Vector Bundles over Classifying Spaces

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Abstract. Let $\mathbb{K}(X)$ denote the Grothendieck group of the monoid of (complex) vector bundles over any given space X. This is not in general the same as the K-theory group $K(X)$. When $X = BG$, the classifying space of a compact Lie group G, then $K(BG)$ has already been described by Atiyah and Segal as a certain completion of the representation ring $R(G)$. The main result described here is that the Grothendieck group $\mathbb{K}(BG)$ also can be described explicitly, in terms of the representation rings of certain subgroups of G, and compared with both $R(G)$ and $K(BG)$.

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A vector bundle over a space X can be thought of as a collection of finite dimensional vector spaces (the fibers), one for each point in X, which are combined together in one topological space. A product $X\times\mathbb{R}^n$ or $X\times\mathbb{C}^n$ is a "trivial" vector bundle over X. The simplest example of a nontrivial vector bundle is the Möbius band, regarded as a vector bundle over the circle with fibers $\mathbb{R}^1$. One standard source of vector bundles is the tangent bundle of a smooth manifold; i.e., the union of the tangent planes at all points of the manifold (given an appropriate topology).

We focus attention here on the case of *complex* vector bundles; i.e., vector bundles whose fibers are complex vector spaces. For any topological space X, let $\mathrm{Vect}(X)$ denote the set of isomorphism classes of complex vector bundles over X. This is a commutative monoid under the operation of direct sum. Define $\mathbb{K}(X)$ to be the Grothendieck group of $\mathrm{Vect}(X)$; i.e., the abelian group of all formal differences $x - x'$ for $x, x' \in \mathrm{Vect}(X)$ (with the obvious relations).

When X is compact (e.g., a finite cell complex), then $K(X) \overset{\text{def}}{=} \mathbb{K}(X)$ is just the K-theory of X. For such X (in fact, for finite dimensional X), these groups define a cohomology theory. In other words, they have nice properties, such as forming exact sequences and Bott periodicity, which provide useful tools for calculating and applying these groups.

When working with non-compact spaces, and in particular with infinite dimensional spaces, the Grothendieck groups of vector bundles do not define a cohomology theory. So a different definition of the K-theory of X is used, one involving classifying spaces. For each n, there is a classifying space $BU(n)$ for the unitary group $U(n)$ which "classifies" bundles, in the sense that the set $\mathrm{Vect}_n(X)$ of n-dimensional complex vector bundles over X is in one-to-one correspondence with the set $[X, BU(n)]$ of homotopy classes of maps $X \to BU(n)$. More generally, for any topological group G, there is a space BG which classifies all fiber bundles with "structure group" G. For example, the structure group of an n-dimensional complex vector bundle is $GL_n(\mathbb{C})$, the group of self-transformations of the fiber $\mathbb{C}^n$ which preserve the vector space structure; but any vector bundle can be given an essentially unique (fiberwise) hermitian product which reduces its structure group to the unitary group $U(n)$.

The classifying space construction is functorial, in the sense that any homomorphism $\rho : G \to G'$ induces a map of spaces $B\rho : BG \to BG'$. The space BG is characterized (up to homotopy equivalence) as being the orbit space of a contractible space EG with a free G-action.

Regard $U(n)$ as a subgroup of $U(n+1)$ by identifying an $n \times n$-matrix A with the $(n{+}1) \times (n{+}1)$-matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$. This allows us to consider $BU(n)$ as a subspace of $BU(n+1)$, and hence to define $BU = \bigcup_{n=1}^{\infty} BU(n)$. For arbitrary X, the K-theory of X is now defined by setting $K(X) \overset{\text{def}}{=} [X, \mathbb{Z} \times BU]$. This agrees with the earlier definition $K(X) = \mathbb{K}(X)$ when X is compact, and defines a cohomology theory for general X. There is a natural homomorphism

$$\beta_X : \mathbb{K}(X) \longrightarrow K(X),$$

which is an isomorphism whenever X is compact, but not in general.

The geometrically defined functor $\mathbb{K}(-)$ can behave very differently from $K(-)$. For example, the sequence

$$\widetilde{\mathbb{K}}(\mathbb{C}P^\infty/\mathbb{R}P^\infty) \longrightarrow \mathbb{K}(\mathbb{C}P^\infty) \xrightarrow{\text{restr}} \mathbb{K}(\mathbb{R}P^\infty)$$

is not exact: if ξ_n (for $n \in \mathbb{Z}$) denotes the line bundle over $\mathbb{C}P^\infty$ with Chern class n times some fixed generator of $H^2(\mathbb{C}P^\infty) \cong \mathbb{Z}$, then $[\xi_1] - [\xi_3]$ lies in the kernel of the above restriction map, but not in the image of $\mathbb{K}(\mathbb{C}P^\infty/\mathbb{R}P^\infty)$. One can also show that Bott periodicity fails for the functor $\mathbb{K}(-)$ (see the remarks after Theorem 1.1 in [JO]).

If G is a compact Lie group and BG is its classifying space, then $\mathbb{K}(BG)$ and $K(BG)$ can both be studied by comparing them with the representation ring $R(G)$ of G. Let $\mathrm{Rep}(G)$ be the commutative monoid of isomorphism classes of complex finite dimensional G-representations (with addition defined by direct sum). Define

$$\alpha'_G : \mathrm{Rep}(G) \longrightarrow \mathrm{Vect}(BG)$$

by sending a complex G-representation V to its Borel construction $(EG \times_G V)$, regarded as a vector bundle over BG. Equivalently, if one thinks of a representation of G as a homomorphism $\rho : G \to GL_n(\mathbb{C})$, then α'_G sends ρ to the vector bundle classified by $B\rho : BG \to BGL_n(\mathbb{C})$. This is a homomorphism of monoids, and upon passing to Grothendieck groups we obtain a homomorphism

$$\alpha_G : R(G) \longrightarrow \mathbb{K}(BG)$$

of abelian groups. The *completion theorem* of Atiyah and Segal [AS] says that the composite

$$R(G) \xrightarrow{\alpha_G} \mathbb{K}(BG) \xrightarrow{\beta_{BG}} K(BG)$$

extends to an isomorphism $\widehat{\alpha}_G : R(G)\widehat{\ } \xrightarrow{\cong} K(BG)$, where

$$R(G)\widehat{\ } = \varprojlim_n \left(R(G)/I^n\right)$$

is completion with respect to the augmentation ideal $I = \operatorname{Ker}[R(G) \to \mathbb{Z}]$.

The main result described here, which was joint work with Stefan Jackowski [JO], is a description of the Grothendieck group $\mathbb{K}(BG)$ itself, also in terms of representations. This in turn grew out of earlier work by the two of us together with Jim McClure ([JMO], [JMO2], [JMO3]) dealing with maps between classifying spaces of arbitrary pairs of compact Lie groups. Note that the monoid $\operatorname{Vect}(BG)$ is the disjoint union of the sets $\operatorname{Vect}_n(BG) \cong [BG, BU(n)]$.

The starting point when computing $\mathbb{K}(BG)$ was to consider the case where G is a finite p-group, or more generally a p-toral group: an extension of a torus by a finite p-group. By theorems of Dwyer-Zabrodsky [DZ] (when G is a finite p-group) and of Notbohm [Nb] (when G is p-toral),

$$[BG, BG'] \cong \operatorname{Hom}(G, G')/\operatorname{Inn}(G')$$

for any p-toral group G and *any* compact Lie group G'. In particular, when $G' = U(n)$, this says that $\operatorname{Vect}_n(BG) \cong \operatorname{Rep}_n(G)$. In other words,

$$\alpha'_G : \operatorname{Rep}(G) \xrightarrow{\cong} \operatorname{Vect}(BG)$$

is an isomorphism whenever G is p-toral, and hence $\mathbb{K}(BG) \cong R(G)$ for such G.

Now let G be an arbitrary compact Lie group. For each p-toral subgroup P of G, consider the composite

$$\operatorname{Vect}(BG) \xrightarrow{\text{restr}} \operatorname{Vect}(BP) \ \xrightarrow[\cong]{\alpha_P'^{-1}} \operatorname{Rep}(P) \subseteq R(P),$$

where $R(P)$ is the complex representation ring of G. These maps combine to define a homomorphism

$$r_G : \operatorname{Vect}(BG) \longrightarrow R_{\mathcal{P}}(G) \overset{\text{def}}{=} \varprojlim_P R(P),$$

where the inverse limit is taken over all p-toral subgroups of G (for all primes p) with respect to inclusion and conjugation of subgroups. The main theorem in [JO] is the following:

Theorem 1. *For any compact Lie group G, r_G extends to an isomorphism*

$$\bar{r}_G : \mathbb{K}(BG) \xrightarrow{\cong} R_{\mathcal{P}}(G).$$

A more precise version of this theorem is given as Theorem 1′ below.

Of particular interest is the question of when the homomorphism $\alpha_G : R(G) \to \mathbb{K}(BG)$ is surjective; i.e., for which groups all vector bundles over BG are induced (stably, at least) by virtual representations of G. This is the case whenever G is finite, or whenever $\pi_0(G)$ is a p-group for some p (in particular if G is connected). In fact, $\mathbb{K}(BG) \cong R(G)$ (α_G is an isomorphism) whenever $\pi_0(G)$ (the group of connected components of G) has the property that all of its elements have prime power order. Note that this property — all elements have prime power order — is held not only by p-groups, but also by other finite groups such as Σ_4 and A_5. These, and other conditions which imply α_G is onto, are shown in [Ol, Corollary 3.11].

In general, α_G is not surjective, but its cokernel always has finite exponent. The simplest example of a group G for which $\alpha_G : R(G) \to \mathbb{K}(BG)$ is not onto, i.e., not all vector bundles are induced by virtual representations, was given by Adams in [Ad, Example 1.4]. For the group $G = (S^1 \times_{C_2} Q(8)) \times C_3$, he constructed a 2-dimensional complex vector bundle $\xi \to BG$ whose class does not lie in the image of $R(G)$. The cokernel of $\bar{\alpha}_G^{\mathbb{C}}$ for arbitrary G is described in [Ol, Lemma 3.8 and Theorem 3.9].

Another consequence of Theorem 1 is that $\beta_{BG} : \mathbb{K}(BG) \to K(BG)$ is always injective. This result follows upon combining the description of $\mathbb{K}(BG)$ as the inverse limit of the representation rings $R(P)$ for p-toral $P \subseteq G$, with a result of Segal [Se, Proposition 3.10] that $R(P)$ injects into $K(BP) \cong R(P)\widehat{\ }$ whenever $\pi_0(P)$ is a p-group (and in particular whenever P is p-toral).

The image of $\beta_{BG} : \mathbb{K}(BG) \rightarrowtail K(BG) \cong R(G)\widehat{\ }$ can be described directly, in terms of the exterior power operations on $K(BG)$. For any space X, homomorphisms $\lambda^k : K(X) \to K(X)$ are defined (for all $k \geq 0$) which send the class of any vector bundle over X to the class of its k-th (fiberwise) exterior power. Adams [Ad] defined and studied the subgroup $FF(BG) \subseteq K(BG)$ generated by the "formally finite dimensional elements"; i.e., those elements $x \in K(BG)$ such that $\lambda^k(x) = 0$ for k sufficiently large. Clearly, the class in $K(BG)$ of any vector bundle over BG satisfies this condition, and hence the image of β_{BG} is contained in $FF(BG)$. The results described here, when combined with those in [Ad], imply that in fact $FF(BG) = \mathrm{Im}(\beta_{BG})$.

The following two examples help describe the difference between the groups $K(BG)$ and $\mathbb{K}(BG)$. Consider first the case $G = T^n$: the n-dimensional torus. Here, $\mathbb{K}(BG) \cong R(G) \cong \mathbb{Z}[t_1, \ldots, t_n]$, where the generators t_i all represent one-dimensional representations. The augmentation ideal (i.e., the ideal of virtual zero-dimensional representations) is thus generated as an ideal by the elements $x_i = t_i - 1$. Since $R(G)$ is also generated as a polynomial algebra by the x_i, we see that $K(BG) \cong R(G)\widehat{\ } \cong \mathbb{Z}[[x_1, \ldots, x_n]]$ is a power series algebra.

As a second example, let G be any finite group. Let r_p (for any prime p) denote the number of conjugacy classes of elements of p-power order. Then

$$\mathbb{K}(BG) \cong \mathbb{Z} \times \prod_p (\mathbb{Z})^{r_p - 1}, \qquad \text{while} \qquad K(BG) \cong \mathbb{Z} \times \prod_p (\widehat{\mathbb{Z}}_p)^{r_p - 1}.$$

(This last statement follows from the proof of [Ja, Theorem 2.2].)

The above discussion has focused on the case of complex bundles, but all of these results are also shown in [JO] to hold for real bundles. In particular, if $\mathbb{KO}(BG)$ denotes the Grothendieck group of the monoid of real vector bundles over BG, and $RO_{\mathcal{P}}(G)$ is the inverse limit over p-toral subgroups $P \subseteq G$ (for all p) of the real representation rings $RO(P)$, then $\mathbb{KO}(BG) \cong RO_{\mathcal{P}}(G)$.

The proof of Theorem 1 is based on the description by Dwyer-Zabrodsky and Notbohm of $\mathbb{K}(BG)$ when G is p-toral (see Proposition 4 below), and a certain decomposition of BG as a homotopy direct limit of classifying spaces of p-toral subgroups. This decomposition is described as follows. Fix a compact Lie group G and a prime p. A p-toral subgroup $P \subseteq G$ is called *p-stubborn* if $N(P)/P$ is finite and contains no nontrivial normal p-subgroup $1 \neq Q \lhd N(P)/P$. Let $\mathcal{R}_p(G)$ denote the category whose objects are the orbits G/P for p-stubborn subgroups $P \subseteq G$, and where $\mathrm{Mor}_{\mathcal{R}_p(G)}(G/P, G/P')$ is the (finite) set of G-maps $G/P \to G/P'$.

Proposition 2. [JMO, Theorem 1.4] *For each prime p, the map*

$$\underset{G/P \in \mathcal{R}_p(G)}{\underrightarrow{\mathrm{hocolim}}} (EG/P) \longrightarrow BG,$$

induced by the projections $EG/P \to EG/G = BG$, is an $\mathbb{F}_p$-homology equivalence.

Here, EG can be any contractible complex with free action of G, and with orbit space BG. Note that $EG/P \simeq BP$ for each P (since the free G-action restricts to a free P-action). Thus, Proposition 2 describes BG, at least p-locally, as a limit of classifying spaces of p-toral subgroups of G.

For any space X, let $X^\wedge_p$ denote the p-adic completion of Bousfield and Kan. This will be used here only when X is 1-connected and its homotopy groups have finite type. So $X^\wedge_p$ can just be thought of as a space together with a map $X \to X^\wedge_p$, which induces isomorphisms $\widehat{\mathbb{Z}}_p \otimes \pi_i(X) \to \pi_i(X^\wedge_p)$ and $\widehat{\mathbb{Z}}_p \otimes H_i(X) \to H_i(X^\wedge_p)$ for all i. Proposition 2 implies that for any such X, the natural maps $EG/P \to BG$ induce a homotopy equivalence

$$\mathrm{Map}(BG, X^\wedge_p) \xrightarrow{\ \simeq\ } \mathrm{Map}\Big(\underset{G/P \in \mathcal{R}_p(G)}{\underrightarrow{\mathrm{hocolim}}} \big(EG/P\big), X^\wedge_p \Big).$$

Thus, in order to study maps to $BU(n)$, it is first necessary to look at maps to the p-completions $BU(n)^\wedge_p$. The following proposition, based on Sullivan's arithmetic pullback square, describes how the information about maps to the p-adic completions can be pieced together to give information about maps to $BU(n)$ itself.

Proposition 3. [JMO3, Proposition 1.2] *Let $T \subseteq G$ be a maximal torus of G, and set $w = |N(T)/T|$. Then the following square is a pullback:*

$$\begin{array}{ccc} [BG, BU(n)] & \longrightarrow & \prod_{p|w}[BG, BU(n)^\wedge_p] \\ {\scriptstyle\text{restr}}\big\downarrow & & {\scriptstyle\text{restr}}\big\downarrow \\ [BT, BU(n)] & \longrightarrow & \prod_{p|w}[BT, BU(n)^\wedge_p]. \end{array}$$

The goal now is to describe maps from BG to $BU(n)$ or $BU(n)^\wedge_p$, by replacing BG by the homotopy direct limit of spaces BP described in Proposition 2. To do this, one must understand, not only the sets $[BP, BU(n)^\wedge_p]$ of homotopy classes of maps, but also the higher homotopy groups of the connected components of the mapping spaces $\mathrm{Map}(BP, BU(n)^\wedge_p)$. These can be described with the help of the next proposition, where we also repeat the description of $[BP, BU(n)]$ mentioned earlier.

For any P-representation V (assumed to have a G-invariant hermitian product), $\mathrm{Aut}(V)$ denotes the group of all unitary automorphisms of V, and $\mathrm{Aut}_P(V)$ the subgroup of all P-equivariant unitary automorphisms.

Proposition 4. [DZ],[Nb] *For any prime p and any p-toral group P, the homomorphism*

$$\alpha'_P : \mathrm{Rep}(P) \xrightarrow{\ \cong\ } \mathrm{Vect}(BP) \cong \coprod_{n=0}^{\infty} [BP, BU(n)]$$

is an isomorphism of monoids. Also, for any P-representation V, corresponding to a homomorphism $\rho : P \to \mathrm{Aut}(V)$, the homomorphism $P \times \mathrm{Aut}_P(V) \xrightarrow{(\rho,\mathrm{incl})} \mathrm{Aut}(V)$ induces (by adjointness) a homotopy equivalence

$$B\,\mathrm{Aut}_P(V)^\wedge_p \xrightarrow{\ \simeq\ } \mathrm{Map}\big(BP, B\,\mathrm{Aut}(V)^\wedge_p\big)_{B\rho}.$$

Here, $\mathrm{Map}(-,-)_{B\rho}$ *denotes the connected component of $B\rho : BP \to B\,\mathrm{Aut}(V)$.*

Proposition 4 is a special case of more general results, which describe mapping spaces $\mathrm{Map}(BP, BG')$ (where P is p-toral and G' is any compact Lie group), or $\mathrm{Map}(BP, BG'^\wedge_p)$ (where G' must be connected.)

The next proposition, due to Wojtkowiak, describes the obstructions which are encountered when trying to compare the set of homotopy classes of maps $\underrightarrow{\mathrm{hocolim}}(X_\alpha) \to Y$ defined on a homotopy direct limit, with the inverse limit of the sets $[X_\alpha, Y]$ of maps defined on the "pieces". These obstructions turn out to be higher derived functors of certain inverse limits over the indexing category.

Proposition 5. [Wo] *Fix a discrete category $\mathcal{C}$, and a (covariant) functor $F : \mathcal{C} \to \mathsf{Top}$. Let Y be any space, and fix maps $f_c : F(c) \to Y$ (for all $c \in \mathrm{Ob}(\mathcal{C})$) whose homotopy classes define an element $\widehat{f} = ([f_c])_{c\in\mathcal{C}} \in \varprojlim[F(-), Y]$. Set*

$$\alpha_n(c) = \pi_n\big(\mathrm{Map}(F(c), Y), f_c\big)$$

for all $c \in \mathrm{Ob}(\mathcal{C})$*. Then the obstructions to constructing a map* $f : \underrightarrow{\mathrm{hocolim}}(F) \to Y$ *such that* $f|_{F(c)} \simeq f_c$ *for each* c *lie in the groups* $\underleftarrow{\lim}^{n+1}(\alpha_n)$ *for* $n \geq 1$*. Also, given two maps* $f, f' : \underrightarrow{\mathrm{hocolim}}(F) \to Y$ *such that* $f|_{F(c)} \simeq f_c \simeq f'|_{F(c)}$ *for each* c*, the obstructions to* f *and* f' *being homotopic lie in the groups* $\underleftarrow{\lim}^{n}(\alpha_n)$ *for* $n \geq 1$*.*

As usual, these obstructions are iterative, in that the i-th obstruction is defined only if the $(i{-}1)$-st vanishes (and the i-th obstruction may depend on choices made in earlier constructions). The above formulation avoids certain technical points involving basepoints of mapping spaces and nonabelian fundamental groups; problems which are dealt with in detail in [Wo].

When applying Proposition 5, we will need to deal with the higher limits of homotopy groups of mapping spaces $\mathrm{Map}\big(EG/P, BU(n)^\wedge_p\big)$. The homotopy groups of these spaces are in general unknown or difficult to compute, and this is one of the reasons for the difficulty in describing precisely the sets $\mathrm{Vect}_n(BG) \cong [BG, BU(n)]$. But since we are interested in the Grothendieck group of vector bundles, and not in the vector bundles themselves, it suffices to handle these mapping spaces and groups after stabilizing: more precisely, after taking certain direct limits over all $V \in \mathrm{Rep}(G)$. Very roughly, the following proposition says that while higher limits *can* influence the monoid $\mathrm{Vect}(BG)$ of vector bundles, they have no effect on the Grothendieck group $\mathbb{K}(BG)$.

Proposition 6. [JO, Proposition 1.5] *For each* $i > 0$*, let* $\Pi_i : \mathcal{R}_p(G) \to \widehat{\mathbb{Z}}_p\text{-}\mathsf{mod}$ *be the functor defined by setting*

$$\Pi_i(G/P) = \underset{V \in \mathrm{Rep}(G)}{\underrightarrow{\lim}} \pi_i\Big(\mathrm{Map}(EG/P, B\,\mathrm{Aut}(V)^\wedge_p), B\rho_V|_{BP}\Big),$$

where $\rho_V : G \to \mathrm{Aut}(V)$ *is induced by the action of* G *on* V*. Then* $\Pi_i \cong \widehat{\mathbb{Z}}_p \otimes K_G^{-i}(-)$ *as functors on* $\mathcal{R}_p(G)$*, and*

$$\underset{\mathcal{R}_p(G)}{\underleftarrow{\lim}^j} \Pi_i = 0$$

for all $i, j > 0$*.*

We are now ready to sketch the proof of Theorem 1. In fact, we prove a somewhat stronger statement. Recall that $r_G : \mathrm{Vect}(BG) \to R_{\mathcal{P}}(G)$ is defined as the inverse limit of the homomorphisms

$$\mathrm{Vect}(BG) \xrightarrow{\text{restr}} \mathrm{Vect}(BP) \cong \mathrm{Rep}(P) \subseteq R(P)$$

for p-toral subgroups $P \subseteq G$, and that $\bar{r}_G : \mathbb{K}(BG) \to R_{\mathcal{P}}(G)$ is induced by r_G upon passing to Grothendieck groups.

To simplify the notation, when V is a G-representation, η_V will denote the vector bundle induced by the Borel construction on V: $\eta_V = \big(EG \times_G V \to BG\big)$.

THEOREM 1′. [JO] *For any compact Lie group G, $\bar{r}_G : \mathbb{K}(BG) \xrightarrow{\cong} R_{\mathcal{P}}(G)$ is an isomorphism of groups. More precisely, the following two statements hold.*

(a) For each pair of bundles $\xi, \xi' \to BG$ such that $r_G(\xi) = r_G(\xi')$, there exists a G-representation V such that $\xi \oplus \eta_V \cong \xi' \oplus \eta_V$.

(b) For each $X \in R_{\mathcal{P}}(G)$, there exist a vector bundle $\xi \to BG$ and a G-representation V such that $r_G(\xi) = X + r_G(\eta_V)$.

Any vector bundle over BG can be embedded as a summand of a bundle η_V for some G-representation V; and $\mathbb{K}(BG)$ can thus be obtained from Vect(BG) *by inverting only those vector bundles coming from G-representations.*

Outline of the proof. The injectivity of $\bar{r}_G$ follows immediately from point (a), and the surjectivity from (b). The last statement also follows easily from (a) and (b).

For any map $f : BG \to BU(n)$, we write ξ_f for the corresponding vector bundle over BG; i.e., for the pullback via f of the universal vector bundle over $BU(n)$. For any (n-dimensional) G-representation V, $f_V : BG \to BU(n)$ denotes the classifying map of the corresponding homomorphism $G \to U(n)$; or equivalently the classifying map of the vector bundle η_V.

We focus attention on the proof of (a). Fix maps $f, g : BG \to BU(n)$ such that $r_G(\xi_f) = r_G(\xi_g)$. In other words, for each p and each p-toral subgroup $P \subseteq G$, $f|_{BP} \simeq g|_{BP}$. We must show that there is a G-representation W for which

$$f \oplus f_W \simeq g \oplus f_W.$$

By Proposition 3, it suffices to show that $(f \oplus f_W)^\wedge_p \simeq (g \oplus f_W)^\wedge_p$ for each prime p. This is a problem only for primes $p \big| |N(T)/T|$ (where T is a maximal torus in G); hence only for a finite number of primes. It thus suffices to find W_p, for each p, such that

$$(f \oplus f_{W_p})^\wedge_p \simeq (g \oplus f_{W_p})^\wedge_p; \tag{1}$$

and then set $W = \bigoplus_{p \big| |NT/T|} W_p$.

Fix a prime $p \big| |N(T)/T|$. For each i, let $\Pi_i^{(f)} : \mathcal{R}_p(G) \to \widehat{\mathbb{Z}}_p\text{-}\mathsf{mod}$ be the functor

$$\Pi_i^{(f)}(G/P) = \pi_i\Big(\mathrm{Map}(EG/P, BU(n)^\wedge_p), f|_{BP}\Big).$$

By Proposition 2,

$$[BG, BU(n)^\wedge_p] \cong \Big[\underset{G/P \in \mathcal{R}_p(G)}{\underrightarrow{\mathrm{hocolim}}} \big(EG/P\big), BU(n)^\wedge_p\Big].$$

So by Proposition 5, the successive obstructions to constructing a homotopy $f \simeq g$ lie in the groups $\varprojlim^i_{\mathcal{R}_p(G)} \Pi_i^{(f)}$ (for all $i \geq 1$). Since $\mathcal{R}_p(G)$ is equivalent to a finite category [JMO, Proposition 1.6], higher derived functors of inverse limits over $\mathcal{R}_p(G)$ can be switched with direct limits over directed categories. Hence for all $i, j \geq 1$,

$$\begin{aligned}\varinjlim_{W \in \mathrm{Rep}(G)} \Big(\varprojlim^j_{\mathcal{R}_p(G)} \Pi_i^{(f \oplus f_W)}\Big) &\cong \varprojlim^j_{\mathcal{R}_p(G)} \Big(\varinjlim_{W \in \mathrm{Rep}(G)} \Pi_i^{(f \oplus f_W)}\Big) \\ &\cong \varprojlim^j_{\mathcal{R}_p(G)} \Big(\varinjlim_{W \in \mathrm{Rep}(G)} \Pi_i^{f_W}\Big) \cong \varprojlim^j_{\mathcal{R}_p(G)} \big(\Pi_i\big) = 0\end{aligned}$$

by Proposition 6.

In other words, each successive obstruction to showing $f_p^\wedge \simeq g_p^\wedge$ vanishes after replacing f and g by $f \oplus f_W$ and $g \oplus f_W$ for a sufficiently large G-representation W. Also, by [JMO2, Proposition 4.11], the higher limits of any functor $\mathcal{R}_p(G) \to \widehat{\mathbb{Z}}_p\text{-}\mathtt{mod}$ vanish in degrees above some fixed $d(G,p)$ (depending only on G and p); and so there are only finitely many obstructions to constructing the homotopy. And this finishes the proof of (1), and hence the proof of point (a).

The proof of (b) is similar, but slightly more complicated at some points. One first shows that $R_{\mathcal{P}}(G)$ is the Grothendieck group of the monoid $\varprojlim \mathrm{Rep}(P)$ (where the limit is taken over all p-toral subgroups $P \subseteq G$, for all p). In other words, it suffices to show that (b) holds for elements $X = (V_P) \in R_{\mathcal{P}}(G)$, where the $V_P \in \mathrm{Rep}(P)$ are actual representations. Set $n = \dim(V_1)$ (where 1 denotes the trivial subgroup). Then $n = \dim(V_P)$ for all P: just note that $(V_P)|_1 \cong V_1$ since the V_P form an element in the inverse limit.

Fix a prime $p \big| |N(T)/T|$. Let $\Pi_i^{(X)} : \mathcal{R}_p(G) \to \widehat{\mathbb{Z}}_p\text{-}\mathtt{mod}$ be the functor

$$\Pi_i^{(X)}(G/P) = \pi_i\Big(\mathrm{Map}(EG/P, BU(n)_p^\wedge), f_{V_P}\Big).$$

As in the proof of (a), we show that

$$\varinjlim_{W \in \mathrm{Rep}(G)} \Big(\varprojlim_{\mathcal{R}_p(G)}{}^j \, \Pi_i^{(X \oplus f_W)} \Big) = 0,$$

with the help of Propositions 4 and 6 and commuting limits. Thus, each successive obstruction to constructing a map $f : BG \to BU(n)$ vanishes after replacing X by $X + r_G(\eta_W)$ for some sufficiently large representation W. Since there are only finitely many nonzero obstructions, we obtain a map $f_p : BG \to BU(n+k)$ such that $f_p|_{BP} \simeq (f_{V_P} \oplus f_W)_p^\wedge$ for some k-dimensional representation W. After stabilizing further, we can arrange that this has been done for all primes $p \big| |N(T)/T|$, and with the same G-representation W. And the pullback square in Proposition 3 can now be used to construct $f : BG \to BU(n+k)$ such that $r_G(\xi_f) = X + r_G(\eta_W)$. □

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