Curvature Contents of Geometric Spaces

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Abstract. We discuss curvature relevant deformations of spaces and indicate the existence of some individual capacity of a manifold (and more general spaces) measuring a maximal amount of curvature that could be carried by this space.

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1. Introduction

It turned out recently that the general tendency is that there is a natural upper bound but definitely *no lower bound* for the curvature on a given manifold or at least certain families of such spaces.
Moreover, roughly speaking, as we approach the maximal curvature "amount" we will often reach a particularly rigid geometry or a singular one (if the manifold cannot exhibit the suitable form of symmetry).
On the other hand, by decreasing the curvature (even uniformly) we gain flexibility: For instance we may combine various geometric conditions. That is certain geometric properties which are *coupled* for higher curvature amounts will become more and more independent the more the curvature melts away. Most notably the curvature and the coarse metric geometry, e.g. volumes, systoles or various radii, will "finally" appear entirely unlinked.
Motivated by this sort of observation we are tempted to think of an *individual curvature content* (or capacity) of a given manifold depending however on the curvature problem under consideration.
From this viewpoint it no longer matters whether this content has a particular sign: there is just a maximal amount of positivity that may be carried by the manifold and this may be positive or not.
We have not made any attempt to state a sharp "definition" of our presently still intuitive notion of a "curvature content" as we do not want to destroy the suggestive flavour of this notion. But certainly such a measure will depend on the context (additional constraints etc.), as we will see below.
We will start our short journey in dimension 2 and give an interpretation of some classical results which allows us to proceed directly to higher dimensions where we will treat the three basic notions of scalar, Ricci and sectional curvature and will concentrate entirely on these.

2. SURFACES

In dimension 2 we have a unique notion of curvature (the Gaussian curvature K) and it shares some properties of both extremes: scalar and sectional curvature. For instance it can be conformally deformed into a metric of constant curvature, but its sign also shows the typical implications of sectional curvature.
Furthermore its behaviour when one tries to increase/decrease the curvature amount lies somewhere in between these two extremes.
We start with a reinterpretation of the uniformization theorem:
First note that by attaching a handle to the Euclidean plane one can construct a complete surface N^2 which contains an open bounded set U where $K < 0$ and is isometric to the Euclidean $\mathbb{R}^2 \setminus B_1(0)$ on $N^2 \setminus U$.
Now we take a two-sphere. If we cut a small disc out of S^2 and instead glue a suitably scaled copy of $U \subset N^2$ in its place we get a surface (a torus) which admits a metric with $K = 0$. Iterating this procedure, that is attaching several copies of such an "island", one gets surfaces allowing $K < 0$-metrics. This is obvious due to the *uniformization theorem* because we know that adding $U \subset N^2$ enlarges the genus by 1 for each copy of $U \subset N^2$ while the geometric properties specified above become redundant.
However the point is that we can also succeed by using purely the available geometry: Cover S^2 (or any other surface) by very small discs $D_r(p_i)$ with an upper bounded covering number independent of the radius r. Next substitute the discs $D_{r/10}(p_i)$ (which can be assumed to be disjoint) by suitably rescaled copies of $U \subset N^2$. This new Riemannian manifold can be deformed using a slight and explicit (conformal) deformation to yield a $K < 0$-metric (cf. [L1]).
Of course, this surface will have quite a high genus, but we have not used the uniformization theorem at all. Actually, we will see (in higher dimensions) that it is quite natural to consider this construction a curvature decreasing "deformation" of a given surface.
Now the second interesting point is the following: there is *no counterpart* for getting more *positively* curved surfaces: Of course, this is clear from Gauss-Bonnet, but let us also have a look at what really happens.
The construction of negative curvature can take place in any open set of arbitrarily small metrical size. On the other hand even in the case where we just reverse this construction (obviously gaining some positivity) we have to start by choosing a closed curve which is *not* homotopic to a constant map. This cannot be accomplished locally.

3. SCALAR CURVATURE

The weakest generalization of Gaussian Curvature in dimension 2 to dimensions $n \geq 3$ is the notion of scalar curvature $Scal$.

In this case we can decrease the curvature locally without any topological changes: we can find metrics on $\mathbb{R}^n$ which satisfy $Scal < 0$ on the unit ball in $\mathbb{R}^n$ and are

Euclidean outside it (i.e. the metrical analogue of $U \subset N^2$ above). Here is a generalized version (cf. [L2]):

Theorem 3.1: *Let $U \subset M$ be an open subset and f any smooth function on M with*
$f < \mathrm{Scal}(g)$ *on* U *and* $f \equiv \mathrm{Scal}(g)$ *on* $M \setminus U$.
Then, for each $\varepsilon > 0$, there is a metric g_ε on M and an ε- neighborhood U_ε of U with

$$g \equiv g_\varepsilon \text{ on } M \setminus U_\varepsilon \text{ and } f - \varepsilon \leq \mathrm{Scal}(g_\varepsilon) \leq f \text{ on } U_\varepsilon.$$

This new metric g_ε can be chosen arbitrarily near to the old one in C^0-topology (using also [L5]). Thus we can basically attach some negative curvature without changing the shape of the manifold.

Next we want to know whether there is any corresponding statement for curvature *increasing* deformations (even admitting topological changes). Here one should take a look at a theorem originating from general relativity, the so-called "positive energy theorem" originally proved by Schoen-Yau and Witten (cf. [SY] and [PT]):

Theorem 3.2: *Let (M, g) be an asymptotically flat manifold with* $\mathrm{Scal}(g) \geq 0$. *Then the energy $E(g)$ is non-negative and $E(g) = 0$ iff (M, g) is flat.*

This already gives a first hint of the existence of a "maximal content" as becomes clear once one realizes that this problem can be solved as follows: In a first step one transforms it to a local one and then one plays this off against some curvature capacity consideration:
(3.2) can be reduced to the (*non*)trivial special case: the *only* complete Riemannian manifold which is Euclidean outside a bounded domain U with $\mathrm{Scal}(g) \geq 0$ on U is the Euclidean space. Now assume the existence of a non-flat manifold of this type. Then one "reverses (3.1)" and constructs manifolds whose positive scalar curvature amount turns out to be actually "too large" for the underlying space thereby proving the non-existence of such a manifold (for details cf. [L2]).
In this context we also meet a recent theorem by Llarull [Ll] for S^n equipped with the round metric g_{round}:

Theorem 3.3: *Let g be any metric on S^n with $g(v, w) \geq g_{round}(v, w)$ for each oriented pair of vectors $v, w \in T_pS^n$, $p \in S^n$ and $Scal(g) \geq 1$. Then $g \equiv g_{round}$.*

Another related subject is the solution of the Yamabe problem (cf. [LP]) claiming that every metric can be conformally deformed into a metric of constant scalar curvature. Recall that the original solution of the Yamabe problem uses the positive energy theorem.

The geometric ingredient of that proof is the following theorem by Aubin and Schoen (cf. [LP]) in dimension $n \geq 3$:

Theorem 3.4: *For every closed manifold M^n, $n \geq 3$ the infimum of the normalized total curvature functional $\boldsymbol{S}(\varphi)$ within a given conformal class $\varphi^{4/n-2} \cdot g$*

$$\begin{aligned}\mathbb{S}(M,g) &:= \inf_{\varphi \not\equiv 0} \boldsymbol{S}(\varphi)\\ &:= \inf_{\varphi \not\equiv 0} \int_M (\|\nabla\varphi\|^2 + \frac{(n-2)}{4(n-1)} \cdot \mathrm{Scal}(g) \cdot \varphi^2) dV_g / \left(\int_M |\varphi|^{2n/(n-2)} dV\right)^{n-2/n}\end{aligned}$$

satisfies: $\mathbb{S} \leq \mathbb{S}(S^n, g_{round})$ *with equality iff M is conformal to the round S^n.*

This means that taking the supremum over all the conformal classes of metrics on a manifold one gets an individual upper bound for a certain kind of scalar curvature content on the manifold. In the case of a torus for instance this is zero and corresponds to a flat metric (cf. [S]).
This is also a good place to check what happens when we approach the supremum by a sequence of smooth metrics (cf. [A]):
Under some additional assumptions *there is a subsequence that converges either to a smooth Einstein metric or to a singular limit consisting of finitely many smooth noncompact Einstein manifolds with cusps.*

4. Ricci Curvature

Now we meet the main candidate for a meaningful notion of curvature content for general smooth manifolds which is the Ricci curvature *Ric.*
We start with a counterpart of (3.1) that is easily derived from [L4] and this time follows from the fact that we can even find metrics which have negative Ricci curvature on a ball in $\mathbb{R}^n$ and are Euclidean outside it:

Theorem 4.1: *Let $U \subset M$ be an open subset and f any smooth function on the unit tangent bundle SM with $f(\nu) < \mathrm{Ric}(g)(\nu)$ on SU and $f \equiv \mathrm{Ric}(g)$ on $SM \setminus SU$.*
Then there is a smooth metric g_f on M with $Vol(M,g) = Vol(M,g_f)$ and

$$g \equiv g_f \text{ on } M \setminus U \text{ and } \mathrm{Ric}(g_f)(\nu) \leq f(\nu) \text{ on } SU.$$

We supplement this theorem with two examples of the "decoupling" effect of curvature dereasing deformations mentioned in the introduction:
For simplicity take a compact manifold M^n and $U = M$, then there are several extensions of the statement above:
We may also prescribe finitely many Laplace eigenvalues (cf. [L3]) *or* we can choose g_f arbitrarily near to g in various geometric topologies (cf. [L5]).

The counterpart of (4.1) for positive curvature (even allowing topological changes) is excluded already by the scalar curvature argument above, however here we may

also use the standard Bochner argument.
Thus we can immediately proceed to the question of "curvature contents" and their application.
Here one has several very recent results by Cheeger and Colding and by Besson, Courtois and Gallot (cf. [Co],[Ga] and references therein). We state some of them as follows:

Theorem 4.2: *There is an $\varepsilon(n) > 0$ such that if M^n is a closed manifold with $Ric(g) \geq (n-1)g$ and $Vol(M^n) \geq Vol(S^n) - \varepsilon(n)$, then M^n is diffeomorphic to S^n.*

Theorem 4.3: *Let M^n be a closed manifold, and assume there are metrics g_0 on M^n with $Sec \equiv -1$ and g on M^n with $Ric(g) \geq -(n-1)g$. Then $Vol(M^n, g) \geq Vol(M^n, g_0)$. If equality holds and $n \geq 3$, then (M^n, g) and (M^n, g_0) are isometric.*

These two results can be reinterpreted as follows: A manifold M^n (with some normalized volume) admits at most as much (lower bounded) Ricci curvature as the sphere and this extremum is reached if and only if M^n is the sphere. Secondly, if M^n carries some sufficiently symmetric geometry, then the Ricci curvature amount cannot exceed the borderline preassigned by that geometry. Thus even these "local maxima" for the present Ricci curvature content are very distinguished.
There is also another observation: For some Ricci (and sectional) curvature content notions the superlevel sets are very thin: *There are only finitely many homotopy tyes of manifolds whose Ricci curvature capacity exceeds certain bounds* in the respective context (a general reference is [P]).

5. Sectional Curvature

The strongest curvature notion is that of sectional curvature *Sec*. There are natural generalizations of this curvature notion to metric spaces, specifically "Alexandrov spaces", which will also be of interest to us.
We will very briefly remind the reader of these general notions. For details cf. [BN] and [BGP].
A locally complete metric space (X, d) is called *a space of curvature $\geq k$ or $\leq k$ respectively* if the following conditions are satisfied at least locally:

(i) Any two points can be joined by a geodesic, i.e. by a curve whose length equals the distance of its endpoints.

(ii) For any geodesic triangle Δpqr and any point z on an arbitrary side pq, we find a triangle ΔPQR and a point Z on the side PQ in the simply connected smooth surface M_k of constant sectional curvature $= k$ with

$d(p,q) = d_{M_k}(P,Q)$, $d(p,r) = d_{M_k}(P,R)$, $d(q,r) = d_{M_k}(Q,R)$ *and*

$d(z,r) \geq$ *(or $\leq$)* $d_{M_k}(Z,R)$, $d(z,p) = d_{M_k}(Z,P)$, $d(z,q) = d_{M_k}(Z,Q)$

Many theorems translate from Riemannian to Alexandrov geometry. For instance, if (X, d) is complete with curvature $\geq k > 0$, then X is compact with $diam(X, d) \leq \pi/\sqrt{k}$. Also, completeness and curvature ≤ 0 imply contractibility of the universal covering.
We will again begin with curvature decreasing deformations. From the preceding remark it is clear that we have to admit "deformations" that alter the topology to some extent. This is what was anticipated in section 2 in the case of surfaces. We will see that we can interpret the concept of "hyperbolization" (cf. [G1], [DJ] and [CD]) as a substitute for adding handles to a given surface locally. A hyperbolization is a process converting a space into a negatively curved one. The known processes work as follows: One starts with some, say PL-manifold M and substitutes (in one or several steps) each simplex by a negatively curved manifold with some smooth boundary. This boundary might be different from the simplex-boundary, but if one carries this out everywhere simultaneously these boundaries fit together and on gets a new PL-manifold $\mathcal{H}(M)$.
This can be achieved in such a way that one obtains a negatively curved Alexandrov space $\mathcal{H}(M)$ and in some obvious sense theses changes are *local* .
Of course, the aim is to find constructions which do not damage the topology too much. This is paraphrased in some kind of axioms in [DJ] and [CD] and the main point is that such processes exist. We state this as a descriptive theorem:

Theorem 5.1: *There is a process that converts a cell complex K into a new polyhedron $\mathcal{H}(K)$ which admits a metric with curvature ≤ -1 such that*

(i) If K is a PL-manifold, then so is $\mathcal{H}(K)$

(ii) There is a map ϕ: $\mathcal{H}(K) \to K$ which induces a surjection on homology and is such that ϕ pulls back the rational Pontryagin classes from K to those of $\mathcal{H}(K)$.

(iii) $\mathcal{H}$ behaves functorial and preserves the local structure. That is a PL-embedding $f : L \to K$ induces an isometric map $\mathcal{H}(f) : \mathcal{H}(L) \to \mathcal{H}(K)$ such that $\mathcal{H}(L) \subset \mathcal{H}(K)$ is totally geodesic and in the case of $L =$ single simplex $\subset K$ the "ambient angles" (more precisely the link) are mapped PL-isomorphically.

Iterating this process or applying it to sufficiently fine triangulations one gets geometries whose curvatures are arbitrarily strongly negative.

Next we turn to the question of whether one can increase curvature at least in the sense of Alexandrov. But it is easily verified that there is no process that satisfies similar axioms to those above but *increases* the curvature since otherwise one may, for instance, construct a complete *non-compact* Alexandrov space with curvature > 1 which is impossible.
Finally in the case of sectional curvature we meet a lot of new reasonable notions of capacities. We select one example: Gromov's Betti number theorem [G2]

Theorem 5.2: *There is a constant $c(n,k)$ such that for a manifold M^n with $Sec \geq k$ and diameter $= 1$: $\sum_{i=0}^{n}$ i-th Betti number $\leq c(n,k)$.*

This gives an obvious form of curvature content: For a normalized diameter and fixed total Betti number there is an upper bound k_0 for the existence of metrics with $Sec \geq k$.
Here we have a nice opportunity to compare Ricci and sectional curvature: For the connected sum of sufficiently many copies of $S^n \times S^m$, $m, n \geq 2$ or $\mathbb{C}P^2$, we find that k_0 becomes arbitrarily strongly negative, while Sha and Yang (cf. [ShY]) resp. Perelman (unpublished) have shown that these manifolds always carry a $Ric > 0$-metric. Thus the maximal amounts of Ricci and sectional curvature on a manifold may differ to any extent.

6. Conclusion

Finally we want to add some general remarks and suggestions.
1. In many cases the curvature capacity is related to lower curvature bounds and correspondingly the "rigid" maximal geometries usually have constant (e.g. Ricci) curvature.
This is not surprising since "nice geometries" should have the "topological" property of not distinguishing between different parts on the manifold. However, when one starts with concrete constructions in topology one frequently breaks up this homogeneity (e.g. in Morse theory).
Then one may still think of curvature contents, this time respecting and/or forcing decompositions of the underlying space which might lead to capacity notions with an own for instance algebraic structure.
2. Motivated by the discussion concerning sectional curvature "deformations" we are led to believe that it will be reasonable in various contexts to include certain types of topological changes in a class of admissable deformations. Sometimes it might even be useful to go one (speculative) step further: consider the space and its geometry as one entity - then such notions of deformations (containing spaces with various topologies) become completely natural.

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