A Filtration of the Set of Integral Homology 3-Spheres

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Abstract. A filtration on the set of integral homology 3-spheres is introduced, based on the universal perturbative invariant (the LMO invariant) or equivalently based on finite type invariants of integral homology 3-spheres. We also survey on these invariants related to quantum invariants.

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In 1989, Witten [14] proposed his famous formula of topological invariants of 3-manifolds, based on Chern-Simons gauge theory. The formula is given by using a path integral over all G connections on a 3-manifold M, where G is a fixed Lie group. Following combinatorial properties of the invariants predicted by Witten's formula, the invariants, what we call the *quantum G invariant*, denoted by $\tau_r^G(M)$, have been rigorously reconstructed by many researchers, say by using surgery presentations of 3-manifolds.

Since we have many Lie groups, we have obtained many quantum invariants of 3-manifolds in this decade. To control these many invariants we consider the following two approaches, where, as for the quantum invariants of knots (or links), we had obtained Kontsevich invariant and Vassiliev invariants by the two approaches.

- Unify them into an invariant.
- Characterize them with a common property.

By the first approach, we expect the existence of the universal invariant among quantum invariants, though the universal quantum invariant of 3-manifolds is not found yet. Instead of universal quantum invariants, we have the universal invariant Ω among perturbative invariants of 3-manifolds, where the perturbative G invariant is obtained from the quantum G invariant by "asymptotic expansion". By the second approach, we obtain the notion of finite type such that the d-th coefficient of a perturbative invariant is of finite type of degree d.

Further we consider how fine these invariants distinguish 3-manifolds; for simplicity we consider integral homology 3-spheres ($\mathbf{Z}$HS's) instead of 3-manifolds, in this manuscript. To describe it, we obtain a filtration on the set of $\mathbf{Z}$HS's by using the invariant Ω, or equivalently by using finite type invariants.

In section 1 we review roles of Kontsevich invariant and Vassiliev invariants to understand quantum invariants of knots (or links). In section 2, as invariants related to quantum invariants of 3-manifolds, we survey on perturbative invariants, the universal perturbative invariant (the LMO invariant) and finite type invariants. Further we introduce a filtration on the set of **Z**HS's based on these invariants in section 3.

1 Invariants of knots related to quantum invariants

We prepare some notations. Let X be a closed (possibly empty) 1-manifold. A *web diagram* on X is a uni-trivalent graph such that each univalent vertex of the graph is on X and a cyclic order of three edges around each trivalent vertex of the graph is fixed; see Figure 1 for examples of web diagrams. We denote by $\mathcal{A}(X;\mathcal{R})$ the quotient vector space over $\mathcal{R}$ spanned by web diagrams on X subject to the AS, IHX and STU relations (see Figure 2), where $\mathcal{R}$ is $\mathbf{C}$ or $\mathbf{Z}$. We define the degree of a web diagram to be half the number of univalent and trivalent vertices of the uni-trivalent graph of the web diagram. We denote by $\mathcal{A}(X;\mathcal{R})^{(d)}$ the subspace spanned by web diagrams of degree d. We denote by $\hat{\mathcal{A}}(X;\mathcal{R})$ the completion of $\mathcal{A}(X;\mathcal{R})$ with respect to the degree. We have a map $\mathcal{A}(X;\mathbf{C}) \to \mathbf{C}$ obtained by "substituting" a Lie algebra $\mathbf{g}$ to dashed lines of web diagrams and a representation R of $\mathbf{g}$ solid lines of web diagrams; we call the map *weight system* and denote it by $W_{\mathbf{g},R}$, and we define $\hat{W}_{\mathbf{g},R} : \hat{\mathcal{A}}(X;\mathbf{C}) \to \mathbf{C}[[h]]$ by $\hat{W}_{\mathbf{g},R}(D) = W_{\mathbf{g},R}(D)h^{\deg(D)}$. For precise definitions of these notations see for example [12].

Figure 1: A web diagram on S^1 and a web diagram on ϕ. For a web diagram on X, the uni-trivalent graph of the web diagram is depicted by dashed lines and X solid lines.

1.1 Kontsevich invariant

The quantum $(\mathbf{g}, R)$ invariant $Q^{\mathbf{g},R}(L)$ of a link L can be constructed by using monodromy of solutions of the Knizhnik-Zamolodchikov equation (the KZ equation) where a Lie algebra $\mathbf{g}$ and its representation R are included in the equation. By considering "universal" version of the KZ equation obtained by replacing $\mathbf{g}$ with a dashed line and R a solid line of web diagrams, we obtain Kontsevich invariant instead of the quantum $(\mathbf{g}, R)$ invariant; see for example [1]. In the following, instead of the original Kontsevich invariant, we use the modified Kontsevich invariant [8], denoted by $\hat{Z}(L) \in \hat{\mathcal{A}}(\coprod^l S^1;\mathbf{C})$ for a framed link L with l components.

The AS relation : $= -$

The IHX relation : $= \qquad -$

The STU relation : $= \qquad -$

Figure 2: Definition of the AS, IHX and STU relations

For a knot K, the modified Kontsevich invariant $\hat{Z}(K)$ can be expressed as the exponential of a linear sum of web diagrams of connected dashed graphs as

$$\hat{Z}(K) = \exp\Big(a_1 \qquad + a_2 \qquad + a_3 \qquad + \cdots\Big),$$

because $\hat{Z}(K)$ is group-like with respect to a Hopf algebra structure of $\hat{\mathcal{A}}(S^1; \mathbf{C})$, and a group-like element can be, in general, expressed as the exponential of a primitive element, which is a linear sum of web diagrams of connected dashed graphs here; see for example [12] for this argument.

By the origin of Kontsevich invariant, any quantum invariant of links recovers from $\hat{Z}$ as

Theorem 1 (see for example [12]) (universality of $\hat{Z}$ among quantum invariants) *For any framed link L, we have*

$$\hat{W}_{\mathbf{g},R}\big(\hat{Z}(L)\big) = Q^{\mathbf{g},R}(L)|_{q=e^h}.$$

1.2 Vassiliev invariants

In this section we review Vassiliev invariants, which characterize quantum invariants in the sense of Corollary 3 below. We introduce Habiro's clasper to describe the weight system of a Vassiliev invariant.

Let $\mathcal{K}$ be the vector space freely spanned by isotopy classes of framed knots. For a knot K and a set C of crossings of K, we put

$$[K, C] = \sum_{C' \subset C} (-1)^{\# C'} K_{C'}$$

where the sum runs over all subset C' of C including the empty set and $K_{C'}$ denotes the knot obtained from K by crossing changes at the crossings of C'. We put $\mathcal{K}_d$ to be the vector subspace of $\mathcal{K}$ spanned by $[K, C]$ such that K is a knot and C is a set of d crossings of K. A linear map $v : \mathcal{K} \to \mathbf{C}$ is called a *Vassiliev invariant* (or a *finite type invariant*) of degree d if $v|_{\mathcal{K}_{d+1}} = 0$.

For a Vassiliev invariant v of degree d, we have a natural linear map $\varphi : \mathcal{A}(S^1; \mathbf{C})^{(d)} \to \mathcal{K}_d/\mathcal{K}_{d+1}$, which is called the *weight system* of v; see for example [1]. Habiro [5] gave a reconstruction of φ as shown in Figure 3 using claspers; see Figure 4 for the definition of Habiro's clasper.

$$\overset{\varphi}{\longmapsto} \qquad\qquad -$$

Figure 3: Habiro's reconstruction of the map $\varphi : \mathcal{A}(S^1; \mathbf{C})^{(d)} \to \mathcal{K}_d/\mathcal{K}_{d+1}$ by his claspers. The image of a web diagram by φ in $\mathcal{K}_d/\mathcal{K}_{d+1}$ does not depend on the choice of a knot (the middle picture) and an embedding of the edges of claspers (in the right picture).

denotes , or alternatively .

Figure 4: Definition of Habiro's clasper: The right picture implies the result obtained by integral surgery along Hopf link (in the picture) with 0 framings.

THEOREM 2 (KONTSEVICH) (universality of $\hat{Z}$ among Vassiliev invariants) *For any positive integer d, any Vassiliev invariant v of degree d is expressed as the composite map*

$$v : \{knots\} \xrightarrow{\hat{Z}} \hat{\mathcal{A}}(S^1; \mathbf{C}) \xrightarrow{projection} \mathcal{A}(S^1; \mathbf{C})^{(\leq d)} \xrightarrow{W} \mathbf{C}$$

with some linear map W. Conversely, for any linear map $W : \mathcal{A}(S^1; \mathbf{C})^{(d)} \to \mathbf{C}$ the above composite map v is a Vassiliev invariant of degree d.

As a corollary of Theorems 1 and 2,[1] we have

COROLLARY 3 (see for example [1, 12]) *The d-th coefficient of $Q^{\mathbf{g},R}(K)|_{q=e^h}$ is a Vassiliev invariant of degree d* (BIRMAN-LIN)*. Further the weight system of the Vassiliev invariant is equal to $W_{\mathbf{g},R}$* (PIUNIKHIN)*.*

[1]Before the theorems Corollary 3 had been directly proved by Birman-Lin and Piunikhin.

2 Invariants of 3-manifolds related to quantum invariants

In this section, as invariants related to quantum invariants of 3-manifolds, we survey on perturbative invariants, the universal perturbative invariant (the LMO invariant) and finite type invariants.

2.1 Perturbative invariants of rational homology 3-spheres

By taking asymptotic expansion of Witten's path integral formula [14] of a quantum invariant, we obtain a power series as an invariant of 3-manifolds. As for the quantum G invariant $\tau_r^G(M)$ reconstructed rigorously, we take "asymptotic expansion" at $r \to \infty$ in the sense below.

For simplicity we consider the case $G = SO(3)$ here. It is known that the quantum $SO(3)$ invariant $\tau_r^{SO(3)}(M)$ belongs $\mathbf{Z}[\zeta]$ for any odd prime r,' where $\zeta = \exp(2\pi\sqrt{-1}/r)$. Further it is known [11] that for a rational homology 3-sphere M there exists the series of $\lambda_n \in \mathbf{Z}[1/2, 1/3, \cdots, 1/(2n+1)]$ satisfying

$$\begin{aligned} \tau_r^{SO(3)}(M) &= \left(\frac{|H_1(M;\mathbf{Z})|}{r}\right) \sum_{i=0}^{N} \lambda_n (\zeta - 1)^n \\ &+ \Big(\text{terms divisible by } (\zeta-1)^{N+1} \text{ in } \mathbf{Z}[\zeta, \tfrac{1}{2}, \tfrac{1}{3}, \cdots, \tfrac{1}{2N+1}]\Big), \end{aligned}$$

for any positive integer N and any odd prime $r > \max\{2N+1, |H_1(M;\mathbf{Z})|\}$, where $\left(\frac{\cdot}{r}\right)$ denotes the Legendre symbol and $|H_1(M;\mathbf{Z})|$ denotes the order of the first homology group $H_1(M;\mathbf{Z})$. Using the series $\{\lambda_n\}$ we define the *perturbative* $SO(3)$ *invariant* of M by

$$\tau^{SO(3)}(M) = \sum_{n=0}^{\infty} \lambda_n (q-1)^n \in \mathbf{Q}[[q-1]].$$

For example the quantum $SO(3)$ invariant of the lens space $L(5,1)$ is expressed as

$$\begin{aligned} \tau_r^{SO(3)}\Big(L(5,1)\Big) &= \left(\frac{5}{r}\right) \zeta^{-3/5} \frac{\zeta^{1/10} - \zeta^{-1/10}}{\zeta^{1/2} - \zeta^{-1/2}} \\ &= \left(\frac{5}{r}\right) \Big(\frac{1}{5} - \frac{3}{5^2}(\zeta - 1) + \frac{11}{5^3}(\zeta-1)^2 + \cdots\Big). \end{aligned}$$

Hence the perturbative $SO(3)$ invariant of $L(5,1)$ is given by

$$\begin{aligned} \tau^{SO(3)}\Big(L(5,1)\Big) &= \frac{1}{5} - \frac{3}{5^2}(q-1) + \frac{11}{5^3}(q-1)^2 + \cdots \\ &= q^{-3/5} \frac{q^{1/10} - q^{-1/10}}{q^{1/2} - q^{-1/2}} \in \mathbf{Q}[[q-1]]. \end{aligned}$$

Perturbative invariants might dominate quantum invariants. This can be partially shown as follows, if we assumed Lawrence's conjecture "$\tau^{SO(3)}(M) \overset{?}{\in}$

$\mathbf{Z}[[q-1]]$ for any integral homology 3-sphere M". In the following diagram, where $F(q)$ is a cyclotomic polynomial, the value of $\tau^{SO(3)}(M)$ determines the value of $\tau_r^{SO(3)}(M)$ for any odd prime r.[2]

$$\begin{array}{cccccccc} & & & & & \mathbf{Z}[[q-1]] & \ni & \tau^{SO(3)}(M) \\ & & & & & \downarrow & & \\ \tau_r^{SO(3)}(M) \in & \mathbf{Z}[\zeta] & \overset{\text{isomorphic}}{\cong} & \mathbf{Z}[q]/F(q) & \overset{\text{injective}}{\hookrightarrow} & \mathbf{Z}[[q-1]]/F(q) & & \end{array}$$

2.2 The LMO invariant of 3-manifolds

The LMO invariant, constructed as below, is universal among perturbative invariants. See also [2] for another approach to construct such a universal invariant among perturbative invariants.

As in [9] there is a series of linear maps $\iota_n : \mathcal{A}(\coprod^l S^1; \mathbf{C}) \to \mathcal{A}(\phi; \mathbf{C})$ obtained by replacing a solid circle including m dashed univalent vertices with some dashed univalent graph with m dashed univalent vertices, such that the degree $\leq n$ part of $\iota_n(\check{Z}(L))$ is invariant under Kirby move KII (handle slide move) on a framed link L, where $\check{Z}(L)$ is a different normalization of the modified Kontsevich invariant $\hat{Z}(L)$. Further, for the 3-manifold obtained from S^3 by integral surgery along L, the degree $\leq n$ part of

$$\Omega_n(M) = \iota_n(\check{Z}(U_+))^{-\sigma_+} \iota_n(\check{Z}(U_-))^{-\sigma_-} \iota_n(\check{Z}(L)) \in \hat{\mathcal{A}}(\phi; \mathbf{C})$$

becomes invariant under Kirby moves KI and KII, where $U_\pm$ is the unknot with ± 1 framing and $\sigma_\pm$ is the number of positive or negative eigenvalues of the linking matrix of L. Hence it becomes a topological invariant of M. Further, since the degree $< n$ part of $\Omega_n(M)$ can be expressed by $\Omega_{n-1}(M)$ as in [9], we put

$$\Omega(M) = \sum_{n=0}^{\infty} \Big(\text{the degree } n \text{ part of } \Omega_n(M)\Big) \in \hat{\mathcal{A}}(\phi; \mathbf{C})$$

and call it the *LMO invariant* (or the *universal perturbative invariant*). Further for a rational homology 3-sphere M we put

$$\hat{\Omega}(M) = \sum_{n=0}^{\infty} |H_1(M; \mathbf{Z})|^{-n} \Big(\text{the degree } n \text{ part of } \Omega_n(M)\Big) \in \hat{\mathcal{A}}(\phi; \mathbf{C}).$$

For example [9], for the 3-manifold $M_{n,k}$ obtained from S^3 by integral surgery along $(2,n)$ torus knot with k framing, the LMO invariant $\Omega(M_{n,k})$ is given by

$$\begin{aligned} \Omega(M_{n,k}) \quad = \quad & \exp\Big(\frac{1}{48}(3n^2 - k^2 + 3k - 5) \\ & + \frac{1}{2^7 \cdot 3^2}(12n^4 - 12kn^3 + 3k^2n^2 - 15n^2 + 12kn - 4k^2 + 4) \end{aligned}$$

[2] This argument was suggested by Thang Le.

$$+(\text{terms of degree} \geq 3)\Big).$$

In general $\Omega(M)$ can be expressed as the exponential of a linear sum of connected web diagrams like the case of $\hat{Z}(K)$ in section 1.1; see [9].

The following theorem is proved for $n=2$ in [13] and for general n by Lev Rozansky and Thang Le.

Theorem 4 (universality of the LMO invariant among perturbative invariants) *For a rational homology 3-sphere M, the perturbative $PSU(n)$ invariant recovers from the LMO invariant as*

$$\tau^{PSU(n)}(M) = |H_1(M;\mathbf{Z})|^{-n(n-1)/2}\hat{W}_{sl_n}(\hat{\Omega}(M)).$$

2.3 Finite type invariants of integral homology 3-spheres

The notion of finite type invariants of integral homology 3-spheres ($\mathbf{Z}$HS's) was introduced in [10] by replacing "crossing change" on knots in the definition of Vassiliev invariants (in section 1.2) with "integral surgery" on $\mathbf{Z}$HS's. See also [3] for recent development of finite type invariants of 3-manifolds.

We call a framed link L in a $\mathbf{Z}$HS M *algebraically split* if the linking number of any two components of L is zero, and call L *unit-framed* if the framing of each component of L is ± 1. Let $\mathcal{M}$ be the vector space over $\mathbf{C}$ freely spanned by homeomorphism classes of $\mathbf{Z}$HS's. For a $\mathbf{Z}$HS M and an algebraically split and unit-framed link L in M, we put

$$[M,L] = \sum_{L' \subset L} (-1)^{\#L'} M_{L'} \in \mathcal{M},$$

where the sum runs over all sublinks L' in L including the empty link, and $M_{L'}$ denotes the $\mathbf{Z}$HS obtained from M by integral surgery along L'. Further we put $\mathcal{M}_d$ to be the vector subspace of $\mathcal{M}$ spanned by $[M,L]$ such that M is a $\mathbf{Z}$HS and L is an algebraically split and unit-framed link with d components in M. Then, as shown in [4], the equalities $\mathcal{M}_{(3d)} = \mathcal{M}_{(3d-1)} = \mathcal{M}_{(3d-2)}$ hold. Hence we put again $\mathcal{M}_d = \mathcal{M}_{(3d)}$. A linear map $v : \mathcal{M} \to \mathbf{C}$ is defined to be of *finite type* of degree d if $v|_{\mathcal{M}_{d+1}} = 0$.

For a finite type invariant v of degree d, we have a linear map $\varphi : \mathcal{A}(\phi;\mathbf{C}) \to \mathcal{M}_d/\mathcal{M}_{d+1}$ associated with v as shown in [4]; the linear map φ is called the *weight system* of v. Habiro [5] gave a reconstruction of φ as shown in Figure 5. By Theorem 5 below, the weight system φ becomes an isomorphic linear map.

Theorem 5 ([7]) (universality of the LMO invariant among finite type invariants) *For any positive integer d, any finite type invariant v of degree d is expressed as the composite map*

$$v : \{\mathbf{Z}HS\text{'s}\} \xrightarrow{\Omega} \hat{\mathcal{A}}(\phi;\mathbf{C}) \xrightarrow{\text{projection}} \mathcal{A}(\phi;\mathbf{C})^{(\leq d)} \xrightarrow{W} \mathbf{C}$$

with some linear map W. Conversely, for any linear map $W : \mathcal{A}(\phi)^{(d)} \to \mathbf{C}$ the above composite map v is a finite type invariant of degree d.

As a corollary of Theorems 4 and 5,[3] we have

[3]Before the theorems Corollary 6 for $n=2$ had beed directly proved by Kricker-Spence [6].

$\overset{\varphi}{\longmapsto}$ $\qquad\qquad\qquad$ $-$

Figure 5: Habiro's reconstruction of the map $\varphi : \mathcal{A}(\phi; \mathbf{C})^{(d)} \to \mathcal{M}_d/\mathcal{M}_{d+1}$ by his claspers. The image of a web diagram by φ in $\mathcal{M}_d/\mathcal{M}_{d+1}$ does not depend on the choice of a **Z**HS (the middle picture) and an embedding of the edges of claspers (in the right picture).

COROLLARY 6 *The d-th coefficient of the perturbative $PSU(n)$ invariant is of finite type of degree d. Moreover the weight system of the finite type invariant is equal to W_{sl_n}.*

3 A FILTRATION OF THE SET OF INTEGRAL HOMOLOGY 3-SPHERES

How fine do quantum invariants distinguish **Z**HS's?[4] Since perturbative invariants might dominate quantum invariants (see section 2.1), this question might be reduced to study of perturbative invariants. Further, by universality of the LMO invariant among perturbative invariants (Theorem 4), the question might be reduced to study of the LMO invariant $\Omega(M)$. Furthermore, since $\Omega(M)$ can be expressed as $\Omega(M) = \exp\Big(\omega(M)\Big)$ (see section 2.2), the question is reduced to the question: how fine does the invariant $\omega(M)$ distinguish **Z**HS's?

On the other hand, noting that $\Omega(M)$ is a universal finite type invariant as in Theorem 5, we also consider the question: how fine do finite type invariants distinguish **Z**HS's? To distinguish them, we define the d-th equivalence relation $\sim_d$ among **Z**HS's as follows. Two **Z**HS's M and M' are *d-th equivalent*, denoted by $M \sim_d M'$, if $v(M) = v(M')$ for any finite type invariant v of degree $< d$. We have the ascending series of the sets of the equivalence classes;

$$\{\mathbf{Z}\text{HS's}\}/\sim_1 \longleftarrow \{\mathbf{Z}\text{HS's}\}/\sim_2 \longleftarrow \{\mathbf{Z}\text{HS's}\}/\sim_3 \longleftarrow \cdots.$$

Such equivalence relations have been originally studied by Habiro by using his claspers; to be precise his definition of the equivalence relations is slightly stronger than ours. Habiro showed that generators of $\{\mathbf{Z}\text{HS's}\}/\sim_d$ are given by web diagrams of degree $< d$ in the sense that the equivalence relation is generated

[4]For simplicity we here discuss for **Z**HS's, not for more general 3-manifolds. To distinguish, say, rational homology 3-spheres in the same way, we might need, not only the invariant ω, but isomorphism classes of cohomology rings.

by the relation obtained by putting the image of the map in Figure 5 to be zero. As a corollary of results of Habiro [5] we have

Theorem 7 (a corollary of results in [5]) *The set $\{\mathbf{Z}HS\text{'s}\}/\sim_d$ becomes a commutative group which is isomorphic to $\mathcal{A}(\phi;\mathbf{Z})^{(<d)}_{conn}$, where $\mathcal{A}(\phi;\mathbf{Z})_{conn}$ denotes the subspace of $\mathcal{A}(\phi;\mathbf{Z})$ spanned by connected web diagrams. By taking the direct limit of these isomorphisms we have the following homomorphism*

$$\{\mathbf{Z}HS\text{'s}\} \longrightarrow \varprojlim_d \{\mathbf{Z}HS\text{'s}\}/\sim_d \; \cong \hat{\mathcal{A}}(\phi;\mathbf{Z})_{conn}. \tag{1}$$

Since the map (1) gives a reconstruction of the invariant ω, Theorem 7 implies

Corollary 8 *The image of $\omega : \{\mathbf{Z}HS\text{'s}\} \to \hat{\mathcal{A}}(\phi;\mathbf{C})_{conn}$ is included in a lattice in $\hat{\mathcal{A}}(\phi;\mathbf{C})_{conn}$ which is isomorphic to $\hat{\mathcal{A}}(\phi;\mathbf{Z})_{conn}$.*

If the map (1) (or the invariant ω) was injective, we would identify $\{\mathbf{Z}$HS's$\}$ with a subset of $\hat{\mathcal{A}}(\phi;\mathbf{Z})_{\mathrm{conn}}$ by the map (1), and all invariants related to quantum invariants would be understood as functions of weight systems on the space $\hat{\mathcal{A}}(\phi;\mathbf{Z})_{\mathrm{conn}}$. Further we would expect that there should be structures on the set $\{\mathbf{Z}$HS's$\}$ induced by combinatorial structures on the space of web diagrams.

Similar arguments are also available for the set of knots. By using Vassiliev invariants, we define the equivalence relation $\sim_d$ on the set of knots in the same way as above. Then we have $\{\text{knots}\}/\sim_d \cong \mathcal{A}(S^1;\mathbf{Z})^{(<d)}_{\mathrm{conn}}$ and the homomorphism

$$\{\text{knots}\} \longrightarrow \varprojlim_d \{\text{knots}\}/\sim_d \; \cong \hat{\mathcal{A}}(S^1;\mathbf{Z})_{\mathrm{conn}},$$

which is a reconstruction of the invariant $z(K) \in \hat{\mathcal{A}}(S^1;\mathbf{C})_{\mathrm{conn}}$, where $z(K)$ is the invariant satisfying $\hat{Z}(K) = \exp\Big(z(K)\Big)$; see section 1.1. We would identify the set of knots and would expect structures on the set of knots in the same sense as the above case of $\mathbf{Z}$HS's.

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