The Geometry of the Seiberg-Witten Invariants

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My purpose in this talk is to describe a curious story about a search for symplectic forms on smooth, compact, 4-dimensional manifolds. However, be aware that at the time of this writing, the story that I relate below has no conclusion.

1 The start of the story

The story starts with the Seiberg-Witten invariants which were introduced just under four years ago by Witten [W1]. These are invariants of compact, smooth, oriented 4-manifolds. (Here, and below, all 4-manifolds will be connected and oriented.) After the choice of orientation for the real line $\det^+ = H^0 \otimes \det(H^1) \otimes \det(H^{2+})$, the Seiberg-Witten invariants constitute a map from the set, $\mathcal{S}$, of $\mathrm{Spin}^{\mathbb{C}}$ structures on the 4-manifold to the integers. There is also an extension of SW in the case where the Betti number b^1 is positive to a map SW: $\mathcal{S} \to \Lambda^* H^1(X;\mathbb{Z})$. (Here and below, $\Lambda^* H^1(X;\mathbb{Z}) = \mathbb{Z} \oplus H^1 \oplus \Lambda^2 H^1 \oplus \cdots \oplus \Lambda^{b_1} H^1$. Note that the projection of the image of SW on the summand $\mathbb{Z}$ reproduces the original map as defined from $\mathcal{S}$ to $\mathbb{Z}$ in [W1].) In either guise, this map, SW, is computed by an algebraic count of solutions to a certain non-linear system of differential equations on the manifold. The equation in question is for a pair of unknowns which consist of a section of a certain $\mathbb{C}^2$ bundle and a connection on this same bundle's determinant line.

The invariant SW and the Seiberg-Witten equations were introduced to the mathematical community by Witten [W1] after his ground breaking work with Seiberg in [SW1], [SW2]. See also [KM], [Mor], [KKM] and [T1]. Few would argue against the assertion that the Seiberg-Witten equations have revolutionized 4-manifold differential topology.

The Seiberg-Witten invariants have proved so useful for questions about compact 4-manifolds because they are at least as powerful as the Donaldson invariants (see, e.g. [DK]) and so much easier to compute. In this regard, note that Witten has conjectured that the two sets of invariants carry the same information about compact 4-manifolds. But, it remains to be seen whether they are equivalent in this context, let alone for their other uses. (An argument for Witten's conjecture is outlined in [PT] and a series of papers by Feehan and Leness begin to address the technical details. See, e.g. [FL].)

However, neither the Seiberg-Witten invariants relation with the Donaldson invariants nor their computability is the subject of this story. Rather, the story I am relating concerns another property of the Seiberg-Witten invariants which is the following: These invariants seem to have a direct, geometric interpretation as an algebraic count of certain distinguished submanifolds with boundary in the given 4-manifold. Moreover, this geometric interpretation suggests a novel approach to the existence question for symplectic forms. Here, I have used the verbs 'seem' and 'suggests' because, as I said at the outset, the story is not finished. In particular, the geometric interpretation is not yet completely worked out except in some special cases. One of these cases consists of symplectic manifolds, and as symplectic notions are anyway central to this story, they form the subject of the next three chapters.

2 Symplectic manifolds

A 4-dimensional manifold X is symplectic when it carries a closed, non-degenerate 2-form. That is, there is a 2-form ω with $d\omega = 0$ and with $\omega \wedge \omega \neq 0$ at all points. In this regard, the convention will be to orient the manifold in question with $\omega \wedge \omega$. Now, not all 4-manifolds can be symplectic. First of all, the Betti number b^{2+}, which is the dimension of the maximum subspace of $H_2(X;\mathbb{Q})$ on which the intersection pairing is positive definite, must be positive since $\omega \wedge \omega$ is positive. Also, there is a classical, mod 2 obstruction which asserts that an oriented X has a symplectic form which reproduces the given orientation only if the Betti number sum $b^1 + b^{2+}$ is odd. For example, this condition rules out the connect sum of an even number of $\mathbb{CP}^2$.

The Seiberg-Witten invariants give additional obstructions [T2], [T3]. For example, there must be a $\text{Spin}^{\mathbb{C}}$ structure for which the associated Seiberg-Witten invariant is ± 1. The latter rules out the connect sum of an odd number larger than 1 of $\mathbb{CP}^2$.

By the way, it was innocently conjectured that an irreducible, simply connected 4-manifold was always symplectic with some choice of orientation. However, Szabo proved this conjecture false [Sz] and subsequently, Fintushel and Stern (who are speaking in this Congress) found a slew of counter examples as homotopy K3 surfaces [FS]. In both cases, the Seiberg-Witten invariants play a prominent role.

Anyhow, it is important to realize that, at the time of this writing, necessary and sufficient conditions for the existence of a symplectic form are not known.

(Although not relevant for this story, the reader might be interested to know that the problem of classifying symplectic manifolds up to diffeomorphism is unsolved except in some special cases where $b^{2+} = 1$. However, Donaldson has made progress recently towards a classification theory for symplectic 4-manifolds up to deformation of the symplectic form and symplectomorphisms.)

3 The Seiberg-Witten invariants on a symplectic manifold

As remarked in [T1], a symplectic manifold has a natural orientation as does the line $\det^+$. Furthermore, there is a canonical identification of the set $\mathcal{S}$ with $H^2(X;\mathbb{Z})$. Thus, on a symplectic 4-manifold, SW can be viewed as a map from $H^2(X;\mathbb{Z})$ to $\mathbb{Z}$, or, more generally, from $H^2(X;\mathbb{Z})$ to $\Lambda^* H^1(X;\mathbb{Z})$.

Meanwhile, a compact symplectic 4-manifold has a second natural map sending $H^2(X;\mathbb{Z})$ to $\mathbb{Z}$, its Gromov invariant, Gr. The map Gr also extends on a

$b^1 > 0$ symplectic 4-manifold to a map from $H^2(X;\mathbb{Z})$ into $\Lambda^* H^1(X;\mathbb{Z})$; the extension is sometimes called the Gromov-Witten invariant, but it will be denoted here by Gr as well. In either guise, Gr, assigns to a class e a certain weighted count of compact, symplectic submanifolds whose fundamental class is Poincare dual to e. In this regard, a submanifold is symplectic when the restriction of the symplectic form to its tangent space is non-degenerate. (More is said about the count for Gr in the next chapter.)

The Gromov invariant was introduced initially by Gromov in [Gr] and then generalized by Witten [W2] and Ruan [Ru]. See also [T4]. (Note that Gr here does not count maps from a fixed complex curve. It differs in this fundamental sense from the Gromov-Witten invariant introduced in [W2].) The precise definition of Gr is provided in [T4]. Here is the main theorem which relates SW to Gr:

Theorem 1: *Let* X *be a compact, symplectic manifold with* $b^{2+} > 1$. *Use the symplectic structure to orient X and the line* $\det^+$*; and use the symplectic structure to define SW as a map from* $H^2(X;\mathbb{Z})$ *to* $\Lambda^* H^1(X;\mathbb{Z})$*. In addition, use the symplectic structure to define Gr:* $H^2(X;\mathbb{Z})$. $\Lambda^* H^1(X;\mathbb{Z})$*. Then SW = Gr.*

Thus, according to Theorem 1, on a symplectic manifold with $b^{2+} > 1$, the smooth invariants of Seiberg-Witten can be interpreted geometrically as a certain count of symplectic submanifolds.

Theorem 1 is proved in [T5]. The equivalence between the Gromov invariant and the original SW map into $\mathbb{Z}$ was announced by the author in [T1]. The proof of Theorem 1 can be divided into three main parts. The first part explains how a non-zero Seiberg-Witten invariant implies the existence of symplectic submanifolds. The second part explains how a symplectic submanifold can be used to construct a solution to a version of the Seiberg-Witten equations. The final part compares the counting procedures for the two invariants. The first and second parts of the proof can be found in [T6] and [T7], respectively and the final part (together with an overview of the whole strategy) is in [T5]. (Some of the early applications of Theorem 1 are also described in [Ko].)

A restricted version of Theorem 1 holds in the case when $b^{2+} = 1$. Here, a fundamental complication is that the Seiberg-Witten invariant depends on more than the differentiable structure. This is to say that there is a dependence on a so called choice of chamber. However, the symplectic form selects out a unique chamber, and with this understood, one has:

Theorem 2: *Let* X *be a compact, oriented 4-manifold with* $b^{2+} = 1$ *and with a symplectic form. Then the symplectic form canonically defines a chamber in which the equivalence SW = Gr holds for classes* $\mathrm{e} \in H^2(X;\mathbb{Z})$ *which obey* $\langle \mathrm{e}, \mathrm{s}\rangle \geq -1$ *when ever the two dimensional homology class* $\mathrm{s} \in H_2(\mathrm{X} : \mathbb{Z})$ *is represented by an embedded, symplectic sphere with self-intersection number -1.*

(Here, $\langle , \rangle$ denotes the pairing between cohomology and homology.) Theorem 2 is also proved in [T5].

By the way, when X is a $b^{2+} = 1$ symplectic manifold and e in $H^2(X;\mathbb{Z})$ is a class for which the conditions of Theorem 2 do not hold, the Seiberg-Witten invariant SW(e) still counts pseudo-holomorphic subspaces [LL]. However, the Gromov invariant as defined in [T4] is not the correct symplectic invariant for such e since the subspaces involved can have singularities. The symplectic invariant in this case is given by McDuff in [Mc1] (An overview of Seiberg-Witten story on symplectic manifolds is also provided in [T8].)

4 Pseudo-holomorphic subvarieties

When calculating Gr, one should follow Gromov [Gr] and introduce an auxilliary structure on X which consists of an almost complex structure J for TX. By definition, the latter is an endomorphism $J : TX \to TX$ which obeys $J^2 = -1$. Such J's exist precisely when the Betti number sum $b^1 + b^{2+}$ is odd. Moreover, given the symplectic form ω, there exists such J which are *compatible* with ω in the sense that the bilinear form $\omega(\cdot, J(\cdot))$ on TX defines a Riemannian metric. (Moreover, Gromov showed that the space of ω-compatible J's is contractible.)

With an almost complex structure J chosen, certain dimension 2 submanifolds are distinguished, namely those for which J preserves their tangent space in TX. Such submanifolds are called *pseudo-holomorphic*. Note that if $C \subset$ X is a pseudo-holomorphic submanifold, then J orients TC and thus the homology class of C is canonically defined. Moreover, if J is ω-compatible, then ω restricts positively to C and so C is symplectic. Also, the homology class of C is never (rationally) zero when C is pseudo-holomorpic. (Note that the restriction to TC of J endows C with the structure of a complex curve, in which case the inclusion map from C to X is a pseudo-holomorphic map in the original sense defined by Gromov.)

The set of pseudo-holomorphic submanifolds form a geometrically distinguished subset of symplectic submanifolds. This subset is well behaved in as much as the deformation theory for a pseudo-holomorphic submanifold is highly constrained. Indeed, the latter is a Fredholm deformation problem. (Among other things, this last assertion implies that the space of pseudo-holomorphic submanifolds in a given homology class has the structure of a finite dimensional variety.) By the way, there is an important converse to the preceding, which is that every symplectic submanifold is pseudo-holomorphic for some ω-compatible J. (A good, general reference about pseudo-holomorphic geometry is the book by McDuff and Salamon [MS].)

With the pseudo-holomorphic submanifolds understood, the first point of this chapter is simply to remark that the invariant Gr 'counts' symplectic submanifolds in a given homology class by actually counting the pseudo-holomorphic representatives with certain weights. Except for tori with zero self-intersection, these weights are ± 1. The weights for the excepted tori are more involved. Note also that Gr counts disconnected submanifolds. In any event, see [T4] for the full story. By the way, one consequence of Theorems 1 and 2 is an existence theorem for pseudo-holomorphic curves in certain homology classes [T6].

The second point of this chapter is to offer, for use in the subsequent chapters, a reasonable definition of pseudo-holomorphic submanifolds and pseudo-holomorphic varieties inside a non-compact symplectic manifold. Consider:

Definition 3: *Let* X *be a smooth, 4-manifold with symplectic form* ω. *A subset* $C \subset$ X *is a pseudo-holomorphic variety when the following conditions are met:*

- *C is closed.*
- *There is a set $\Lambda \subset C$ with at most countably many elements and no accumulation points in* X *such that $C - \Lambda$ is a submanifold of* X.
- *J maps $TC|_{C-\Lambda}$ to itself in TX.*
- $\int_c \omega < \infty$.

In previous articles, I have sometimes distinguished amongst those $C \subset$ X which satisfy the first three conditions above, but not the final condition. When C also satisfies the final condition, one can say that TC has '*finite energy*'.

Note that the singularities of a pseudo-holomorphic variety (the points of Λ) are essentially those of complex curves in $\mathbb{C}^2$. (See, e.g. [Mc2], [PW], [Ye], [Pa], [MS].)

5 When no symplectic form is handy

Suppose now that X is a compact, oriented 4-manifold which has no known symplectic form. Here is a suggestion for the next best thing: Put a Riemannian metric on X. Among other things, the latter defines a decomposition of the bundle of 2-forms into a direct sum of two three dimensional bundles, $\Lambda^+ \oplus \Lambda^-$. These are the bundles of self-dual and anti-self-dual 2-forms. (Fix an oriented orthonormal frame $\{\mathrm{e}^i\}_{1\le i\le 4}$ for T^*X at a given point, and then $\Lambda^+ =$ Span $\{e^1 \wedge e^2 + e^3 \wedge e^4, e^2 \wedge e^3 + e^1 \wedge e^4, e^3 \wedge e^1 + e^2 \wedge e^4\}$.) More to the point, when $b^{2+} \ge 1$, then Hodge-DeRham theory provides a self-dual, closed form ω. That is, ω is a section of Λ^+ and $d\omega = 0$. In particular, this implies that $\omega \wedge \omega = |\omega|^2$ dvol and so ω is symplectic where non-zero.

As this story is about the search for symplectic forms, the preceding suggests an investigation into the zero set of a closed self-dual form. For this purpose, let ω be such a form. In as much as ω is a section of an $\mathbb{R}^3$ bundle, one might expect the zero set to be 1-dimensional in some generic sense. This turns out to be the case. Both Honda [Ho] and LeBrun [Le] offer proofs of the following:

Theorem 4: *Fix a compact, oriented 4-manifold* X *with* $b^{2+} \ge 1$. *If the metric on* X *is suitably generic (chosen from a Baire subset of smooth metrics), then there is a closed, self-dual 2-form ω which vanishes transversally as a section of Λ^+. In particular, the zero set of w is a finite, disjoint union of embedded circles.*

With the preceding understood, assume from here on that each given closed, self-dual 2-form vanishes as a transversal section of Λ^+.

Work of Carl Luttinger [Lu] demonstrates that, by themselves, the zero circles of any given closed, self-dual ω carry very little in the way of information about the obstruction to finding a symplectic form. In fact, Luttinger shows that there are closed forms which are self-dual for some metric and have arbitrarily many zero set components. Conversely, Luttinger showed how to modify any given ω so that the result is closed and self-dual for some metric, yet has only one component of its zero set. (In both cases, Luttinger's arguments are essentially local in nature. One constructs explicit models in $\mathbb{R}^4$ of 1-parameter families of closed, self-dual forms for which the topology of the zero set changes either by birth or death of an isolated circle, or by two components melding to one or one component splitting into two. One can then argue using appropriate coordinate charts that these local models can be 'spliced' into any manifold.)

Note however, that the zero circles of ω do carry one small bit of obstruction data. Indeed, Gompf [Go] has shown how data near the zero circle can be used to compute the parity of $b^1 + b^{2+}$. In this regard, it is important to realize that in some neighborhood of a point where ω is zero, there are coordinates $(t, \mathrm{x}) \in \mathbb{R} \times \mathbb{R}^3$ so that $\mathrm{x} = 0$ corresponds to $\omega^{-1}(0)$ and so that

$$\omega = dt \wedge (\mathrm{x}^T) A \cdot d\mathrm{x} + *_3(\mathrm{x}^T A \cdot d\mathrm{x}) + \mathcal{O}(|\mathrm{x}|^2). \tag{1}$$

Here, x^T denotes the transpose of the vector x, while $A = A(t)$ is a 3×3 symmetric (non-degenerate) matrix. Also, $*_3$ denotes the standard Hodge star operator on $\mathbb{R}^3$. Note that the condition $d\omega = 0$ requires that A be both symmetric and traceless.

By changing the sign of the t coordinate, one can then assume that $\det(A) < 0$. That is, A has one negative eigenvector at each t and two positive eigenvectors. In particular, as one moves around any given component of $\omega^{-1}(0)$, the negative eigenspaces of A fit together to yield a line bundle over the circle. The latter can be either oriented or not, and Gompf's observation is that the parity of $b^1 + b^{2+}$ is the opposite of the parity of the number of components of $\omega^{-1}(0)$ for which the aforementioned negative eigenbundle is oriented. (Note: There is no misprint here with the use of 'oriented'.)

6 PSEUDO-HOLOMORPHIC SUBVARIETIES IN X - Z.

As just seen, $\omega^{-1}(0)$ carries by itself little information about the existence of symplectic forms on X. However, this is not to say that $\omega^{-1}(0)$ is completely irrelevant to the story. Indeed, at least some non-trivial data seems to be stored as configurations of certain kinds of symplectic surfaces in $\mathrm{X} - \omega^{-1}(0)$ which bound $\omega^{-1}(0)$. A digression is required to be more precise in this regard.

To start the digression, introduce as short hand $Z \equiv \omega^{-1}(0)$. By definition, ω restricts to X - Z as a symplectic form. Moreover, if $g : TX \to T^*\mathrm{X}$ denotes the given metric, then the endomorphism $J = \sqrt{2}g^{-1}\omega/|\omega|$ defines an ω-compatible almost complex structure for X - Z. Note that the latter is singular along Z. Indeed, when ω vanishes transversely, then the first Chern class of the associated canonical bundle has degree 2 on all linking 2-spheres of Z. Even so, one can use J to define pseudo-holomorphic subvarieties in X - Z. The pseudo-holomorphic subvarieties in X - Z might be curiosities were it not for the following theorem [T9]:

THEOREM 5: *Suppose that* X *is a compact, oriented, Riemannian manifold with* $b^{2+} \geq 1$ *and a non-zero Seiberg-Witten invariant. Let* ω *be a closed, self-dual 2-form whose zero set, Z, is cut out transversally by* ω*. Then, there exists a pseudo-holomorphic subvariety in* $\mathrm{X} - Z$ *which homologically bounds* Z *in the sense that it has intersection number 1 with every linking 2-sphere of* Z*.*

In the preceding, when $b^{2+} = 1$, the Seiberg-Witten invariants in the statement of the theorem are from a certain chamber which is specified by ω. By the way, the statement of the previous theorem can be strengthened in the following direction: Given a specific $\mathrm{Spin}^{\mathbb{C}}$ structure on X with non-zero Seiberg-Witten invariant, there exists a pseudo-holomorphic subvariety in X - Z with homological boundary Z whose relative homology class in X - Z can be foretold in terms of the given $\mathrm{Spin}^{\mathbb{C}}$ structure. Note that Theorem 5 suggests the following likely conjecture:

- The Seiberg-Witten invariants of X can be computed via a specific algebraic count of the pseudo-holomorphic subvarieties in X - Z which bound Z.

Theorem 1 affirms this conjecture in the case where $Z = \emptyset$. Moreover, through work of Hutchings and Lee [HL] and Turaev [Tu], this conjecture has been confirmed also in the case where $Z = S^1 \times M$ where M is a compact, oriented 3-manifold with $b^1 > 0$. (This last case is discussed further in a subsequent chapter.)

Remark that the pseudo-holomorphic subvarieties which arise in the context of Theorem 4 have a well defined Fredholm deformation theory; and this last fact supplies further evidence for the validity of the preceding conjecture.

7 A regularity theorem

Since the almost complex structure J in Theorem 5 is singular along Z, the behavior near Z of a pseudo-holomorphic variety in X - Z is problematic. (As remarked previously, away from Z, such a variety is no more singular than a complex subvariety of $\mathbb{C}^2$.) Indeed, as ω near Z vanishes as the distance to Z, it is not apriori clear that such varieties have finite area. However, it turns out that the fourth condition in Definition 3 is stronger than it looks, and in particular, some (partial) regularity results can be proved, at least under some restrictive hypothesis about the form of ω near its zero set. The results are summarized in Theorem 6, below. However, first comes a digression to explain the restrictions. The restriction on ω is as follows: Near each point in Z, there should exist coordinates (t, x, y, z) such that Z coincides with the set x = y = z = 0 and such that in this coordinate patch,

$$\begin{aligned} \omega &= dt \wedge (\mathrm{x}d\mathrm{x} + y dy - 2z dz) + \mathrm{x} dy \wedge dz + y dz \wedge d\mathrm{x} - 2z d\mathrm{x} \wedge dy, \\ g &= dt^2 + d\mathrm{x}^2 + dy^2 + dz^2. \end{aligned} \tag{2}$$

Note that this is a rather special version of the general form for ω which is given by Eq.(1). However, on any $b^{2+} > 0$ manifold, there are metrics and self-dual forms which satisfy these restrictions. In fact, given any metric and closed self-dual form ω with non-degenerate zeros, both can be modified solely in a given neighborhood of $\omega^{-1}(0)$ so that the resulting form is closed and self-dual for the resulting metric and has the same zero set as the original and obeys the restrictions in Eq.(2). The following summarizes what is presently known about regularity near Z:

Theorem 6: *Let* X *be a smooth, compact, oriented, Riemannian 4-manifold and let* ω *be a closed, self-dual form which is described near* Z *by Eq.(2). Let* $C \subset$ X *be a pseudo-holomorphic subvariety. Then,* C *has finite area. Moreover, except for possibly a finite subset of points on* Z*, every point on* Z *has a ball neighborhood which intersects* C *in a finite number of components. And, the closure of each component in such a ball neighborhood is a real analytically embedded half disk whose straight edge coincides with* Z*.*

(Note that the behavior of C near each of the singular points can also be described.) Theorem 6 is proved in [T10]. I expect that a very similar theorem holds without the special restriction in Eq.(2). Moreover, I expect that the finite number of singular points are 'removable by perturbation' in the sense that these singularities are, in some well defined sense, codimension one phenomena. (Hofer, Wysocki and Zehnder have an alternative approach to studying X - Z as a symplectic manifold. See, e.g. [HWZ].)

8 An illustrative example

An example with much food for thought has X $= S^1 \times M$ where M is a compact, oriented, 3-manifold with $b^1 > 0$. The set of M for which the corresponding X is symplectic remains (as of this writing) mysterious. However, it is known that X is symplectic when M admits a fibering $f : M \to S^1$, and for all we know at present, these are the only 3-manifolds for which $S^1 \times M$ is symplectic. (The latest results on this question are due to Kronheimer [Kr].) In any event, X $= S^1 \times M$ does have closed, self-dual 2-forms. For example, if one uses a product metric on X (the Euclidean metric on $S^1 = \mathbb{R}/\mathbb{Z}$ plus a metric on M), then all closed, self-dual 2-forms have the form $\omega = dt \wedge \nu + *_3\nu$, where ν is a harmonic 1-form on M and where $*_3$ is the Hodge star operator on $\Lambda^* M$. In particular, one can find harmonic 1-forms which equal df where $f : M \to \mathbb{R}/\mathbb{Z}$ is a non-zero cohomology class. This last case is instructive in as much as one can see that $Z = \omega^{-1}(0)$ is given by $Z = S^1 \times \mathrm{Crit}(f)$, where $\mathrm{Crit}(f)$ is the set of critical points of f. Moreover, for a suitably generic choice of metric on M, the $\mathbb{R}/\mathbb{Z}$-valued function f will have only non-degenerate critical points (see, e.g. [Ho]), and in this case, the corresponding ω will have a transversal zero set in the sense of Theorem 4.

In this last example, subsets of X given by $S^1\times$ (gradient flow lines of ∇f) are pseudo-holomorphic submanifolds. In fact, when the metric on M is suitably generic, then the pseudo-holomorphic submanifolds promised by Theorem 5 have the form $S^1 \times \Gamma$ where Γ is a finite union of gradient flow lines of ∇f having the following properties: First, each flow line $\gamma \in \Gamma$ is complete and has bounded length. Second, each critical point of f is an end point of one and only one flow line in Γ. (By the way, Hutchings and Lee [HL] have found an intrinsic count of such Γ which computes a certain Alexander polynomial of the associated $\mathbb{Z}$-cover of M. Meanwhile Meng and the author [MT] have a theorem to the effect that such Alexander polynomials essentially determine the Seiberg-Witten invariants of X. See also [Tu].) One lesson from the preceding example is the following: In the S^1 invariant context on X $= S^1 \times M$, self-dual, symplectic geometry on X is nothing more than Morse theory with $\mathbb{R}/\mathbb{Z}$ valued functions on M. This is to say that the problem of eliminating component circles of the zero set of an S^1 invariant, closed, self-dual 2-form on X is that of eliminating the critical points of a harmonic function on M.

9 A dictionary?

The previous example suggests that there may exist a dictionary which translates Morse theoretic notions in 3-manifold topology to notions which involve closed, self-dual 2-forms on 4-manifolds and their associated pseudo-holomorphic varieties. (Below, I call the second subject 'self-dual symplectic geometry'.) Some of the dictionary has already been established, and some is conjectural. This dictionary is reproduced below. In the dictionary, M is a 3-dimensional Riemannian manifold with $b^1 > 0$ and X is a 4-dimensional Riemannian manifold with $b^{2+} > 0$.

Morse Theory on M	Self-dual symplectic geometry on X
Critical points of an $\mathbb{R}/\mathbb{Z}$-valued harmonic function f	The zero set Z of the closed, self-dual 2- form ω.
Gradient flow lines of ∇f	Pseudo-holomorphic varieties in X - Z with boundary on Z.
Milnor torsion/Alexander polynomial	Seiberg-Witten invariants.
Whitney disk	Lagrangian disk with a boundary piece on a pseudo-holomorphic subvariety.
Self-indexing Morse function	?
Handle sliding	?
Morse-Smale cancellation lemma	?

Here are some comments about the preceding table:

- The appearance of Lagrangian disks as the analog to Whitney disks in 3-dimensional Morse theory is closely related to observations of Donaldson about the appearance of Lagrangian 2-spheres in his study of symplectic Lefschetz pencils. In any event, the point here is that a symplectic submanifold can be symplectically deformed via 'finger moves' along Lagrangian disks which have a part of their boundary on the submanifold in question.

- The Morse-Smale cancellation lemma asserts that a pair of critical points of f can be cancelled (without introducing new critical points or disturbing the configuration of gradient flow lines) if there is a unique, stable minimal energy gradient flow line between them. (The energy of a flow line is simply the drop in f between the start and the finish. A flow line is stable if it persists under perturbation of the gradient flow or the function f.) On the 4-dimensional side, the analogous lemma might be something like the following: Let Z_0 be a component of Z. If the smallest energy, pseudo-holomorphic variety in X - Z with Z_0 as a boundary component is suitably stable, is unique and is a disk, then the form ω can be altered to produce a new closed form which is self-dual for some metric on X, has zero set with fewer components, and has a less complicated set of bounding, pseudo-holomorphic varieties. Note that the local 'melding' procedure of Luttinger which joins all components of Z into one circle appears to increase the genus of the bounding pseudo-holomorphic varieties unless suitable Lagrangian disks are present.

- In fact, there is a self-dual symplectic analog of handle sliding, but the details are still somewhat obscure (to the author, anyway).

10 Summary

The following two as yet unanswered questions aim at the heart of the matter:

- How much is self-dual symplectic geometry like 3-dimensional Morse theory?
- More to the point, can self-dual symplectic geometry shed any light on 4-manifold differential topology?

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