# Curvature-Decreasing Maps are Volume-Decreasing

(on joint work with G. Besson and G. Courtois)

S. Gallot

Abstract. Giving a lower bound of the minimal volume of a manifold in terms of the simplicial volume, M. Gromov obtained a generalization of the Gauss-Bonnet-Chern-Weil formulas and conjectured that the minimal volume of a hyperbolic manifold is achieved by the hyperbolic metric. We proved this conjecture via an analogue of the Schwarz's lemma in the non complex case: if the curvature of X is negative and not greater than the one of Y, then any homotopy class of maps from Y to X contains a map which contracts volumes. We give a construction of this map which, under the assumptions of Mostow's rigidity theorems, is an isometry, providing a unified proof of these theorems. It moreover proves that the moduli space of Einstein metrics, on any compact 4-dimensional hyperbolic manifold reduces to a single point.

Assuming that X is a compact negatively curved locally symmetric manifold, and without any curvature assumption on Y, another version of the real Schwarz's lemma provides a sharp inequality between the entropies of Y and X . This answers conjectures of A. Katok and M. Gromov. It implies that Y and X have the same dynamics iff they are isometric.

This also ends the proof of the Lichnerowicz's conjecture : any negatively curved compact locally harmonic manifold is a quotient of a (noncompact) rank-one-symmetric space.

## 1. A real Schwarz lemma :

As was remarked by Pick, the classical Schwarz lemma may be rewritten in the language of the hyperbolic geometry (i. e. on the disk $B^2$ endowed with the hyperbolic metric $g_o = \frac{4}{(1-\|x\|^2)^2}\ \big((dx_1)^2 + (dx_2)^2\big)$) as follows :

1.1. Schwarz lemma. - *Any holomorphic map $f : B^2 \to B^2$, is a contracting map from $(B^2, g_o)$ to $(B^2, g_o)$.*

Considering now holomorphic maps between compact Kählerian manifolds of higher dimension, there have been many generalizations of this Schwarz lemma (due in particular to L. Ahlfors, S. T. Yau, N. Mok, ...). For example, the following one, which may be found in [Mok] :

1.2. Proposition.- *Let* $X, Y$ *be compact Kählerian manifolds of the same dimension whose Kählerian metrics are denoted by* $g_X$ *and* $g_Y$. *If* $Ricci_{g_Y} \geq -C^2 \geq Ricci_{g_X}$, *then any holomorphic map* $F : Y \to X$ *satisfies* $|JacF| \leq 1$. *Moreover, if* $|JacF| = 1$ *at some point y, then* $d_yF$ *is isometric.*

Let us recall that $Ricci_g$ *is the Ricci curvature tensor of the metric* g, and that *the assumption* $Ricci_g \geq -C^2$ *means that* $Ricci_g(u, u) \geq -C^2.g(u, u)$ *at any point and for any tangent vector u at this point.*
In its homotopy class, when the target-space has negative sectional curvature, a holomorphic map is unique ([Ha]) and is a good candidate for contracting the measure. Holomorphic maps are a particular case of harmonic maps between Riemannian manifolds. As, by the negativity of the curvature, each $C^0$ homotopy class of maps contains exactly one harmonic map (J. Eells and J. H. Sampson, [E-S]), one may ask whether it contracts volumes. Though unsuccessful, this idea underlies the attempts for a unified proof of the Mostow's rigidity theorem, where the method of harmonic maps fits very well to the hermitian cases and moreover improves Mostow's theorem (works of Y. T. Siu, K. Corlette, J. Jost and S. T. Yau, M. Gromov and R. Schoen ..., see for instance [Mok] and [Jo]), but still gives nothing in the real hyperbolic case. Substituting another canonical map to the harmonic one, we prove that the contracting property is not particular to complex manifolds and holomorphic maps :

1.3. Theorem ([B-C-G 3]).- *Let* $(Y^n, g_Y)$, $(X^m, g_X)$ *be complete riemannian manifolds satisfying* $3 \leq dim(Y) \leq dim(X)$, *let us assume that* $Ricci_{g_Y} \geq -(n-1)\, C^2$ *and that the sectional curvature of X satisfy* $K_{g_X} \leq -C^2$ *for some constant* $C \neq 0$. *Then any continuous map* $f : Y \to X$ *may be deformed to a family of* $C^1$ *canonical maps* $F_\epsilon$ $(\epsilon \to 0_+)$ *such that* $Vol[F_\epsilon(A), g_X] \leq (1+\epsilon)\ Vol(A, g_Y)$ *for any measurable set A in Y. Moreover*
*(i) if* $Y$, $X$ *are compact of the same dimension and if* $Vol(Y) = |degf|\ Vol(X)$, *then* $Y, X$ *have constant sectional curvature and the* $F_\epsilon' s$ *converge, when* $\epsilon \to 0$, *to a riemannian covering F (an isometry when* $|degf| = 1$*).*
*(ii) If* $Y$, $X$ *are compact, homotopically equivalent, of the same dimension, and if* $K_{g_Y} < 0$, *then any homotopy equivalence f may be deformed to a smooth (canonically constructed) map F such that* $|JacF| \leq 1$ *at every point y of Y. Moreover, if* $|JacF| = 1$ *at some point y, then* $d_yF$ *is isometric.*

1.4 Remarks : (1) Contrary to the above result of J. Eells and J. H. Sampson on harmonic maps, the theorem 1.3 is not only an existence theorem, but moreover a direct construction of the maps $F_\epsilon$ and $F$.
(2) The property (ii) remains valid when $dim(Y) < dim(X)$ and when $X$ is noncompact (however, we must assume that $\pi_1(X)$ acts on the universal covering $\tilde{X}$ in a "convex cocompact" way, i. e. that $X$ retracts to a compact submanifold with convex boundary). *In this case, any homotopy equivalence* $Y \to X$ *is homotopic to some (canonical) map F such that* $|JacF| \leq 1$; *moreover* $|JacF| \equiv 1$ *iff F is an isometric and totally geodesic embedding* (cf [B-C-G 3]).

2. Applications to minimal (and maximal) volume :
Let $M$ be a compact connected manifold ; its *minimal volume* (denoted by $MinVol(M)$) is defined by M. Gromov ([Gr 1]) as the infimum of the volumes of all the metrics g on M whose sectional curvature $K_g$ satisfies $-1 \leq K_g \leq 1$. Similarly, when the manifold admits some metric with strictly negative sectional curvature, one may define the *maximal volume* of M as the supremum of $Vol(g)$, for all the metrics $g$ which satisfy $K_g \leq -1$.

In dimension 2, the Gauss-Bonnet formula gives $\int_M K_g dv_g = 2\pi\chi(M)$, where $\chi(M)$ is the Euler characteristic of $M$. When $\chi(M) < 0$, this immediately implies that $MinVol(M) = 2\pi|\chi(M)| = MaxVol(M)$ and that the minimal and the maximal volumes are achieved for (and only for) metrics with constant sectional curvature $-1$.
In the higher even dimensional case, the Allendœrfer-Chern-Weil formulas also provide a lower bound of the minimal volume in terms of the Euler characteristic, however this bound is not sharp.
The simplicial volume (denoted by $SimplVol$), is defined as the infimum of $\|c\|_1 = \sum|\lambda_i|$ for all the linear real combinations of simplices $c = \sum \lambda_i \sigma_i$ which are closed chains $c$ representing the fundamental n-class. Substituting this notion to the Euler characteristic, M. Gromov obtained the:

2.1. Theorem (M. Gromov, [Gr1]).- *For any compact manifold M, one has $MinVol(M) \geq C_n\ SimplVol(M)$, where $C_n$ is a universal constant.*

For any compact manifold which admits a *hyperbolic metric* (i. e. a metric, denoted by $g_o$, whose sectional curvature is constant and equal to $-1$), an exact computation of the simplicial volume has been given by M. Gromov and W. Thurston ([Gr1]). By the theorem 2.1, it implies that $MinVol(X) \geq C'_n\ Vol(X, g_o)$. However, this estimate was also not sharp and justifies the

2.2. Theorem ([B-C-G 1,3]).- *Let $X$ be a compact manifold with dimension $n \geq 3$. If $X$ admits a hyperbolic metric $g_o$, then*
*(i) $MinVol(X) = Vol(g_o) = MaxVol(X)$.*
*(ii) A metric $g$ on $X$ (such that $|K_g| \leq 1$) realizes the minimal volume iff it is isometric to $g_o$.*
*(iii) For any other riemannian manifold $(Y^n, g)$ satisfying $Ricci_g \geq -(n-1).g$ and any map $f : Y^n \to X^n$, one has $Vol(Y, g) \geq |deg(f)| Vol(X, g_o)$*

This theorem answers a conjecture of M. Gromov and provides the first exact computations of (non trivial) minimal volumes in dimension $n \geq 3$.

*Proof* : We first apply the theorem 1.3 to the map $id_X : (X, g) \to (X, g_o)$. It implies the existence of homotopic maps $F_\epsilon$, of degree 1 (and thus surjective), such that $(1+\epsilon)Vol(g) \geq Vol\ (F_\epsilon(X), g_o) = Vol(g_o)$. Making $\epsilon \to 0$, we deduce the first equality of (i).

If $Vol(g) = Vol(g_o)$, the equality case in the theorem 1.3 (i) proves that the $F'_\epsilon s$ converge to an isometry. This proves (ii).
The same proof also gives (iii) if one notices that the integral on $Y$ of the Jacobian of the $F'_\epsilon s$ provides an upper bound for the degree of $f$.
On the other hand, if $K_g \leq -1$, the second equality of (i) is proved by applying the theorem 1.3 (ii) to the map $id_X : (X, g_o) \to (X, g)$. $\diamond$

3. Applications to Einstein manifolds :
*An Einstein manifold* is a Riemannian manifold whose Ricci curvature tensor is proportional to the metric. As the moduli space of Einstein metrics on a given compact manifold $Y$ may also be characterized as the set of critical metrics for the functional $g \to$ *total scalar curvature of* $g$, the main problem is thus to describe this moduli space. In dimensions 2 and 3, it reduces to metrics of constant sectional curvature, so this problem is relevant only when the dimension is at least 4. However, in the non Kählerian case, very little is known. Even the simplest questions :

3.1. - *Does every n-manifold admit at least one Einstein metric?*

3.2. - *If a n-manifold X admits a negatively curved locally symmetric metric, is it the only Einstein metric on X (modulo homotheties)?*

are still conjectures in dimension $n \geq 5$. In dimension 4, there were some answers to the problem 3.1, involving the Euler characteristic $\chi(Y)$, the signature $\tau(Y)$ and the simplicial volume :

3.3. - *In the 3 following cases, a 4-dimensional compact manifold Y does not admit any Einstein metric :*
*(i) If* $\chi(Y) < 0$ (M. Berger, [Bes2]),
(ii) *If* $\chi(Y) - \frac{3}{2}|\tau(Y)| < 0$ (J. Thorpe, [Bes2] p 210),
(iii) *If* $\chi(Y) < \frac{1}{2592\pi^2}.SimplVol(Y)$ (M. Gromov, see [Bes2] theorem 6.47).

In dimension 4, nothing was known about the problem 3.2.
If true, the conjecture 3.2 would give a strong version of the Mostow's rigidity theorem. In fact, when the sectional curvature is a negative constant, the possible local models are all homothetic. On the contrary, for negative Einstein manifolds, the possible local models are not homothetic (see [Bes 2]). Thus, one must previously find the topological (or global) reason which excludes all the possible local models except one.

Let us thus assume that $(Y, g)$ is a Einstein 4-dimensional manifold with $Ricci_g = (n-1)k.g$. The Allendœrfer-Chern-Weil formulas for the Euler characteristic and the signature give $\frac{4\pi^2}{3}\left(\chi(Y) \pm \frac{3}{2}\tau(Y)\right) = \int_Y P_\pm(R_g)dv_g$, where $P_\pm$ is a quadratic form in $R_g$, which satisfies $P_\pm(R_g) \geq k^2$ when $g$ is Einstein, the equality being achieved when $g$ has constant sectional curvature $k$ (see for instance [Bes 2] or [Bes 3]). From this comes :

(3.4) $\frac{4\pi^2}{3}\left(\chi(Y) - \frac{3}{2}|\tau(Y)|\right) \geq k^2\ Vol(Y, g)$,

the equality being achieved when $g$ has constant sectional curvature $k$. This is the classical proof of the theorems 3.3 (i) and (ii).
Let us now assume that there exists some map f of nonzero degree from $Y$ to some hyperbolic 4-dimensional manifold $X$. The corollary 2.2 (iii) and the equality-case of (3.4) imply

(3.5) $Max(0, -k)^2\ Vol(Y, g) \geq |degf|\ Vol(X, g_o) = \frac{4\pi^2}{3}|degf|\left(\chi(X) - \frac{3}{2}|\tau(X)|\right)$

This implies that $k < 0$. *If* $\chi(Y) - \frac{3}{2}|\tau(Y)| = \chi(X) - \frac{3}{2}|\tau(X)|$ (for example if $Y$ is homotopically equivalent to $X$), the inequalities (3.4) and (3.5) are equalities, thus $|degf| = 1$ and $Vol(Y, g) = Vol(X, g_o)$. We thus are in the equality case of the theorem 1.3 (i) and $(Y, g)$ is isometric to $(X, g_o)$. This applies in particular to the case where $Y = X$ and $f = id_X$ and proves the

3.6. Theorem ([B-C-G 1]).- *Let $X$ be a compact 4-dimensional manifold which admits a real hyperbolic metric, then this is (modulo homotheties) the only Einstein metric on $X$.*

*If* $\chi(Y) - \frac{3}{2}|\tau(Y)| < |deg\ f|\ (\chi(X) - \frac{3}{2}|\tau(X)|)$, inequalities (3.4) and (3.5) are contradicted and $Y$ *does not admit any Einstein metric* (A. Sambusetti, [Sam]), providing new answers to the conjecture 3.1 : in fact, from theorem 3.3 (ii), one might conjecture that any manifold $Y$ which satisfies $\chi(Y) - \frac{3}{2}|\tau(Y)| > 0$ (or some other relation between $\chi$ and $\tau$) admits an Einstein metric. M. Gromov's theorem 3.3 (iii) provided some counter-examples ([Bes 2] example 6.48); a complete answer is the :

3.7. Proposition (A. Sambusetti, [Sam]).- *To every possible values $k$ and $t$ of the Euler characteristic and of the signature corresponds an infinity of (non homeomorphic) 4-dimensional manifolds $Y_i$ which satisfy $\chi(Y_i) = k$ and $\tau(Y_i) = t$ and which admit no Einstein metric.*

The $Y_i$'s are obtained by gluing, to any compact hyperbolic manifold $X$ (such that $\chi(X) > k$), copies of $\pm CP^2$, $S^2 \times S^2$ or $S^2 \times \mathbf{T}^2$, in order to obtain the prescribed signature and Euler characteristic. One then apply the above Sambusetti's obstruction to the map of degree one : $Y_i \to X$.
These results may be compared to those obtained simultaneously by C. LeBrun ([LeB 1,2]), using Seiberg-Witten invariants, in particular the :

3.8. Theorem (C. LeBrun, [LeB 1]).- *Let $X$ be a compact 4-dimensional manifold which admits a complex hyperbolic metric, then this is (modulo homotheties) the only Einstein metric on $X$.*

4. Sketch of the proof of the real Schwarz lemma (see [B-C-G 1,2,3] for a complete proof) :
Rescaling the metrics $g_Y$ and $g_X$ of the theorem 1.3, we may assume that $Ricci_{g_Y} \geq -(n-1)\ g_Y$ and $K_{g_X} \leq -1$.
Let us consider the riemannian universal coverings $(\tilde{Y}, \tilde{g}_Y)$ and $(\tilde{X}, \tilde{g}_X)$ of the compact riemannian manifolds $(Y, g_Y)$ and $(X, g_X)$, whose riemannian distance and riemannian volume-measure are denoted by $\rho_{\tilde{Y}}$, $\rho_{\tilde{X}}$ and $dv_{\tilde{g}_Y}$, $dv_{\tilde{g}_X}$. Let $\mu_y^c$ be the measure on $\tilde{Y}$ defined by $\mu_y^c = e^{-c\rho_{\tilde{Y}}(y,\bullet)} dv_{\tilde{g}_Y}$.
The infimum $h_Y$ of the values $c$ such that this measure is finite is called the *entropy* of $(Y, g_Y)$. Another definition is $h_Y = \lim_{R\to+\infty}\left(\frac{1}{R} Log(Vol\ \tilde{B}(y,R))\right)$, where $\tilde{B}(y,R)$ is the ball of $(\tilde{Y}, \tilde{g}_Y)$ centered at $y$ and of radius $R$.

Let us consider positive measures $\mu$ on $\tilde{X}$ which are absolutely continuous w. r. t. the riemannian measure and such that the function $D_\mu(x) = \int_{\tilde{X}} \rho_{\tilde{X}}(x,z)\ d\mu(z)$ is finite. Following an idea of H. Furstenberg ([Fu], see also [D-E]), the *barycentre* $bar(\mu)$ is defined as the unique point where the function $D_\mu$ achieves its minimum (the existence comes from the triangle inequality and the uniqueness from the convexity of $\rho_{\tilde{X}}$). The barycentre is thus given by the implicit equation $(dD_\mu)_{|_{bar(\mu)}} = 0$.

Let $\tilde{f} : \tilde{Y} \to \tilde{X}$ be the lift of $f$, *we define* $\tilde{F}_c$ *by* $\tilde{F}_c(y) = bar(\tilde{f}_*\mu_y^c)$, where $\tilde{f}_*\mu_y^c$ is the push-forward by $\tilde{f}$ of the measure $\mu_y^c$. If $\rho = [f]$ is the induced representation $\pi_1(Y) \to \pi_1(X)$, $\tilde{f}$ (and thus $\tilde{f}_*$ also) satisfies the equivariance property $\tilde{f} \circ \gamma = \rho(\gamma) \circ \tilde{f}$ for any deck-transformation $\gamma \in \pi_1(Y)$. The invariance of the distance and of the riemannian measure by deck-transformations implies that $bar\,(\rho(\gamma)_*\mu) = \rho(\gamma)\,(bar(\mu))$ and $\mu^c_{\gamma.y} = \gamma_*\mu_y^c$. Thus $\tilde{F}_c$ is equivariant w. r. t. the same representation $\rho = [f]$, and goes down to a map $F_c : Y \to X$ which is homotopic to f.
Let $c = (1+\epsilon)\ h_Y$, we want to prove that, when $\epsilon \to 0_+$, $F_c$ answers theorem 1.3.
Let us define $\Delta : \tilde{X} \times \tilde{Y} \to \mathbf{R}$ by $\Delta(x,y) = D_{\tilde{f}_*\mu_y^c}(x)$ and let $\partial^1$ (resp. $\partial^2$) be the derivatives w. r. t. the first (resp. the second) parameter.
By the definition of $\tilde{F}_c$ and by the variational characterization of the barycentre, $\tilde{F}_c$ is defined by the implicit equation : $\partial^1\Delta_{|_{(\tilde{F}_c(y),y)}} = 0$.
By derivation, we get $\partial^1\partial^1\Delta_{|_{(\tilde{F}_c(y),y)}}(d\tilde{F}_c(u), v) = -\ \partial^2\partial^1\Delta_{|_{(\tilde{F}_c(y),y)}}(u,v)$ for any $u \in T_y\tilde{Y}$ and $v \in T_{\tilde{F}_c(y)}\tilde{X}$. This writes

$$(4.1) \qquad \int_{\tilde{Y}} Dd\rho_{\tilde{X}_{|_{(\tilde{F}_c(y),\tilde{f}(z))}}}(d\tilde{F}_c(u), v)\ d\mu_y^c(z)$$
$$= c\int_{\tilde{Y}} d\rho_{\tilde{X}_{|_{(\tilde{F}_c(y),\tilde{f}(z))}}}(v)\ d\rho_{\tilde{Y}_{|_{(y,z)}}}(u)\ d\mu_y^c(z) \leq c\ \tilde{g}_X(H_y(v),v)^{1/2}\ \tilde{g}_Y(K_y(u),u)^{1/2},$$

where the tensor $Dd\rho_{\tilde{X}}$ is computed by derivation w. r. t. the first parameter and where $H_y$ (resp.$K_y$) is the symmetric endomorphism of $T_{\tilde{F}_c(y)}\tilde{X}$ (resp. of $T_y\tilde{Y}$) associated to the quadratic form $v \to \int_{\tilde{Y}}(d\rho_{\tilde{X}_{|_{(\tilde{F}_c(y),\tilde{f}(z))}}}(v))^2 d\mu_y^c(z)$ (resp.

to the quadratic form $u \to \int_{\tilde{Y}} (d\rho_{\tilde{Y}_{|_{(y,z)}}}(u))^2 d\mu^c_y(z)$ ).

As the gradient of $\rho_{\tilde{X}}(\bullet, \tilde{f}(z))$ is a unit vector normal to the geodesic spheres centered at $\tilde{f}(z)$, the second fundamental form of these spheres is equal to $Dd\rho_{\tilde{X}_{|_{(\bullet,\tilde{f}(z))}}}$. As $K_{g_X} \leq -1$, the Rauch's comparison theorem provides the lower bound *coth* $\rho_{\tilde{X}}(\bullet, \tilde{f}(z))$ for the principal curvatures of these spheres, and thus implies that $\tilde{g}_X - d\rho_{\tilde{X}} \otimes d\rho_{\tilde{X}}$ is a lower bound for $Dd\rho_{\tilde{X}}$.

First plugging this in (4.1), replacing $c$ by its value and then writing the induced inequality for determinants, we obtain :

$$(4.2)\quad \tilde{g}_X \left( (Id - H_y) \circ d_y \tilde{F}_c(u), v \right) \leq (1+\epsilon)\, h_Y\, \tilde{g}_X\left(H_y(v), v\right)^{1/2} \tilde{g}_y\left(K_y(u), u\right)^{1/2},$$

$$(4.3)\qquad (1+\epsilon)^{-n} h_Y^{-n}\, \frac{det(Id - H_y)}{(det H_y)^{1/2}}\, |det(d_y\tilde{F}_c)| \leq (det\ K_y)^{1/2} \leq \left(\tfrac{1}{n} Trace\ K_y\right)^{n/2}$$

As $\|d\rho_{\tilde{Y}}\| = 1 = \|d\rho_{\tilde{X}}\|$, we have Trace $K_y = 1 = Trace H_y$. On the other hand, *the function* $\delta : A \to \frac{det(I-A)}{(det A)^{1/2}}$ (defined on the set of symmetric positive definite $n \times n$ matrices ($n \geq 3$) whose trace is equal to 1) achieves its minimum at the unique point $A_o = \frac{1}{n} I$.

Plugging this in (4.3) gives : $|det(d_y\tilde{F}_c)| \leq (1+\epsilon)^n \left(\frac{h_Y}{n-1}\right)^n$. We end the proof of the general inequality of the theorem 1.3 by applying the comparison theorem of R. L. Bishop : i. e. the assumption $Ricci_{g_Y} \geq -(n-1)$ implies that $h_Y \leq n-1$. $\diamond$

When $K_{g_Y} < 0$, one may identify $\tilde{Y}$ with a ball and compactify it by addition of the sphere, called the *ideal boundary* and denoted $\partial\tilde{Y}$. One may then extend continuously $\tilde{f}$ to a map $\bar{f} : \partial\tilde{Y} \to \partial\tilde{X}$.

Let us fix an origin $y_o$ in $\tilde{Y}$. A sequence of measures $(\mu^{c_n}_{y_o}(\tilde{Y}))^{-1}\mu^{c_n}_y$ converges, on the compact set $\tilde{Y} \cup \partial\tilde{Y}$ (when $c_n \to h_Y$), to a measure $\mu_y$, with support in $\partial\tilde{Y}$, which is known as *the Patterson-Sullivan measure* and satisfies $\mu_y = e^{-h_Y B_{\tilde{Y}}(y,\bullet)}\mu_{y_o}$, where $B_{\tilde{Y}}(y,\theta) = \lim_{t\to+\infty}[\rho_{\tilde{Y}}(c_\theta(t), y) - t]$ and where $c_\theta$ is the normal geodesic-ray from $y_o$ to $\theta$.

Mimicking the previous proof (just replacing $\rho_{\tilde{Y}}$ and $\rho_{\tilde{X}}$ by $B_{\tilde{Y}}$ and $B_{\tilde{X}}$), *we define* $\tilde{F}$ *by* $\tilde{F}(y) = bar(\bar{f}_*\mu_y)$ and prove the inequality of the theorem 1.3 (ii) : $|det(d_y\tilde{F})| \leq \left(\frac{h_Y}{n-1}\right)^n \leq 1$.

When $|det(d_y\tilde{F})| \geq \left(\frac{h_Y}{n-1}\right)^n$, and a fortiori when $|Jac\tilde{F}| = 1$, the analogues of the inequalities (4.3) are equalities which imply that $K_y = \frac{1}{n} I$ and that $\delta$ achieves its minimum at the point $H_y$, which is thus equal to $A_o = \frac{1}{n} I$. Plugging this in (4.2) and replacing $v$ by $d_y\tilde{F}(u)$, we deduce that $d_y\tilde{F}$ is a contracting map whose determinant is equal to 1, thus it is isometric (see [B-C-G 2,3] for more explanations).

On the contrary, when $K_{g_Y}$ may take both signs, we have to prove that the $F_c$'s admit a limit when $c \to h_Y$, that this limit is a contracting map and that the property of preserving global volumes implies that it is isometric (see [B-C-G 1] sections 7 and 8). $\diamond$

5. Another version of the real Schwarz lemma :
The present version of the real Schwarz lemma is adapted to the case where the target-space is a compact quotient of a hyperbolic space modelled on the real or complex or quaternionic or Cayley field (the canonical basis of the field beeing denoted by $\{1, J_1, ..., J_d\}$).

5.1. Theorem ([B-C-G 1,2,3]).- *Let $(X, g_X)$ be a compact locally symmetric manifold with negative curvature and $(Y, g_Y)$ be any compact riemannian manifold such that dim $X$ = dim $Y \geq 3$, then any continuous map $f : Y \to X$ may be deformed to a family of $C^1$ maps $F_\epsilon (\epsilon \to 0_+)$ such that $|Jac\ F_\epsilon| \leq \left(\frac{h_Y + \epsilon}{h_X}\right)^n$. In particular, one has $(h_Y)^n Vol(Y) \geq |degf|(h_X)^n Vol(X)$. Moreover, if $(h_Y)^n Vol(Y) = |degf|(h_X)^n Vol(X)$, then $Y$ is also locally symmetric and $f$ is homotopic to a riemannian covering $F$ (an isometry when $|degf| = 1$).*

5.2. Remarks.- (1) This theorem proves conjectures of A. Katok and M. Gromov about the minimal entropy.
(2) When $(Y, g_Y)$ has negative curvature and f is a homotopy equivalence, the following proof provides a direct construction of $F : Y \to X$ which satisfies $|Jac\ F(y)| \leq \left(\frac{h_Y}{h_X}\right)^n$ and $d_y F$ is isometric in the equality case.

*Sketch of the proof* : We already proved the theorem 5.1 and the remark 5.2 (2) when $(X, g_X)$ is (locally) real hyperbolic (see section 4). In the other locally symmetric cases, the proof is exactly the same, except for the fact that, expliciting the new expression of $Dd\rho_{\tilde{X}}$, we have to prove that the function $A \to \frac{det(I - A - \sum_i J_i A J_i)}{(detA)^{1/2}}$ still achieves its minimum at the unique point $A_o = \frac{1}{n} I$. This comes from the log-concavity of the determinant which reduces the problem to minimizing the previous function $\delta$ (see [B-C-G 1]). $\diamond$

5.3. Corollary (G.D. Mostow).- *Let $(X, g_X)$ and $(Y, g_Y)$ be two compact negatively curved locally symmetric manifolds such that $dimX = dimY \geq 3$, then any homotopy-equivalence $f : Y \to X$ is homotopic to an isometry.*

*Proof* : Let $g : X \to Y$ such that $g \circ f \sim id_Y$. By the remark 5.2 (2), there exist $F \sim f$ and $G \sim g$ such that $|Jac(G \circ F)| \leq \left(\frac{h_X}{h_Y}\right)^n \left(\frac{h_Y}{h_X}\right)^n$. As the degree of $G \circ F$ is equal to 1, this inequality is an equality and we are in the equality case of the remark 5.2 (2), thus F is an isometry. $\diamond$

This provides a unified proof for the Mostow's rigidity theorem. Moreover, the isometry $F$ is explicitely constructed (see section 4)

6. Application to dynamics and Lichnerowicz's conjecture :
Let $\phi_t^Y : \dot{c}(0) \to \dot{c}(t)$ (for any geodesic $c$) be the geodesic flow of Y. Two riemannian manifolds $Y$ and$X$ are said to *have the same dynamics* iff there exists a

$C^1$-diffeomorphism $\Phi$ between their unitary tangent bundles $UY$ and $UX$ which exchanges their geodesic flows, i. e. $\Phi \circ \phi_t^Y = \phi_t^X \circ \Phi$. The fundamental question is : *two riemannian manifolds having the same dynamics are they isometric?* This is generally false, for there exists non isometric manifolds all of whose geodesic are closed with the same period (see [Bes 1]). C. B. Croke and J. P. Otal proved this conjecture to be true for negatively curved surfaces. In any dimension, we get the

6.1. Theorem ([B-C-G 1]).- *Any Riemannian manifold which has the same dynamics as a negatively curved locally symmetric one is isometric to it.*

*Proof* : As $UY \approx UX$ and $n \geq 3$, the manifolds under consideration $Y$ and $X$ are homotopically equivalent. As the volume and the entropy are invariants of the dynamics, the assumption implies that $h_Y = h_X$ and $Vol(Y) = Vol(X)$; we thus are in the equality case of the theorem 5.1 and $Y$ and $X$ are isometric. $\diamond$

A riemannian manifold is said to be *locally harmonic* when all geodesic spheres of its universal covering have constant mean curvature. Any locally symmetric manifold of rank one is locally harmonic. A. Lichnerowicz asked for the converse question : *Consider any locally harmonic manifold, is it locally symmetric of rank one?.*

When the universal covering $\tilde{X}$ is compact, this conjecture was proved by Z. Szabo ([Sz]). In the case where $\tilde{X}$ is noncompact, the geodesics have no conjugate points ([Bes 1]), and the conjecture is not significantly changed when assuming the sectional curvature to be negative. A counter-example (admitting no compact quotient) was given by E. Damek and F. Ricci ([D-R]). Assuming that $\tilde{X}$ admits a compact quotient, we get the

6.2. Corollary ([B-C-G 1]).- *Any compact negatively curved locally harmonic manifold is locally symmetric of rank one.*

*Proof* : Under these assumptions, P. Foulon and F. Labourie ([F-L]) proved that the manifold has the same dynamics as a negatively curved locally symmetric manifold. We conclude by applying the theorem 6.1. $\diamond$

## References


[B-P] R. Benedetti, C. Petronio, *Lectures on Hyperbolic Geometry*, Universitext, Springer 1993.

[Bes 1] A. L. Besse, *Manifolds all of whose geodesics are closed*, Ergebnisse der Math., Springer 1978.

[Bes 2] A. L. Besse, *Einstein Manifolds*, Ergebnisse der Math., Springer 1987.

[Bes3] A. L. Besse, *Géométrie riemannienne en dimension 4*, Séminaire Arthur Besse, Texte Mathématiques, Cedic-Nathan.

[B-C-G 1] G. Besson, G. Courtois, S. Gallot, *Entropies et rigidités des espaces localement symétriques de courbure négative*, Geom. And Funct. Anal.5 (1995), pp. 731-799.

[B-C-G 2] G. Besson, G. Courtois, S. Gallot, *Minimal Entropy and Mostow's rigidity theorems*, Ergod. Theory Dynam. Syst. 16 (1996), pp. 623-649.

[B-C-G 3] G. Besson, G. Courtois, S. Gallot, *Lemme de Schwarz réel et applications géométriques*, prépublications Centre Math. Ec. Polytechnique (98-7).

[D-R] E. Damek, F. Ricci, *A class of non-symmetric harmonic riemannian spaces*, Bull. Amer. Math Soc. 27 (1992), pp. 139-142.

[D-E] E. Douady, C. Earle, *Conformally natural extension of homeomorphisms of the circle*, Acta Math. 157 (1986), pp. 23-48.

[E-S] J. Eells, J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. Journ. Math. 86 (1964), pp. 109-160.

[F-L] P. Foulon, F. Labourie, *Sur les variétés compactes asymptotiquement harmoniques*, Invent. Math. 109 (1992), pp. 97-111.

[Fu] H. Furstenberg, *A Poisson formula for semi-simple Lie groups*, Annals of Maths 77 (1963), pp. 335-386.

[Gr1] M. Gromov, *Volume and Bounded Cohomology*, Publ. Math. I. H. E. S. 56 (1981), pp : 213-307.

[Ha] P. Hartman, *On homotopic harmonic maps*, Canad. J. Math. 19 (1967), pp. 673-687.

[Jo] J. Jost, *Riemannian Geometry and Geometric Analysis*, Universitext, Springer 1998.

[LeB 1] C. LeBrun, *Einstein metrics and Mostow rigidity*, Math. Res. Lett. 2 (1995), pp. 1-8.

[LeB 2] C. LeBrun, *Four-manifolds without Einstein metrics*, Math. Res. Lett. 2 (1996), pp. 133-147.

[Mok] N. Mok, *Metric Rigidity Theorems on Hermitian locally symmetric Manifolds*, Series in pure Maths 6, World Scientific.

[Sam] A. Sambusetti, *An obstruction to the existence of Einstein metrics on 4-Manifolds*, to appear in Math. Annalen.

[Sz] Z. Szabo, *The Lichnerowicz conjecture on harmonic manifolds*, J. Diff. Geom. 31 (1990), pp. 1-28.

Sylvestre Gallot
Institut Fourier Maths Pures
UMR 5582 CNRS-UJF
B. P. 74
38402 Saint Martin d'Heres Cedex
France