Dimension Theory and Large Riemannian Manifolds

A. N. Dranishnikov

Abstract. In this paper we discuss some recent applications of dimension theory to the Novikov and similar conjectures. We consider only geometrically finite groups i.e. groups Γ that have a compact classifying space $B\Gamma$. It is still unknown whether all such groups admit a sphere at infinity [B]. In late 80s old Alexandroff's problem on the coincidence between covering and cohomological dimensions was solved negatively [Dr]. This brought to existence a locally nice homology sphere which is infinite dimensional. In the beginning of 90s S. Ferry conjectured that if such homology sphere can be presented as a sphere at infinity of some group Γ, then the Novikov conjecture is false for Γ. Here we discuss the development of this idea. We outline a reduction of the Novikov conjecture to dimension theoretic problems. The pionering work in this direction was done by G. Yu [Yu]. He found a reduction of the Novikov Conjecture to the problem of finite asymptotic dimensionality of the fundamental group Γ. Our approach is based on the hypothetical equivalence between asymptotical dimension of a group and the covering dimension of its Higson corona. The slogan here is that most of the asymptotic properties of Γ can be expressed in terms of topological properties of the Higson corona $\nu\Gamma$. At the end of the paper we compare existing reductions of the Novikov conjecture in terms of the Higson corona.

§1. Dimension theory of compacta

The *covering dimension* $\dim X$ of a compact metric space X can be defined as the smallest number n such that for any $\epsilon > 0$ there is an ϵ- covering $\{U_1, ..., U_k\}$ of X of order $\leq n+1$. The definition does not depend on the metric on X. There are many equivalent reformulations of this property and not all of them are exactly obvious. Here we give two of them.

Ostrand Theorem. $\dim X \leq n \Leftrightarrow$ *for any positive* ϵ *there exist* $n+1$ *discrete families* $\mathcal{U}_i$ *of mesh* $< \epsilon$ *such that the union* $\cup \mathcal{U}_i$ *forms a cover of* X.

Alexandroff-Hurewicz Theorem. $\dim X \leq n \Leftrightarrow$ *for every map* $\phi : A \to S^n$ *of a closed subset* $A \subset X$ *there is an extension* $\bar{\phi} : X \to S^n$.

The *cohomological dimension* $\dim_{\mathbf{Z}} X$ is the smallest n such that $\check{H}_c^{n+1}(U) = 0$ for all open sets $U \subset X$. The notion of cohomological dimension was introduced by P.S. Alexandroff in late 20s in homology language. Since then until late 80s there was an open problem on the coincidence of dim and $\dim_{\mathbf{Z}}$. In early 30s Alexandroff, collaborating with H. Hopf, proved the following.

Alexandroff Theorem. *For finite dimensional compacta* $\dim X = \dim_{\mathbf{Z}} X$.

In 70s R.D. Edwards connected the Alexandroff problem with the following more geometric problem: *Can a cell-like map of a manifold raise dimension?* We recall that a map $f : X \to Y$ is called *cell-like* if all fibers $f^{-1}(y)$ have trivial shape. Edwards proved the following

Resolution Theorem [Wa]. *For every compactum X there is a compactum Y of* $\dim Y \leq \dim_{\mathbf{Z}} X$ *and a cell-like map* $f : Y \to X$.

In particular the Resolution Theorem allowed to extend the equality $\dim = \dim_{\mathbf{Z}}$ on classes of countable dimensional compacta, ANR-compacta and compacta with C-property [A]. The C-property is a generalization of finite dimensionality in the direction of the Ostrand theorem. A space X has *C-property* if for any sequence $\{\mathcal{U}_i\}$ of covers of X there is a sequence of disjoint families $\{\mathcal{O}_i\}$ such that $\mathcal{O}_i$ is inscribed in $\mathcal{U}_i$ and the union $\cup\mathcal{O}_i$ forms a cover of X.

The Alexandroff problem was solved by a counterexample [Dr]. That counterexample in view of the Resolution Theorem gives a cell-like map $f : S^7 \to X$ with $\dim X = \infty$. The space X is a homology manifold which is locally connected in all dimensions. Every cell-like map of a manifold induces an isomorphism of homotopy groups, homology groups and cohomology groups. It turns out to be that this fails for K-theory.

Theorem 1 [D-F]. *For any p there is a cell-like map $f : S^7 \to X$ such that* $KerK_*(f) \neq 0$ *for* $\bmod p$ *complex homology K-theory.*

Corollary. *The homology sphere X does not admit a map of degree one onto* S^7.

§2. Novikov Conjecture

Let G^k_n be the Grassmanian space of k-dimensional oriented vector subspaces in n-space with the natural topology. There is the natural imbedding $G^k_n \subset G^k_{n+1}$. Then one can define the space $G^k_\infty = \lim_\to G^k_n$. The natural imbedding $G^k_\infty \subset G^{k+1}_\infty$ leads to the definition of the space $BO = G^\infty_\infty = \lim_\to G^k_\infty$. The tangent bundle of an n-dimensional manifold N can be obtained as the pull-back from the natural n-bundle over the space G^n_∞. Let $f_\tau : N \to BO$ be a map which induces the tangent bundle on N. The cohomology ring $H^*(BO;\mathbf{Q})$ is a polynomial ring generated by some elements $a_i \in H^{4i}(BO;\mathbf{Q})$. The rational Pontryagin classes of a manifold N are the elements $p_i = f^*(a_i) \in H^{4i}(BO;\mathbf{Q})$. Novikov proved [N] that the rational Pontryagin classes are topological invariants. It was known that they are not homotopy invariants. Hirzebruch found polynomials $L_k(p_1,\dots,p_k) \in H^{4k}(N;\mathbf{Q})$ which do not depend on N and such that the signature of every closed (oriented) $4k$ manifold N can be defined as the value of L_k on the fundamental class of N. Note that the signature is homotopy invariant and even more, it is bordism invariant. For non-simply connected manifolds Novikov defined the higher signature as follows. Let Γ be the fundamental group of a closed oriented manifold N, let $g : N \to B\Gamma = K(\Gamma,1)$ be a map classifying the universal cover of N and let $b \in H^*(K(\Gamma,1);\mathbf{Q})$. Then he defines the b-signature as $sign_b(N) =$

$\langle L_k \cap g^*(b), [N]\rangle$, here $4k + \dim(b) = \dim N$. These rational numbers $sign_b(N)$ are called *higher signatures*. The higher signature are the only possible homotopy invariants [M]. The Novikov conjecture states that they are homotopy invariant.

NOVIKOV CONJECTURE. *Let $h : N \to M$ be an orientation preserving homotopy equivalence between two close oriented manifolds, then $sign_b(N) = sign_b(M)$ for any $b \in H^*(K(\Gamma, 1); \mathbf{Q})$.*

We say that the Novikov conjecture holds for a group Γ if it holds for every manifold with the fundamental group Γ.

A tangent bundle can be defined for a topological manifold N as well. This bundle is classified by a map $f : M \to BTOP$ where $TOP = \lim_{\to} TOP_n$ and TOP_n is a topological group of homeomorphisms $f : \mathbf{R}^n \to \mathbf{R}^n$ with $f(0) = 0$. Since the natural map $BO \to BTOP$ induces an isomorphism of rational cohomology groups, one can define Pontryagin classes and higher signatures for M. In the TOP category there is a functorial 4-periodic surgery exact sequence:

$$\ldots \to \mathcal{S}_n(N) \xrightarrow{\eta} H_n(N; \mathbb{L}) \xrightarrow{\alpha} L_n(\Gamma) \to \mathcal{S}_{n-1}(N) \to \ldots,$$

where Γ is the fundamental group of X, $L_n(\Gamma)$ are Wall's groups, $\mathbb{L}$ is a periodic spectrum generated by G/TOP, and $\mathcal{S}_n(N)$ is the group of manifold structures on N with possible summand $\mathbf{Z}$. The group $\mathcal{S}_n(N)$ can be defined as the group of classes of homotopy equivalences $q : M \to \bar{N}$ with $q\mid_{\partial M} = 1_{\partial \bar{N}}$, here $\bar{N}$ is a regular neighborhood of N in some euclidean space of dimension $n + 4l$ [We]. This sequence is defined for any finite polyhedron. One can consider the lost tribe manifolds [B-F-M-W] to avoid possible extra $\mathbf{Z}$s in the definition of $\mathcal{S}_*(N)$.

The *higher L-genus* of an n-manifold N with the fundamental group Γ is an element $g_*(L(N) \cap [N]) \in H_*(B\Gamma; \mathbf{Q}) = \oplus H_i(B\Gamma; \mathbf{Q})$. This notion is dual to the higher signatures. The Novikov conjecture is equivalent to the statement that for any homotopy equivalence $h : M \to N$ the higher L-genuses of N and M are equal. Note that $H_n(X; \mathbb{L}) \otimes \mathbf{Q} = \oplus_{i=n \mod 4} H_i(X; \mathbf{Q})$. The morphism η takes a homotopy equivalence $q : M \to N$ to the difference $L(M) \cap [M] - L(N) \cap [N]$. Assume that $B\Gamma$ is a finite complex i.e. Γ is geometrically finite, then the map $g : N \to B\Gamma$ and the periodic surgery exact sequence produce the diagram

$$\begin{array}{ccccc} S_*(N) & \xrightarrow{\eta} & H_*(N; \mathbf{Q}) & \xrightarrow{\alpha} & L_*(\Gamma) \otimes \mathbf{Q} \\ & & {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle =} \\ & & H_*(B\Gamma; \mathbf{Q}) & \xrightarrow{A} & L_*(\Gamma) \otimes \mathbf{Q} \end{array}$$

So g_* takes the image of the class of a homotopy equivalence q to the difference of the higher signatures of M and N. Thus, the injectivity of the *assembly map* $A : H_*(B\Gamma; \mathbf{Q}) \to L_*(\Gamma) \otimes \mathbf{Q}$ implies the Novikov conjecture. The opposite is also true [K-M].

In the case of geometrically finite Γ it makes sense to ask whether the integral assembly map $A : H_*(B\Gamma; \mathbb{L}) \to L_*(\Gamma)$ is a split monomorphism. This is called the *integral Novikov conjecture*. By Davis' trick with Coxeter groups, it follows that every finite aspherical complex is a retract of a closed aspherical manifold. A diagram chasing shows that in the class of geometrically finite groups for studying the Novikov conjecture it suffices to consider the case when $B\Gamma$ is a manifold. In that case the universal cover $E\Gamma = X$ is a contractible manifold. Without loss of generality we may assume that X homeomorphic to the euclidean space.

A special case of the Novikov conjecture is the following:

Gromov-Lawson Conjecture. *An aspherical manifold cannot carry a metric of a positive scalar curvature.*

An open n-dimensional riemannian manifold X is called *hypereuclidean* if there is a Lipschitz map $f : X \to \mathbf{R}^n$ of degree one. The Gromov-Lawson conjecture holds true for hypereuclidean manifolds [G-L]. A metric space X is called *uniformly contractible* if for every $R > o$ there is $S > o$ such that any R-ball $B_R(x)$ centered at x can be contracted to a point in $B_S(x)$ for any $x \in X$. A typical example of a uniformly contractible manifold is a universal cover of a closed aspherical manifold with the lifted metric. A positive answer to the following problem [G2] would imply the Gromov-Lawson conjecture.

Is every uniformly contractible manifold hypereuclidean?

There is also an analytic approach to the Novikov conjecture which reduces the problem to the question of an injectivity of an analytic assembly map $\mathcal{A} : K_*(B\Gamma) \to K_*(C^*(\Gamma))$, where the right part is an algebraic K-theory of some C^*-algebra. This assembly map can be defined in terms of a universal cover $E\Gamma$ [B-C]. Then the assembly map and the conjecture can be to extended to general metric spaces [R1], [H-R].

Coarse Baum-Connes Conjecture [R1],[R2]. *For every uniformly contractible bounded geometry metric space X the assembly map $A : K_*(X) \to K_*(C^*X)$ is a monomorphism (isomorphism).*

A metric space has a *bounded geometry* if for any $\epsilon > 0$ for every $R > 0$ there is m such that every R-ball contains an ϵ-net consisting of $< m$ points. It is clear that every finitely presented group has a bounded geometry. Without this restriction the coarse Baum-Connes conjecture is not true [D-F-W]. A description of the C^*-algebra $C^*(X)$ can be found in [H-R],[R2]. We note that the coarse Baum-Connes conjecture implies the Gromov-Lawson conjecture [R1] and the isomorphism version of it implies the Novikov conjecture [R2].

A fascinating result in the coarse approach to the Novikov Conjecture was obtained by Yu [Yu]. He proved the following.

Theorem [Yu]. *If a proper uniformly contractible metric space X has a finite asymptotic dimension, then the coarse Baum-Connes conjecture holds for X.*

The definition of asymptotic dimension is given in the next section where we also sketch the idea how to prove Yu's theorem.

§3. Coarse topology

A metric space (X, d) is called *proper* if every closed ball $B_r(x_0) = \{x \in X \mid d(x, x_0) \leq r\}$ is compact. A map between proper metric spaces $f : (X, d_X) \to$

(Y, d_Y) is called a *coarse morphism* [R2] if it is proper and uniformly expansive i.e. $f^{-1}(C)$ is compact for every compact C and for any $R > 0$ there is $S > 0$ such that $d_Y(f(x), f(x')) < S$ if $d_X(x, x') < R$. Note that every Lipschitz map is a coarse morphism. Vice versa, for a geodesic metric space there are $R > 0$ and $\lambda > 0$ such that $d_Y(f(x), f(x')) < \lambda d_X(x, x')$ for all x, x' with $d_X(x, x') \geq R$. Such maps are called *coarsely Lipschitz*.

In this section we consider a category $\mathcal{C}$ of proper metric spaces with proper coarsely Lipschitz maps as morphisms. The Coarse category is the quotient of $\mathcal{C}$ by the equivalence stating that any two morphisms, which are in a finite distance from each other, are equivalent. We consider only uniformly contractible metric spaces. In the case of general proper metric spaces one should consider morphisms which are not necessarily continuous and the properness should be replaced by the following: $f^{-1}(B)$ is bounded for every bounded set B. In many cases a general type metric space (X, d) admits a uniformly contractible filling $X' \supset X$ with $(X, d' \mid_X)$ coarsely equivalent to (X, d). Thus geometrically finite groups Γ with word metric d have a filling called a universal cover of $B\Gamma$ with lifted metric d'.

Note that a closed subspace $Y \subset X$ of a proper metric space X with the induced metric is an object of $\mathcal{C}$. We define the notion of an absolute extensor in $\mathcal{C}$ as usual: $X \in AE(\mathcal{C})$ if for any $Z \in \mathcal{C}$ and for any closed $A \subset Z$ and a morphism $\phi : A \to X$ there is an extension $\bar{\phi} : Z \to X$.

Let $\mathbf{R}^n_+$ denote the halfspace of dimension n with the induced metric.

Theorem 2. *$\mathbf{R}^n_+ \in AE(\mathcal{C})$ for all n.*

Note that $\mathbf{R}^n$ is not AE.

We define a *coarse neighborhood* W of $Y \subset X$ as a subset of X with $\lim dist(y, X \setminus W) = \infty$ as $y \in Y$ approaches infinity. Define a finite open cover of (X, d) as a finite cover of X by open coarse neighborhoods with the Lebesgue function $\lambda(x)$ tending to infinity as x approachs infinity.

Note that, $\mathbf{R}^{n+1}$ is obtained from $\mathbf{R}^n$ by the operation analogous to the suspension. By analogy with Alexandroff-Hurewicz theorem we define a coarse dimension $\dim^c(X, d)$ as follows:

$\dim^c(X, d) \leq n$ *if and only if for every closed subspace $A \subset X$ and any coarse morphism $\phi : A \to \mathbf{R}^{n+1}$ there is an extension to a coarse morphism $\bar{\phi} : X \to \mathbf{R}^{n+1}$.*

Here we use $\mathbf{R}^{n+1}$ as an analog of S^n in order to have the equality $\dim^c \mathbf{R}^n = n$. By Pontryagin-Nobeling theorem every n-dimensional compactum can be embedded in the cube I^{2n+1}. Then the following problem is quite natural.

EMBEDDING PROBLEM. *Does a metric space with* $\dim^c(X, d) \leq n$ *have a coarse embedding in the space* $\mathbf{R}^{2n+2}_+$?

M. Gromov defined [G1] the notion of asymptotic dimension using a coarse analog of the Ostrand theorem. By the definition $\operatorname{asdim}(X, d) \leq n$ *if for any $R > 0$ there are $n+1$ R-disjoint uniformly bounded families $\mathcal{U}_i$ such that the union forms a cover of X.* The inequality $\operatorname{asdim}(X, d) \leq n$ means that X is coarse equivalent to a simplicial complex K_R of dimension $\leq n$ with all simplices with edges of the length R for an arbitrary large R. This property leads to the notion of *anti Čech approximation* of X by simplicial complexes. Metric spaces that admit an anti-Čech approximation by finite simplicial complexes are called *spaces of bounded geometry*. We note that universal covers of classifying spaces of geometrically finite groups Γ supplied with a Γ-invariant metric are spaces of bounded geometry. J. Roe defined coarse homology (cohomology) of a metric space using anti-Čech approximation. This leads to the definition of asymptotical cohomological dimension of a metric space. Another approach to the cohomological dimension is the following. Since we are already have n-cells, one can define CW-complexes in the coarse category. Coarse homotopy groups we define below. Then we can construct a coarse Eilenberg-MacLane complexes $K(\mathbf{Z}, n)$ and define $\operatorname{asdim}_{\mathbf{Z}} X \leq n$ if every partial map on X to $K(\mathbf{Z}, n)$ can be extended.

The following is an analog of Kuratowski-Dugundji theorem.

Proposition 1. *Let X be uniformly contractible proper metric space with* $\operatorname{asdim} X < \infty$*, then* $X \in ANE(\mathcal{C})$.

Following Gromov's idea, we define a homotopy in the coarse category as a morphism of the set $D_X = \{(x,t) \in X \times \mathbf{R} \mid |t| \leq d(x, x_0)\}$ where $x_0 \in X$ is a based point. Note that the subspaces $D_X^+ = \{(x, d(x, x_0)\} \subset D_X$ and $D_X^- = \{(x, -d(x, x_0)\} \subset D_X$ are coarsely isomorphic to X. It is possible to show that coarse homotopic maps induce the same homomorphism of coarse homology (cohomology) groups. The next natural notion is *coarse homotopy type*. Thus, $\mathbf{R}^n$ and $\mathbb{H}^n$ have the same coarse homotopy type. It turns out to be that the coarse Baum-Connes conjecture is invariant under coarse homotopy equivalence [R2]. It is possible to show that the coarse Baum-Connes conjecture holds for coarse polyhedra [R2] and hence for metric spaces which are coarse homotopy equivalent to polyhedra. Now Yu's theorem would follow from Proposition 1 and a coarse analog of the West theorem: ANE-space is homotopy equivalent to a polyhedron. The following straightforward proposition allows to give a simpler approach.

Proposition 2. *Let a metric space X be coarse homotopically dominated by a space Y. Assume that the Baum-Connes conjecture holds for Y, then it holds for X as well.*

Let $f_0 : \mathbf{R}_+ \to X$ be a coarse morphism. A *coarse loop* $\phi : \mathbf{R}^2_+ \to X$ is a morphism such that $\phi\mid_{\mathbf{R}_+} = f_0 = \phi\mid_{-\mathbf{R}_+} \circ (-1)$ where $\mathbf{R}_+$ is naturally imbedded in the first factor of $\mathbf{R}^2_+ = \mathbf{R} \times \mathbf{R}_+$. The product of two coarse loop can be defined by compression of two $\mathbf{R}^2_+$ to quadrants and gluing two quadrants together.

This leads to the definition of the coarse fundamental group and higher dimensional coarse homotopy groups. Since we have the notion of the standard n-simplex in $\mathcal{C}$ we can define singular coarse homology (cohomology) of metric spaces. We expect that all theorems of classical algebraic topology hold here.

§4. Higson corona

Let (X, d) be a metric space and let $f : X \to \mathbf{R}$ be a function on X. An *r-variation* of f at $x \in X$ is the following number $V_r(f(x)) = \sup\{|f(x) - f(y)| \mid y \in B_r(x)\}$. Let $B(X)$ be the set of all bounded functions $f : X \to \mathbf{R}$ with $\lim_{x\to\infty} V_r(f(x)) = 0$ for ant $r > 0$. We define the Higson compactification of X as the closure $\bar{X}$ of

X embedded in $I^{B(X)}$ by the family $\{f_b \mid b \in B(X)\}$. The remainder $\nu X = \bar{X} \setminus X$ of the Higson compactification is called the *Higson corona* [H],[R1].

The Higson corona is an invariant of a coarse isometry. Hence the Higson corona of a discrete finitely generated group Γ is a group invariant, i.e. $\nu\Gamma$ does not depend on choice of a word metric on Γ. Thus, two metric spaces in a finite distance in the Gromov-Hausdorff metric space have the same Higson coronas. Moreover, the Higson corona is a functor $\nu : \mathcal{C} \to \mathcal{C}omp$ from the coarse category to the category of compact Hausdorff spaces, taking embeddings to embeddings.

There is a partial order on compactifications of a given (locally compact) space X. A compactification cX is dominated by a compactification $c'X$ if there is a continuous map $f : c'X \to cX$ with $f|_X = 1_X$. A compactification, dominated by the Higson compactfication, we call *Higson dominated*.

Many asymptotic properties of metric spaces can be formulated in terms of the Higson corona. We give two examples of such properties. A notion of small action of a discrete group Γ at infinity of a universal cover X of $B\Gamma$ appears naturally in the combinatorial group theory. Thus, Bestvina takes that property as an axiom of his Z-boundary of a group [B]. An action of Γ is *small at infinity* for a given compactification $\bar{X}$ of X if for every $x \in \bar{X} \setminus X$ and a neighborhood U of x in $\bar{X}$, for every compact set $C \subset X$ there is a smaller neighborhood V such that $g(C) \cap V \neq \emptyset$ implies $g(C) \subset U$ for all $g \in \Gamma$. We consider a Γ-invariant metric on X. Since $B\Gamma$ is a finite complex, the Higson corona of X does not depend on choice of metric and coincides with the Higson corona of Γ.

Proposition 3. *The action of* Γ *on* X *is small at infinity for a compactification* $\bar{X}$ *if and only if* $\bar{X}$ *is Higson dominated.*

Existence of such compactification is crucial in all cases were the Novikov conjecture is proved.

Another property is also related to the Novikov Conjecture.

Theorem [R1]. *An open* n*-manifold* M *is hypereuclidean if and only if there is a map* $f : \nu M \to S^{n-1}$ *of degree one.*

Since a dimension is an important invariant in the coarse theory we establish the following.

Theorem 3. $\dim \nu X = \dim^c(X, d)$ *for a proper metric space* (X, d)*.*

Theorem 4. $\dim \nu X = \operatorname{asdim} X$ *if* $\operatorname{asdim} X < \infty$*.*

Conjecture 1. $\dim \nu X = \operatorname{asdim} X$ *for all* X*.*

Note that the inequality $\dim \nu X \leq \operatorname{asdim} X$ always holds [D-K-U]. The proof of this inequality makes plausible that $\nu\Gamma$ has the C-property for geometrically finite group Γ. This together with Ancel's theorem (§1), Conjecture 1 and the following conjecture define another approach to the Novikov Conjecture for all geometrically finite groups.

Conjecture 2. $\dim_{\mathbf{Z}} \nu X \leq \operatorname{asdim}_Z X$*.*

The following conjecture is somewhat weaker of the rational Gromov-Lawson conjecture and it is equivalent to Gromov-Lawson's for even dimensional manifolds [D-F].

Weinberger Conjecture. *For every uniformly contractible n-manifold X the boundary homomorphism $\partial : \check{H}^{n-1}(\nu X; \mathbf{Q}) \to H^n_c(X; \mathbf{Q}) = \mathbf{Q}$ is an epimorphism.*

If X is an universal cover of finite $B\Gamma$ and the homomorphism ∂ in the Weinberger Conjecture is equivariantly split, then the Novikov conjecture for Γ holds true. The following theorem shows that there is a room for a $n-1$-cocycle in νX.

Theorem 5. *For every uniformly contractible open n-manifold X^n, $\dim \nu X^n \geq n$.*

The exact sequence of pair implies that the Weinberger Conjecture would hold for X^n if the Higson compactification $\bar{X}$ has trivial rational cohomology: $H^n(\bar{X}; \mathbf{Q}) = 0$. The following theorem sets limits to this approach.

Theorem 6 [D-F]. $H^n(\overline{\mathbf{R}^n}; \mathbf{Q}) \neq 0$ *and* $H^n(\overline{\mathbb{H}^n}; \mathbf{Q}) = 0$ *for all* $n > 1$.

Note that $H^n(\overline{\mathbf{R}^n}; \mathbf{Q}) \neq H^n(\overline{\mathbb{H}^n}; \mathbf{Q})$ despite on the fact that $\mathbf{R}^n$ and $\mathbb{H}^n$ are coarse homotopy equivalent.

The following example gives a negative answer to the integral version of Gromov's problem.

Example [D-F-W]. *There exists a uniformly contractible riemannian metric d on $\mathbf{R}^8$ such that $(\mathbf{R}^8, d)$ is not hypereuclidean.*

This space $(\mathbf{R}^8, d)$ is coarsely isomorphic to an open cone over a homology sphere X from Theorem 1 (§1). We note that in this example $\dim \nu(\mathbf{R}^8, d) = \infty$ and $\dim_{\mathbf{Z}} \nu(\mathbf{R}^8, d) < \infty$ (see [D-K-U]). Although this example is not of bounded geometry, the Weinberger conjecture holds for it.

§5 Descent principle

In this section we compare some of the conditions which enable to prove the Novikov conjecture for certain groups. Let Γ be geometrically finite group and let $X = E\Gamma$ be equipped with a Γ-invariant metric. Each of the following four conditions implies the Novikov conjecture:

(CPI) [C-P]. *There is an equivariant rationally acyclic metrizable compactification $\hat{X}$ of X such that the action of Γ is small at infinity.*

(CPII) [C-P2]. *There is an equivariant rationally acyclic (possibly nonmetrizable) compactification $\hat{X}$ of X with a system of covers α of $Y = \hat{X} \setminus X$ by boundedly saturated sets such that the projection to the inverse limits of the nerves of α induces an isomorphism $H_*(Y; \mathbf{Q}) \to H_*(\lim_{\leftarrow} N(\alpha); \mathbf{Q})$.*

(FW) [F-W], [D-F]. *There is an equivariant Higson dominated compactification $\hat{X}$ of X such that the boundary homomorphism $H^{lf}_*(X; \mathbf{Q}) \to H_{*-1}(\hat{X} \setminus X; \mathbf{Q})$ is an equivariant split injection.*

(HR) [R1]. *There is an equivariant rationally acyclic Higson dominated compactification $\hat{X}$ of X.*

Here H_* stands for the Steenrod homology or its extension for nonmetrizable spaces. An open set $U \subset Y = \hat{X} \setminus X$ is called *boundedly saturated* if for for every closed set $C \subset \hat{X}$ with $C \cap Y \subset U$ the closure of any r-neighborhood $N_r(C \cap X)$ satisfies $\overline{N_r(C \cap X)} \cap Y \subset U$. The homomorphism $H_*(Y; \mathbf{Q}) \to H_*(\lim_{\leftarrow} N(\alpha); \mathbf{Q})$ in CPII is an isomorphism if the system $\{\alpha\}$ is cofinal. We introduce the condition.

($CPII'$). *There is an equivariant rationally acyclic (possibly nonmetrizable) compactification $\hat{X}$ of X with a cofinal system of covers α of $Y = \hat{X} \setminus X$ by boundedly saturated sets.*

We denote by CPI' the condition CPI without an assumption of metrizability of $\hat{X}$. Note that the conditions CPI' and $CPII'$ imply the Novikov conjecture as well.

Theorem 7. *$CPII' \Rightarrow CPI' \Leftrightarrow CPI \Leftrightarrow HR \Rightarrow FW \Leftarrow CPII$.*

Note that $CPI' \Leftrightarrow HR$ by Proposition 3.

In the integral case one should replace the rational homology by the $\mathbb{L}$-homology. The conditions CPI, II remain without changes, in FW and HR we have to add a metrizability of the corona. Then all four would imply the integral Novikov conjecture. It is not clear whether Theorem 7 holds in the integral case. The problem is in the implication $CPI' \Rightarrow CPI$ which can be reduced to the following.

Problem. *Is a $\mathbb{L}_*$-acyclicity equivalent to a $\mathbb{L}^*$-acyclicity for compact Hausdorff spaces?*

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Alexander N. Dranishnikov
Department of Mathematics
University of Florida
358 Little Hall, PO Box 118105
Gainesville, FL 32611-8105
USA
dranish@math.ufl.edu