussed in (B.) and (C.) below. The easiest example of a Shimura curve is $\mathcal{M}_{1,0}$, the moduli stack of elliptic curves. In this case, the projectivization of the rank two bundle on $\mathcal{M}_{1,0}$ defined by the first de Rham cohomology module of the universal elliptic curve on $\mathcal{M}_{1,0}$ gives rise (when equipped with the Gauss-Manin connection) to the canonical indigenous bundle on $\mathcal{M}_{1,0}$. Moreover, it is well-known that the *p-curvature* (a canonical invariant of bundles with connection in positive characteristic which measures the extent to which the connection is compatible with Frobenius) of this bundle has the following property:

The p-curvature of the canonical indigenous bundle on $\mathcal{M}_{1,0}$ (reduced mod p) is square nilpotent.

It was this observation that was the key to the development of the theory of [Mzk1-4].

(B.) The Stack of Nilcurves

Let p be an *odd* prime. Let $\mathcal{N}_{g,r} \subseteq (\mathcal{S}_{g,r})_{F_p}$ denote the closed algebraic substack of indigenous bundles with square nilpotent p-curvature. Then one has the following key result ([Mzk1], Chapter II, Theorem 2.3):

Theorem 3. *The natural map $\mathcal{N}_{g,r} \to (\mathcal{M}_{g,r})_{F_p}$ is a finite, flat, local complete intersection morphism of degree p^{3g-3+r}. Thus, up to "isogeny" (i.e., up to the fact that this degree is not equal to one), $\mathcal{N}_{g,r}$ defines a canonical section of the Schwarz torsor $(\mathcal{S}_{g,r})_{F_p} \to (\mathcal{M}_{g,r})_{F_p}$ in characteristic p.*

It is this stack $\mathcal{N}_{g,r}$ of *nilcurves* – i.e., hyperbolic curves in characteristic p equipped with an indigenous bundle with square nilpotent p-curvature – which is the central object of study in the theory of [Mzk1-4].

Once one has the above Theorem, next it is natural to ask if one can say more about the fine structure of $\mathcal{N}_{g,r}$. Although many interesting and natural questions concerning the structure of $\mathcal{N}_{g,r}$ remain unsolved at the time of writing, a certain amount can be understood by analyzing certain *substacks, or strata,* of $\mathcal{N}_{g,r}$ defined by considering the loci of nilcurves whose p-curvature vanishes to a certain degree. For instance, nilcurves whose p-curvature vanishes identically are called *dormant*. The locus of dormant nilcurves is denoted $\mathcal{N}_{g,r}[\infty] \subseteq \mathcal{N}_{g,r}$. If a nilcurve is not dormant, then its p-curvature vanishes on some divisor in the curve. We denote by $\mathcal{N}_{g,r}[d] \subseteq \mathcal{N}_{g,r}$ the locus of nilcurves for which this divisor is of degree d. The zeroes of the p-curvature are referred to as *spikes*. Now we have the following result (cf. Theorems 1.2, 1.6 of the Introduction of [Mzk4]):

Theorem 4. *The $\mathcal{N}_{g,r}[d]$ are all smooth over F_p and either empty or of dimension $3g-3+r$. Moreover, $\mathcal{N}_{g,r}[0]$ is* affine.

It turns out that *this affineness of $\mathcal{N}_{g,r}[0]$, interpreted properly, gives a new proof of the connectedness of $(\mathcal{M}_{g,r})_{F_p}$* (for p large relative to g). This fact is interesting (relative to the claim that this theory is a p-adic version of Teichmüller theory) in that one of the first applications of classical complex Teichmüller theory is to prove the connectedness of $\mathcal{M}_{g,r}$. Also, it is interesting to note that F. Oort has succeeded in giving a proof of the connnectedness of the moduli stack of principally polarized abelian varieties by using affineness properties of certain natural substacks of this moduli stack in characteristic p.

Despite the fact that the $\mathcal{N}_{g,r}[d]$ are smooth and of the same dimension as $\mathcal{N}_{g,r}$, we remark that in most cases $\mathcal{N}_{g,r}$ *is not reduced at* $\mathcal{N}_{g,r}[d]$. In fact, roughly speaking, the larger d is, the less reduced $\mathcal{N}_{g,r}$ is at $\mathcal{N}_{g,r}[d]$. In order to give sharp quantitative answers to such questions as:

> *How reduced is $\mathcal{N}_{g,r}$ at the generic point of $\mathcal{N}_{g,r}[d]$? Or, what is the generic degree of $\mathcal{N}_{g,r}[d]$ over $(\mathcal{M}_{g,r})_{F_p}$?*

it is necessary to study what happens to a nilcurve as the underlying curve degenerates to a *totally degenerate stable curve* (i.e., a stable curve each of whose irreducible components is P^1, with a total of precisely three marked points/nodes). To do this, one must formulate the theory (using "log structures") in such a way that it applies to stable curves, as well.

Once one formulates the theory for stable curves, one sees that the answers to the questions just posed will follow as soon as one:

(i.) Classifies all *molecules* – i.e., nilcurves whose underlying curve is a totally degenerate stable curve.

(ii.) Understands how molecules *deform*.

The answer to (i.) and (ii.) depends on an extensive analysis of molecules (cf. [Mzk2-4]), and, although combinatorially quite complicated, is, in some sense, complete. Although we do not have enough space here to discuss this answer in detail, we pause to remark the following: It turns out that the answer to (i.) consists of regarding molecules as concatenations of *atoms* – i.e., *toral nilcurves* (a slight generalization of nilcurves) whose underlying curve is P^1 with three marked points – and then classifying atoms. The difference between a toral nilcurve and a (nontoral) nilcurve is that unlike the nontoral case, where the "radii" at the three marked points are assumed to be zero, in the toral case, one allows these radii to be arbitrary elements of $F_p/\{\pm 1\}$ (i.e., the quotient of the set F_p obtained by identifying λ and $-\lambda$ for all $\lambda \in F_p$). Then it turns out that considering the three radii of an atom defines a natural bijection between the isomorphism classes of atoms and the set of (ordered) triples of elements of $F_p/\{\pm 1\}$.

The reason that we digressed to discuss the theory of atoms is that it is interesting (relative to the analogy with classical complex Teichmüller theory) in that it is reminiscent of the fact that a Riemann surface may be analyzed by decomposing it into *pants* (i.e., Riemann surfaces which are topologically isomorphic to $P^1 - \{0, 1, \infty\}$). Moreover, the isomorphism class of a "pants" is completely determined by the radii of its three holes.

(C.) Canonical Liftings

So far, we have been discussing the characteristic p theory. Ultimately, however, we would like to know if the various characteristic p objects discussed in (B.) lift canonically to objects which are flat over Z_p. Unfortunately, it seems that it is unlikely that $\mathcal{N}_{g,r}$ itself lifts canonically to some sort of natural Z_p-flat object. If, however, we consider the open substack – called the *ordinary locus* – $(\mathcal{N}_{g,r}^{\text{ord}})_{F_p} \subseteq \mathcal{N}_{g,r}$ which is the étale locus of the morphism $\mathcal{N}_{g,r} \rightarrow (\mathcal{M}_{g,r})_{F_p}$, then (since the étale site is invariant under nilpotent thickenings) we get a canonical lifting, i.e., an étale morphism

$$\mathcal{N}_{g,r}^{\text{ord}} \rightarrow (\mathcal{M}_{g,r})_{Z_p}$$

of p-adic formal stacks. Over $\mathcal{N}_{g,r}^{\text{ord}}$, one has the sought-after canonical p-adic splitting of the Schwarz torsor (cf. Theorem 0.1 of the Introduction of [Mzk1]):

Theorem 5. *There is a canonical section $\mathcal{N}_{g,r}^{\text{ord}} \rightarrow \mathcal{S}_{g,r}$ of the Schwarz torsor over $\mathcal{N}_{g,r}^{\text{ord}}$ which is the unique section having the following property: There exists a lifting of Frobenius $\Phi_{\mathcal{N}} : \mathcal{N}_{g,r}^{\text{ord}} \rightarrow \mathcal{N}_{g,r}^{\text{ord}}$ such that the indigenous bundle on the tautological hyperbolic curve over $\mathcal{N}_{g,r}^{\text{ord}}$ defined by the section $\mathcal{N}_{g,r}^{\text{ord}} \rightarrow \mathcal{S}_{g,r}$ is* invariant *with respect to the Frobenius action defined by $\Phi_{\mathcal{N}}$.*

Moreover, it turns out that the *Frobenius lifting* $\Phi_{\mathcal{N}} : \mathcal{N}_{g,r}^{\text{ord}} \rightarrow \mathcal{N}_{g,r}^{\text{ord}}$ (i.e., morphism whose reduction modulo p is the Frobenius morphism) has the special property that $\frac{1}{p} \cdot \mathrm{d}\Phi_{\mathcal{N}}$ induces an isomorphism $\Phi_{\mathcal{N}}^* \Omega_{\mathcal{N}_{g,r}^{\text{ord}}} \cong \Omega_{\mathcal{N}_{g,r}^{\text{ord}}}$. Such a Frobenius lifting is called *ordinary.* It turns out that any ordinary Frobenius lifting (i.e., not just $\Phi_{\mathcal{N}}$) defines a set of *canonical multiplicative coordinates* in a formal neighborhood of any point α valued in an algebraically closed field k of characteristic p, as well as a *canonical lifting* of α to a point valued in $W(k)$ (Witt vectors with coefficients in k). Moreover, there is a certain analogy between this general theory of ordinary Frobenius liftings and the theory of *real analytic Kähler metrics* (which also define canonical coordinates). Relative to this analogy, the canonical Frobenius lifting $\Phi_{\mathcal{N}}$ on $\mathcal{N}_{g,r}^{\mathrm{ord}}$ may be regarded as corresponding to the *Weil-Petersson metric* on complex Teichmüller space (a metric whose canonical coordinates are the coordinates arising from the Bers uniformization of Teichmüller space). Thus, $\Phi_{\mathcal{N}}$ is, in a very real sense, a p-adic analogue of the Bers uniformization in the complex case. Moreover, there is, in fact, a canonical ordinary Frobenius lifting on the "ordinary locus" of the tautological curve over $\mathcal{N}_{g,r}^{\mathrm{ord}}$ whose relative canonical coordinate is analogous to the canonical coordinate arising from the Köbe uniformization of a hyperbolic curve.

Next, we observe that Serre-Tate theory for ordinary (principally polarized) abelian varieties may also be formulated as arising from a certain canonical ordinary Frobenius lifting. Thus, the Serre-Tate parameters (respectively, Serre-Tate canonical lifting) may be identified with the canonical multiplicative parameters (respectively, canonical lifting to the Witt vectors) of this Frobenius lifting. That is to say, in a very concrete and rigorous sense, Theorem 5 may be regarded as the analogue of Serre-Tate theory for hyperbolic curves. Nevertheless, we remark that it is *not* the case that the condition that a nilcurve be ordinary (i.e., defines a point of $(\mathcal{N}_{g,r}^{\mathrm{ord}})_{F_p} \subseteq \mathcal{N}_{g,r}$) either implies or is implied by the condition that its Jacobian be ordinary. Although this fact may disappoint some readers, it is in fact very natural when viewed relative to the general analogy between ordinary Frobenius liftings and real analytic Kähler metrics discussed above. Indeed, relative to this analogy, we see that it corresponds to the fact that, when one equips $\mathcal{M}_g$ with the Weil-Petersson metric and $\mathcal{A}_g$ (the moduli stack of principally polarized abelian varieties) with its natural metric arising from the Siegel upper half-plane uniformization, *the Torelli map* $\mathcal{M}_g \to \mathcal{A}_g$ *is not isometric.*

Next, we remark that $(\mathcal{N}_{g,r}^{\mathrm{ord}})_{F_p} \subseteq \mathcal{N}_{g,r}[0]$. Thus, the other $\mathcal{N}_{g,r}[d]$'s are left out of the theory of canonical liftings arising from Theorem 5. Nevertheless, in [Mzk2,4], a more general theory of canonical liftings is developed that includes arbitrary $\mathcal{N}_{g,r}[d]$. In this more general theory, instead of getting local uniformizations by multiplicative canonical parameters, i.e., uniformizations by $\widehat{G}_{\mathrm{m}}$, we get uniformizations by more general types of *Lubin-Tate groups*, or twisted products of such groups. Roughly speaking, the more "spikes" in the nilcurves involved – i.e., the larger the d of $\mathcal{N}_{g,r}[d]$ – the more Lubin-Tate the uniformization becomes.

Finally, we remark that once one develops these theories of canonical liftings, one also gets accompanying canonical (crystalline) Galois representations of the arithmetic fundamental group of the tautological curve over $\mathcal{N}_{g,r}^{\mathrm{ord}}$ (and its Lubin-Tate generalizations) into PGL_2 of various complicated rings with Galois action. It turns out that *these Galois representations are the analogues of the canonical representation* $\rho_{\mathcal{X}}$ (of §1, (A.)) – which was the starting point of our entire discussion.

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