Harmonic Analysis on Semisimple p-Adic Lie Algebras

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Abstract. Certain topics in harmonic analysis on semisimple groups arise naturally when one uses the Arthur-Selberg trace formula to study automorphic representations of adele groups. These topics, which fall under the heading of "comparison of orbital integrals," have been surveyed in Waldspurger's article in the the proceedings of ICM94. As in Harish-Chandra's work, many questions in harmonic analysis on the group can be reduced to analogous questions on its Lie algebra, and in particular this is the case for "comparison of orbital integrals." We will discuss some recent work of this type.

1991 Mathematics Subject Classification: 22E35, 22E50, 11F85
Keywords and Phrases: Orbital integrals, p-adic, Lie algebra

1 Harmonic analysis on $\mathfrak{g}(F)$

Let F be a local field of characteristic 0, let $\bar{F}$ be an algebraic closure of F, and let Γ be the Galois group $\mathrm{Gal}(\bar{F}/F)$. Let G be a connected reductive F-group, and let $\mathfrak{g}$ be its Lie algebra. We begin by recalling Harish-Chandra's viewpoint on the relationship between harmonic analysis on $G(F)$ and harmonic analysis on $\mathfrak{g}(F)$ (see [4] for example).

Harmonic analysis on $G(F)$ is the study of (conjugation) invariant distributions on $G(F)$. Orbital integrals (integrals over conjugacy classes) are such distributions as are the characters of irreducible representations of $G(F)$. Harmonic analysis on $\mathfrak{g}(F)$ is the study of invariant distributions on $\mathfrak{g}(F)$ (invariant under the adjoint action of $G(F)$). Orbital integrals on $\mathfrak{g}(F)$ (integrals over orbits for the adjoint action) are of course the Lie algebra analogs of orbital integrals on the group. One of Harish-Chandra's basic insights was that the Lie algebra analogs of irreducible characters on $G(F)$ are the distributions on $\mathfrak{g}(F)$ obtained as Fourier transforms (in the distribution sense) of orbital integrals.

Harish-Chandra exploited this analogy in two ways. First, he often proved theorems in pairs, one on the group and one on the Lie algebra. For example he showed that both irreducible characters and Fourier transforms of orbital integrals are locally integrable functions, smooth on the regular semisimple set. Second, using the exponential map, he reduced questions in harmonic analysis in a neighborhood of the identity element in $G(F)$ to questions in harmonic analysis in a neighborhood of 0 in $\mathfrak{g}(F)$.

2 Endoscopy for $\mathfrak{g}(F)$

In this section we assume the field F is p-adic. In recent years Waldspurger [21, 22, 23] has significantly broadened the scope of the analogy discussed above by developing a Lie algebra analog of the theory of endoscopy. As is the case on the group, the theory of endoscopy on $\mathfrak{g}(F)$ is not yet complete, but Waldspurger's results give extremely convincing evidence that such a theory exists and go a long way towards establishing the theory by reducing everything to the Lie algebra analog of the "fundamental lemma." We now summarize these results.

We begin with the basic definitions. We fix a non-trivial (continuous) additive character $\psi : F \to \mathbf{C}^\times$ and a non-degenerate G-invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}(F)$, and we use $\langle \cdot, \cdot \rangle$, ψ to identify $\mathfrak{g}(F)$ with its Pontryagin dual. We let dX denote the unique self-dual Haar measure on $\mathfrak{g}(F)$ with respect to $\psi\langle \cdot, \cdot \rangle$. Let f belong to $C_c^\infty(\mathfrak{g}(F))$, the space of locally constant, compactly supported functions on $\mathfrak{g}(F)$, and define the Fourier transform $\hat{f}$ of f by $\hat{f}(Y) = \int_{\mathfrak{g}(F)} f(X)\psi\langle X, Y\rangle dX$. A distribution D on $\mathfrak{g}(F)$ is simply a $\mathbf{C}$-linear map $D : C_c^\infty(\mathfrak{g}(F)) \to \mathbf{C}$, and its Fourier transform $\hat{D}$ is the distribution with the defining property that $\hat{D}(f) = D(\hat{f})$ for all $f \in C_c^\infty(\mathfrak{g}(F))$.

We now define normalized orbital integrals. The Haar measure dX on $\mathfrak{g}(F)$ determines a Haar measure dg on $G(F)$ (impose compatibility under the exponential map in a neighborhood of the origin). Now let $X \in \mathfrak{g}(F)$ be a regular semisimple element and let T be its centralizer in G. Choose a Haar measure dt on $T(F)$. Let $D_G(X) = \det(\mathrm{Ad}(X); \mathfrak{g}/\mathfrak{t})$, where $\mathfrak{t}$ is of course the Lie algebra of T. The normalized orbital integral I_X is defined by

$$I_X(f) = |D_G(X)|^{1/2} \int_{T(F)\backslash G(F)} f(\mathrm{Ad}(g^{-1})(X))\, dg/dt \qquad (f \in C_c^\infty(\mathfrak{g}(F)).$$

We also need the stable orbital integral SI_X defined by $SI_X = \sum_{X'} I_{X'}$, the sum being taken over a set of representatives for the $G(F)$-orbits in the set of elements $X' \in \mathfrak{g}(F)$ that are *stably conjugate* to X in the sense that there exists $g \in G$ such that $\mathrm{Ad}(g)(X) = X'$; we use the (canonical) F-isomorphism $\mathrm{Ad}(g)$ from T to the centralizer T' of X' to transport our measure dt over to $T'(F)$.

We let $C_c^\infty(\mathfrak{g}(F))^{\mathrm{unst}}$ denote the subspace of $C_c^\infty(\mathfrak{g}(F))$ consisting of all elements f such that $SI_X(f) = 0$ for all regular semisimple $X \in \mathfrak{g}(F)$. A distribution D on $\mathfrak{g}(F)$ is said to be *stably invariant* if $D(f) = 0$ for all $f \in C_c^\infty(\mathfrak{g}(F))^{\mathrm{unst}}$. Clearly any stably invariant distribution is in fact invariant.

Waldspurger [23] has shown that the Fourier transform of a stably invariant distribution is again stably invariant. It follows that the Fourier transform carries $C_c^\infty(\mathfrak{g}(F))^{\mathrm{unst}}$ onto itself, and hence induces an automorphism of the quotient space $SC_c^\infty(\mathfrak{g}(F)) := C_c^\infty(\mathfrak{g}(F))/C_c^\infty(\mathfrak{g}(F))^{\mathrm{unst}}$.

Now let (H, s, ξ) be an endoscopic triple for G (see [10]). Thus H is a quasi-split F-group, s is a Γ-fixed element in the center of the Langlands dual group $\hat{H}$ of H, and ξ is an L-embedding of ${}^L H$ into ${}^L G$ that identifies $\hat{H}$ with the identity component of the centralizer in $\hat{G}$ of the image of s. If the derived group of G is not simply connected, objects slightly more general than (H, s, ξ) are needed [10], but let us ignore this minor complication.

For any maximal torus T_H in H there is a canonical G-conjugacy class of embeddings $\mathfrak{t}_H \to \mathfrak{g}$. We say that $Y \in \mathfrak{t}_H$ is G-regular if its image under any of these embeddings is regular in $\mathfrak{g}$. Any such embedding that is defined over F is called an *admissible embedding*. If G is not quasi-split, admissible embeddings need not exist. Given G-regular $X_H \in \mathfrak{t}_H(F)$, one says that $X_G \in \mathfrak{g}(F)$ is an *image* of X_H if there is an admissible embedding mapping X_H to X_G. The set of images of X_H in $\mathfrak{g}(F)$ is either empty or a single stable conjugacy class, called the image of the stable class of X_H.

Waldspurger [22], [23] defines transfer factors for $\mathfrak{g}(F)$ analogous to those of Langlands-Shelstad [10]. These are non-zero complex numbers $\Delta(X_H, X_G)$, defined whenever $X_H \in \mathfrak{h}(F)$ is G-regular and X_G is an image of X_H. The transfer factor depends only on the stable conjugacy class of X_H, and for any stable conjugate X'_G of X_G, we have the simple transformation law

$$\Delta(X_H, X'_G) = \Delta(X_H, X_G) \cdot \langle \mathrm{inv}(X_G, X'_G), s_{T_G}\rangle^{-1}, \tag{1}$$

in which the last factor has the following meaning. Let T_G (respectively, T_H) denote the centralizer of X_G (respectively, X_H) in G (respectively, H). We identify T_H and T_G using the unique admissible embedding that carries X_H to X_G. Choose $g \in G$ such that $\mathrm{Ad}(g)(X'_G) = X_G$. Then $\sigma \mapsto g\sigma(g)^{-1}$ is a 1-cocycle of Γ in T_G, whose class we denote by $\mathrm{inv}(X_G, X'_G)$. The element s appearing in our endoscopic data is a Γ-fixed element in the center of $\hat{H}$, and thus can be regarded as a Γ-fixed element s_{T_G} of $\hat{T}_H = \hat{T}_G$. We then pair $\mathrm{inv}(X_G, X'_G)$ with s_{T_G} using the Tate-Nakayama pairing

$$\langle \cdot, \cdot \rangle : H^1(F, T_G) \times \hat{T}_G^{\Gamma} \to \mathbf{C}^{\times},$$

where $\hat{T}_G^{\Gamma}$ denotes the group of fixed points of Γ in $\hat{T}_G$.

Matching Conjecture. See [22]. For every $f \in C_c^{\infty}(\mathfrak{g}(F))$ there exists $f^H \in C_c^{\infty}(\mathfrak{h}(F))$ such that for every G-regular semisimple element $X_H \in \mathfrak{h}(F)$

$$SI_{X_H}(f^H) = \sum_{X_G} \Delta(X_H, X_G) I_{X_G}(f), \tag{2}$$

where the sum ranges over a set of representatives X_G for the $G(F)$-orbits in the set of images of X_H in $\mathfrak{g}(F)$. Here we are using Haar measures on the centralizers T_H, T_G of X_H, X_G that correspond to each other under the unique admissible isomorphism $T_H \simeq T_G$ that carries X_H to X_G.

Since the G-regular semisimple elements in $\mathfrak{h}(F)$ are dense in the set of all regular semisimple elements in $\mathfrak{h}(F)$, the stable regular semisimple orbital integrals of f^H are uniquely determined, and therefore f^H is uniquely determined as an element in $SC_c^{\infty}(\mathfrak{h}(F))$. Assume the matching conjecture is true. Then $f \mapsto f^H$ is a well-defined linear map $C_c^{\infty}(\mathfrak{g}(F)) \to SC_c^{\infty}(\mathfrak{h}(F))$, and dual to this is a linear map (endoscopic induction, generalizing parabolic induction) i_H^G from stably invariant distributions on $\mathfrak{h}(F)$ to invariant distributions on $\mathfrak{g}(F)$, defined by $i_H^G(D)(f) = D(f^H)$ (where D is a stably invariant distribution on $\mathfrak{h}(F)$ and $f \in C_c^\infty(\mathfrak{g}(F))$). By its very definition endoscopic induction carries SI_{X_H} into $\sum_{X_G} \Delta(X_H, X_G) I_{X_G}$.

Waldspurger [22] observes that the analogous matching conjecture on the group [10] implies the matching conjecture on the Lie algebra (via the exponential map) and that the matching conjecture for all reductive Lie algebras simultaneously implies the matching conjecture for all reductive groups simultaneously (via the theory of descent and local transfer developed by Langlands-Shelstad [11]).

Consider for a moment the case of endoscopy on a real group $G(\mathbf{R})$. Then Shelstad [18] has proved that endoscopic induction carries certain stable combinations of irreducible characters on $H(F)$ to unstable linear combinations of irreducible characters on $G(F)$. The coefficients in these unstable linear combinations can be regarded as spectral analogs of transfer factors. Langlands [12] has conjectured that there is similar theory for p-adic groups as well.

Now we return to endoscopy on $\mathfrak{g}(F)$. Let X be a regular semisimple element in $\mathfrak{g}(F)$. Recall that the Fourier transform $\hat{I}_X$ of the normalized orbital integral I_X is the Lie algebra analog of an irreducible tempered character Θ on $G(F)$. Waldspurger remarks that the Lie algebra analog of the L-packet of Θ is the set of distributions $\hat{I}_{X'}$ where X' ranges through the stable class of X. By Waldspurger's theorem that the Fourier transform preserves stability, the Fourier transform $\widehat{SI}_X = \sum_{X'} \hat{I}_{X'}$ is a stably invariant linear combination of the members in the "L-packet" of $\hat{I}_X$. Waldspurger then makes the following transfer conjecture, analogous to Shelstad's character identities for real groups. The conjecture involves the Fourier transform on $\mathfrak{h}(F)$, which must therefore be normalized properly, using the same additive character ψ as before and using a symmetric bilinear form on $\mathfrak{h}(F)$ deduced from the one of $\mathfrak{g}(F)$. In order to state this conjecture one must assume the truth of the matching conjecture.

Weak transfer conjecture. See [22]. There is non-zero constant $c \in \mathbf{C}$ (which Waldspurger specifies precisely) such that for all G-regular semisimple elements $X_H \in \mathfrak{h}(F)$ and for all $f \in C_c^\infty(\mathfrak{g}(F))$

$$\widehat{SI}_{X_H}(f^H) = c \sum_{X_G} \Delta(X_H, X_G) \hat{I}_{X_G}(f), \tag{3}$$

where X_G ranges over a set of representatives for the $G(F)$-orbits in the set of images of X_H in $\mathfrak{g}(F)$. Equivalently, the Fourier transform commutes with the map $f \mapsto f^H$ from $C_c^\infty(\mathfrak{g}(F))$ to $SC_c^\infty(\mathfrak{h}(F))$, up to the scalar factor c.

If the matching and weak transfer conjectures are both true, then endoscopic induction commutes with the Fourier transform, up to the scalar c. It should be emphasized that the transfer factors appearing in the weak transfer conjecture are the same transfer factors as before. Thus, for groups there are both "geometric" and "spectral" transfer factors, while on Lie algebras the transfer factors $\Delta(X_H, X_G)$ play a double role.

Waldspurger also reformulates the weak transfer conjecture in such a way that it makes sense without assuming the matching conjecture. Assume for the moment that the matching conjecture holds, so that endoscopic induction is defined. Let D_H be a stably invariant distribution on $\mathfrak{h}(F)$ and assume that D_H is a locally integrable function Θ_H on $\mathfrak{h}(F)$, locally constant on the regular semisimple set. The stable invariance of D_H is equivalent to the condition that Θ_H be constant on stable conjugacy classes. It follows from the Lie algebra analog of the Weyl integration formula that the invariant distribution $D_G = i_H^G(D_H)$ is a locally integrable function $i_H^G(\Theta_H)$ on $\mathfrak{g}(F)$, locally constant on the regular semisimple set, and that the value of the function $i_H^G(\Theta_H)$ on a regular semisimple element $X_G \in \mathfrak{g}(F)$ is given by

$$|D_G(X_G)|^{1/2} i_H^G(\Theta_H)(X_G) = \sum_{X_H} \Delta(X_H, X_G) \cdot |D_H(X_H)|^{1/2} \Theta_H(X_H), \qquad (4)$$

where the sum ranges over a set of representatives X_H for the stable conjugacy classes in $\mathfrak{h}(F)$ whose image in $\mathfrak{g}(F)$ is the stable conjugacy class of X_G.

By Harish-Chandra's fundamental local integrability theorem these considerations apply to Fourier transforms of orbital integrals. Thus $\hat{I}_{X_G}$ and $\widehat{SI}_{X_G}$ are given by locally integrable functions Θ_{X_G} and $S\Theta_{X_G}$. Then, assuming the matching conjecture holds, the weak transfer conjecture is equivalent to the following conjecture.

Transfer conjecture. See [22], [23]. There is non-zero constant $c \in \mathbf{C}$ (the same as before) such that for all G-regular semisimple elements $X_H \in \mathfrak{h}(F)$

$$i_H^G(S\Theta_{X_H}) = c \sum_{X_G} \Delta(X_H, X_G) \Theta_{X_G}, \qquad (5)$$

where X_G ranges over the $G(F)$-orbits in the set of images of X_H in $\mathfrak{g}(F)$.

Not only can the transfer conjecture be formulated without the matching conjecture, but in fact Waldspurger [22] VIII.7(8) shows that the transfer conjecture implies the matching conjecture. (The proof uses Harish-Chandra's theorem [5] that Shalika germs appear as coefficients in the Lie algebra analog of his local character expansion. Thus the transfer conjecture implies that κ-Shalika germs on $\mathfrak{g}(F)$ can be expressed in terms of stable Shalika germs on $\mathfrak{h}(F)$, and this, by work of Langlands-Shelstad [11], implies matching for both $G(F)$ and $\mathfrak{g}(F)$.)

There is one last conjecture to discuss, the Lie algebra analog of the fundamental lemma (see [12] for a discussion of the fundamental lemma on the group). For this we assume that G and H are unramified, and we let f, f^H denote the characteristic functions of hyperspecial parahoric subalgebras of $\mathfrak{g}(F)$, $\mathfrak{h}(F)$ respectively. The fundamental lemma (a conjecture, not a theorem) asserts that these particular functions satisfy equation (2) (for suitably normalized transfer factors). Using global methods (the Lie algebra analog of the trace formula), Waldspurger has shown that the fundamental lemma implies the transfer conjecture.

Theorem 1 (Waldspurger [23]) *Suppose that the fundamental lemma holds for all p-adic fields and all unramified G and H. Then the transfer conjecture is true, and hence the matching conjecture is true as well.*

3 Transfer factors in the quasi-split case

Once again we allow F to be any local field. We are going to give a simple formula for transfer factors on Lie algebras in the quasi-split case. So suppose that G is quasi-split and fix an F-splitting $(B_0, T, \{X_\alpha\})$ for G. Thus B_0 is a Borel F-subgroup of G, T is a maximal F-torus in B_0, and $\{X_\alpha\}$ is a collection of simple root vectors $X_\alpha \in \mathfrak{g}_\alpha$, one for each simple root α of T in the Lie algebra of B_0, such that $X_{\sigma\alpha} = \sigma(X_\alpha)$ for all $\sigma \in \Gamma$. (As usual, for any root β of T in $\mathfrak{g}$ we write $\mathfrak{g}_\beta$ for the corresponding root subspace of $\mathfrak{g}$.)

Waldspurger's factors are analogous to the transfer factors $\Delta(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G)$ [10] with the factor Δ_{IV} removed. On the quasi-split group G Langlands and Shelstad also define transfer factors $\Delta_0(\gamma_H, \gamma_G)$ (see p. 248 of [10]). These depend on the chosen F-splitting. The transfer factors $\Delta_0(X_H, X_G)$ we consider now are complex roots of unity, analogous to $\Delta_0(\gamma_H, \gamma_G)$ with the factor Δ_{IV} removed, and they too depend on the choice of F-splitting.

We write $\mathfrak{b}_0$ for the Lie algebra of B_0. For each simple root α we define $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ by requiring that $[X_\alpha, X_{-\alpha}]$ be the coroot for α, viewed as element in the Lie algebra of T. We put $X_- := \sum_\alpha X_{-\alpha}$, where α runs over the set of simple roots of T in B_0. Of course X_- lies in $\mathfrak{g}(F)$ and depends on the choice of F-splitting. The following theorem is proved in [9].

Theorem 2 *The factor $\Delta_0(X_H, X_G)$ is 1 whenever X_G lies in $\mathfrak{b}_0(F) + X_-$.*

Kostant [6] proved that every stable conjugacy class of regular semisimple elements in $\mathfrak{g}(F)$ meets the set $\mathfrak{b}_0(F) + X_-$. Since the values of $\Delta_0(X_H, X_G)$ and $\Delta_0(X_H, X'_G)$ are related by a simple Galois-cohomological factor (see (1)) whenever X_G and X'_G are stably conjugate, this theorem also yields a simple formula for the value of $\Delta_0(X_H, X_G)$ for arbitrary X_G. The methods used to prove the theorem are variants of ones used in [10], [13], [19]. In particular Proposition 5.2 in [13] plays a key role. What is new is the connection with Kostant's set $\mathfrak{b}_0(F)+X_-$.

4 Stability for nilpotent orbital integrals

It is especially interesting to study nilpotent orbital integrals from the point of view of endoscopy. The first question that comes to mind is which linear combinations of nilpotent orbital integrals are stably invariant. One can ask the same question for unipotent orbital integrals on the group. Let u be a unipotent element in $G(F)$. The stable conjugacy class of u is by definition the set of F-rational points on the G-conjugacy class of u. Assume for the moment that F is p-adic and that G is classical. Then Assem [1] made two conjectures concerning the stability of linear combinations of orbital integrals for orbits in the stable class of u. His first conjecture is that there are no non-zero stable combinations unless u is special in Lusztig's sense. Now assume that u is special. The second conjecture asserts that the set of conjugacy classes within the stable class of u can be decomposed as a disjoint union of "stability packets," in such a way that a suitable linear combination of the unipotent orbital integrals for the orbits within a single stability packet is stable, and moreover any stable linear combination is obtained as a sum of these basic ones. The definition of stability packets involves Lusztig's quotient group of the component group of the centralizer of u. For split classical groups Assem further predicted that the relevant linear combination was simply the sum. Waldspurger has announced a proof of Assem's conjectures and has determined the linear combinations needed in the quasi-split case.

Now consider the real case. We work with nilpotent orbital integrals on a quasi-split Lie algebra. Then there is a formula [7, 8] for the dimension of the space of stable linear combinations of nilpotent orbital integrals for nilpotent orbits within a given stable class. The formula involves constructions of Lusztig and is valid for all quasi-split real groups, even the exceptional ones. We now state the formula precisely for split simple real groups.

Let $\mathbf{O}$ be a nilpotent $G(\mathbf{C})$-orbit in $\mathfrak{g}_{\mathbf{C}}$ and let $r_{\mathbf{O}}$ be the number of $G(\mathbf{R})$-orbits in $\mathfrak{g}(\mathbf{R}) \cap \mathbf{O}$. The linear span of the corresponding orbital integrals is an $r_{\mathbf{O}}$-dimensional space $\mathcal{D}_{\mathbf{O}}$ of (tempered [15]) invariant distributions on $\mathfrak{g}(\mathbf{R})$. The formula we are going to give for the dimension $s_{\mathbf{O}}$ of the subspace $\mathcal{D}_{\mathbf{O}}^{\mathrm{st}}$ of stably invariant elements in $\mathcal{D}_{\mathbf{O}}$ should be thought of as the stable analog of Rossmann's formula for $r_{\mathbf{O}}$, which we now recall.

Let T be a maximal torus in G and let W denote its Weyl group in $G(\mathbf{C})$. Then complex-conjugation, denoted σ, acts on W, and we consider the group W^σ of fixed points of σ on W. Inside W^σ we have the subgroup $W_{\mathbf{R}}$ consisting of elements in W that can be realized by elements in $G(\mathbf{R})$ that normalize T. Let R_I denote the set of imaginary roots of T (roots α such that $\sigma\alpha = -\alpha$). We define a sign character ϵ_I on W^σ in the usual way: $\epsilon_I(w) = (-1)^b$, where b is the number of positive roots $\alpha \in R_I$ such that $w\alpha$ is negative. By restriction we also regard ϵ_I as a character on $W_{\mathbf{R}}$.

Via Springer's correspondence [20] the nilpotent orbit $\mathbf{O}$ determines an irreducible character $\chi_{\mathbf{O}}$ of the abstract Weyl group W_a of $G(\mathbf{C})$. For example the trivial orbit $\mathbf{O} = \{0\}$ corresponds to the sign character ϵ of W_a. Of course we can also think of $\chi_{\mathbf{O}}$ as an irreducible character of the Weyl group W of any T as above, so that we can consider the multiplicity $m_{W_{\mathbf{R}}}(\epsilon_I, \chi_{\mathbf{O}})$ of ϵ_I in the restriction of $\chi_{\mathbf{O}}$ to the subgroup $W_{\mathbf{R}}$. Rossmann [17] proved that

$$r_{\mathbf{O}} = \sum_T m_{W_{\mathbf{R}}}(\epsilon_I, \chi_{\mathbf{O}}),$$

where T runs over the maximal $\mathbf{R}$-tori in G, up to $G(\mathbf{R})$-conjugacy.

Theorem 3 *The integer $s_{\mathbf{O}}$ is given by*

$$s_{\mathbf{O}} = \sum_T m_{W^\sigma}(\epsilon_I, \chi_{\mathbf{O}}),$$

where the index set for the sum is the same as in Rossmann's formula and where $m_{W^\sigma}(\epsilon_I, \chi_{\mathbf{O}})$ denotes the multiplicity of ϵ_I in the restriction of $\chi_{\mathbf{O}}$ to W^σ.

The proof of this theorem uses the Fourier transform, and one must check that the Fourier transform of a stably invariant tempered distribution on $\mathfrak{g}(\mathbf{R})$ is stably invariant, as in the p-adic case [23], and this can be done using another theorem of Rossmann [16]. The Fourier transforms of nilpotent orbital integrals are locally integrable functions on $\mathfrak{g}(\mathbf{R})$, and it is easy to recognize when such a locally integrable function represents a stably invariant distribution. Moreover, a description of the image of $\mathcal{D}_{\mathbf{O}}$ under the Fourier transform is implicit in the literature. (I am very much indebted to V. Ginzburg for this remark.) The theorem follows from these observations (see [8]).

In case G is quasi-split the right-hand side of our formula for $s_{\mathbf{O}}$ can be expressed (see [3], [7]) in terms of Lusztig's quotient groups. For simplicity we limit ourselves here to the case of simple split real groups. We define an integer $m(\chi)$ for any irreducible character χ on the abstract Weyl group W_a of $G(\mathbf{C})$ by the right-hand side of our formula for $s_{\mathbf{O}}$, but with $\chi_{\mathbf{O}}$ replaced by χ. Thus by the previous theorem $s_{\mathbf{O}} = m(\chi_{\mathbf{O}})$.

We need to review some results of Lusztig [14]. The set $W_a^\vee$ of isomorphism classes of irreducible representations of W_a is a disjoint union of subsets, called *families*. Associated to a family $\mathcal{F}$ is a finite group $\mathcal{G} = \mathcal{G}_{\mathcal{F}}$. If R is classical, then $\mathcal{G}$ is an elementary abelian 2-group. If R is exceptional, then $\mathcal{G}$ is one of the symmetric groups S_n $(1 \le n \le 5)$.

Associated to any finite group $\mathcal{G}$ is a finite set $\mathcal{M}(\mathcal{G})$, defined as follows (see [14]). Consider pairs (x, ρ), where x is an element of $\mathcal{G}$ and ρ is an irreducible (complex) representation of the centralizer $\mathcal{G}_x$ of x in $\mathcal{G}$. There is an obvious conjugation action of $\mathcal{G}$ on this set of pairs, and $\mathcal{M}(\mathcal{G})$ is by definition the set of orbits for this action. The set $\mathcal{M}(\mathcal{G})$ has an obvious basepoint $(1,1)$, the first entry being the identity element of $\mathcal{G}$ and the second entry being the trivial representation of $\mathcal{G}$.

We define a function η on $\mathcal{M}(\mathcal{G})$ by

$$\eta(x,\rho) = \sum_{s\in S} d(s,\rho),$$

where S is a set of representatives for the $\mathcal{G}_x$-conjugacy classes of elements $s \in \mathcal{G}_x$ such that $s^2 = x$, and $d(s,\rho)$ denotes the dimension of the space of vectors in ρ fixed by the centralizer $\mathcal{G}_s$ of s in $\mathcal{G}$. (Note that $\mathcal{G}_s$ is a subgroup of $\mathcal{G}_x$, since $s^2 = x$.) In particular $\eta(x,\rho)$ is always a non-negative integer.

If $\mathcal{G}$ is an elementary abelian 2-group, then η is very simple: its value at the basepoint in $\mathcal{M}(\mathcal{G})$ is equal to the order of $\mathcal{G}$ and all remaining values are 0.

Let $\mathcal{F}$ be a family of representations of W_a, and let $\mathcal{G}$ be the associated finite group. Then Lusztig defines (case-by-case) an injection

$$\mathcal{F} \hookrightarrow \mathcal{M}(\mathcal{G}).$$

The image of the unique special representation in $\mathcal{F}$ under this injection is the base point $(1,1) \in \mathcal{M}(\mathcal{G})$.

Theorem 4 *Let E be an irreducible representation of the Weyl group W_a, with character χ. Let $x_E \in \mathcal{M}(\mathcal{G})$ denote the image of E under the injection $\mathcal{F} \hookrightarrow \mathcal{M}(\mathcal{G})$. Then $m(\chi)$ is equal to $\eta(x_E)$ if E is non-exceptional and is equal to 1 if E is exceptional. In particular if E is special and non-exceptional, then $m(\chi)$ is the* *number of conjugacy classes of involutions in $\mathcal{G}$. Moreover if R is classical, so that $\mathcal{G}$ is necessarily an elementary abelian 2-group, then $m(\chi) = 0$ if E is non-special, and $m(\chi) = |\mathcal{G}|$ if E is special.*

See [2] for the notion of exceptional representations of Weyl groups; they occur only for E_7 and E_8. This theorem was proved for classical groups in [7] and for exceptional groups by Casselman in [3]. Note that if $\mathbf{O}$ is a special non-exceptional nilpotent orbit, then the last two theorems together tell us that the dimension of the space of stable linear combinations of $G(\mathbf{R})$-invariant measures on the $G(\mathbf{R})$-orbits in $\mathbf{O} \cap \mathfrak{g}(\mathbf{R})$ is equal to the cardinality of the Galois cohomology set $H^1(\mathbf{R}, \mathcal{G})$ (where $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ operates trivially on $\mathcal{G}$); this should be compared with Conjecture C in the introduction to Assem's paper [1].

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