# Topological Methods in Representation Theory

Kari Vilonen[1]

1991 Mathematics Subject Classification: 22, 55, 14

A few years ago Beilinson and Bernstein introduced a localization technique to representation theory of semisimple Lie groups. Their method allows one to translate questions in representation theory to questions in complex algebraic geometry. Beilinson reported on this work at the Warsaw congress [B]. Consequently, in [K2], Kashiwara initiated a research program, in the form of a series of conjectures, that expands the Beilinson-Bernstein picture. In this survey we will report on work inspired by this point of view. The resulting geometry is no longer complex algebraic; it rather involves real (semi-)algebraic sets. Thus, the methods used will be largely topological. A crucial technique is supplied by the characteristic cycle construction of Kashiwara [K1], which amounts to a version of Morse theory. The majority of the results presented here constitute joint work with Wilfried Schmid.

## 1. Introduction.

Let $G_{\mathbb{R}}$ denote a semisimple Lie group which we assume, for simplicity, to be linear and connected. For example, one can take $G_{\mathbb{R}}$ to be any of the classical groups: $SL_n(\mathbb{R})$, $SO(n,\mathbb{R})$, $SO(p,q)$, $\dots$ . To provide motivation for things to come, let us consider one of the outstanding problems in representation theory: the determination of the unitary dual $\hat{G}_{\mathbb{R}}$, i.e., the determination of the set of isomorphism classes of irreducible unitary representations of $G_{\mathbb{R}}$. Ideally, at least from the geometric point of view, the solution of the problem would have the following form. There should exist a manifold $X$ with a $G_{\mathbb{R}}$-action such that $\hat{G}_{\mathbb{R}}$ is in bijection with a certain set of $G_{\mathbb{R}}$-equivariant "objects" on $X$ (they could be sheaves, for example). Let $\mathcal{F}$ be such a $G_{\mathbb{R}}$-equivariant object. Because the group $G_{\mathbb{R}}$ acts both on $X$ and on $\mathcal{F}$, it also acts on the cohomology groups $\mathrm{H}^*(X,\mathcal{F})$. These groups should have a canonical structure of a Hilbert space such that the group $G_{\mathbb{R}}$ acts continuously and via unitary operators on them. At this time there is not even a precise conjecture as to what the set $\hat{G}_{\mathbb{R}}$ ought to be in general. However, the orbit method of Kirillov-Kostant suggests that we should take as $X$ the space $\mathfrak{g}_{\mathbb{R}}^*$, the dual of the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ of $G_{\mathbb{R}}$. In other words, we should be able to associate unitary representations to the coadjoint orbits (the $G_{\mathbb{R}}$-orbits on $\mathfrak{g}_{\mathbb{R}}^*$), or more precisely, to collections of coadjoint orbits together with some extra data. Given such a set of data, Kirillov has proposed that there is a specific formula – a "universal character formula" – for the character of the representation attached to the data.

[1]Partially supported by NSF, NSA, and a Guggenheim fellowship

As we pointed out at the beginning, the above discussion was included only as motivation. In this survey we will work with the class of admissible (finite length) representations. They include all irreducible unitary representations. For this larger class of representations we will:

a) parametrize and exhibit them geometrically
b) give a geometric character formula in the spirit of Kirillov's formula
c) analyze the nilpotent invariants attached to them

2. Geometric parametrization of representations.

Let $G$ denote the complexification of $G_{\mathbb{R}}$ and let $X$ be the flag manifold of $G$. The group $G_{\mathbb{R}}$ acts on $X$ with finitely many orbits. Let us assume, for the moment, that $G_{\mathbb{R}}$ is compact. Then $G_{\mathbb{R}}$ acts transitively on $X$ and there is only one orbit. All the irreducible representations of $G_{\mathbb{R}}$ are finite dimensional. As is well know, they can be classified and exhibited explicitly as follows. To each $\lambda \in \mathrm{H}^2(X,\mathbb{Z})$ corresponds a complex line bundle $\mathcal{O}(\lambda)$ on $X$. This line bundle is holomorphic and $G_{\mathbb{R}}$-homogenous, i.e., the action of $G_{\mathbb{R}}$ on $X$ lifts to an action of $G_{\mathbb{R}}$ on $\mathcal{O}(\lambda)$ (strictly speaking, this it true only if $G_{\mathbb{R}}$ is simply connected; if this is not the case, then $\lambda$ must lie in a sublattice of $\mathrm{H}^2(X,\mathbb{Z})$). Thus we get a representation of $G_{\mathbb{R}}$ on the vector spaces $\mathrm{H}^k(X,\mathcal{O}(\lambda))$. All irreducible representations of $G_{\mathbb{R}}$ arise in this fashion. Furthermore, if we restrict the parameter $\lambda$ to lie in the dominant cone in $\mathrm{H}^2(X,\mathbb{Z})$ then each irreducible representation occurs exactly once among the representations $\mathrm{H}^0(X,\mathcal{O}(\lambda))$.

When the group $G_{\mathbb{R}}$ is not assumed to be compact, the situation is more complicated. First of all, we have to allow the "twisting" parameter $\lambda$ to lie in $\mathrm{H}^2(X,\mathbb{C})$, not just in the lattice $\mathrm{H}^2(X,\mathbb{Z})$. To each such $\lambda \in \mathrm{H}^2(X,\mathbb{C})$ we associate the "twisted" $G$-equivariant sheaf $\mathcal{O}_X^{\mathrm{an}}(\lambda)$ of holomorphic functions on $X$. This is an "ordinary" sheaf on $X$ only if $\lambda$ is integral. The second complication arises because the action of $G_{\mathbb{R}}$ on $X$ is not transitive. As a first approximation, we can construct representations $\mathrm{H}^k(S,\mathcal{O}_X^{\mathrm{an}}(\lambda))$ associated to each $G_{\mathbb{R}}$-orbit $S$ and the parameter $\lambda \in \mathrm{H}^2(X,\mathbb{C})$. This construction yields all the "standard representations" but not all the irreducible (admissible) representations of $G_{\mathbb{R}}$.

To get all the representations, we have to allow combinations of $G_{\mathbb{R}}$-orbits and we have to allow the orbits to "interact" with each other. This can be accomplished purely topologically: we consider $G_{\mathbb{R}}$-equivariant (complexes of) $\mathbb{C}$-sheaves on $X$ whose stalks are finite dimensional over $\mathbb{C}$. Note that the category of $G_{\mathbb{R}}$-sheaves on a $G_{\mathbb{R}}$-orbit $S$ is equivalent to the category of (finite dimensional) complex representations of the component group $(G_{\mathbb{R}})_x/(G_{\mathbb{R}})_x^0$ of $(G_{\mathbb{R}})_x$. Here $(G_{\mathbb{R}})_x$ denotes the stabilizer group of any particular $x \in S$. A general $G_{\mathbb{R}}$-equivariant sheaf is "glued" together from such local systems on the various $G_{\mathbb{R}}$-orbits. Technically, these sheaves should be twisted, with twist $-\lambda$, and we should consider them in the context of derived categories, i.e., we should view them as elements in the $G_{\mathbb{R}}$-equivariant derived category of $\mathbb{C}$-sheaves with twist $-\lambda$. For the purposes of this survey, this technical point can be ignored and one can think of them just as sheaves with a $G_{\mathbb{R}}$-action. In particular, one can assume that $\lambda = 0$, in which case $\mathcal{O}(\lambda)$ is the trivial line bundle on $X$. We define functors

$$\{G_{\mathbb{R}}\text{-equivariant sheaves on } X\} \longrightarrow \{G_{\mathbb{R}}\text{-representations}\} \tag{1.1a}$$

by

$$\text{(1.1b)} \qquad \mathcal{F} \quad \longmapsto \quad \mathrm{H}^k(X, \mathcal{F} \otimes_{\mathbb{C}} \mathcal{O}_X^{\mathrm{an}}(\lambda))\,.$$

In [KSd] it is shown that the cohomology groups $\mathrm{H}^k(X, \mathcal{F} \otimes_{\mathbb{C}} \mathcal{O}_X^{\mathrm{an}}(\lambda))$ carry a natural Fréchet topology such that the action of $G_{\mathbb{R}}$ is continuous. The topology is induced from the natural topology on $\mathcal{O}_X^{\mathrm{an}}(\lambda)$. In representation theoretic terms the choice of the parameter $\lambda$ amounts to fixing the infinitesimal character of the representations in (1.1): the space $\mathrm{H}^2(X, \mathbb{C})$ can be identified[2] with the dual of a Cartan $\mathfrak{t}$ in $\mathfrak{g}$.

The representations produced by the functor (1.1) are admissible. In the rest of this paper, the term $G_{\mathbb{R}}$-representation stands for an admissible representation (of finite length). Recall that a $G_{\mathbb{R}}$-representation $V$ is called admissible if, when viewed as a representation of a maximal compact subgroup $K_{\mathbb{R}}$ of $G_{\mathbb{R}}$, each irreducible representation of $K_{\mathbb{R}}$ appears in it with finite multiplicity. We consider admissible representation modulo infinitesimal equivalence. In other words, we identify representations if they are "the same" except for the topology that we put on the representation space. When we work up to infinitesimal equivalence the functor (1.1) is onto. The infinitesimal equivalence class of a $G_{\mathbb{R}}$-representation $V$ is captured by its Harish-Chandra module. Recall that the Harish-Chandra module $M$ of the representation $V$ consists of all vectors $v \in V$ such that $K_{\mathbb{R}} \cdot v$ generates a finite dimensional subspace of $V$. Both the the lie algebra $\mathfrak{g}_{\mathbb{R}}$ and the group $K_{\mathbb{R}}$, and hence their complexifications $\mathfrak{g}$ and $K$, act compatibly on $M$. Harish-Chandra modules are algebraic objects and are not equipped with a topology.

Let us continue to consider a particular $G_{\mathbb{R}}$-representation which is associated to the parameter $\lambda \in \mathrm{H}^2(X, \mathbb{C})$ and a $G_{\mathbb{R}}$-sheaf $\mathcal{F}$. To construct the Harish-Chandra module associated to this representation geometrically, we appeal to the work of Beilinson-Bernstein. Slightly paraphrased, they constructed functors

$$\text{(1.2a)} \qquad \{K\text{-equivariant sheaves on } X\} \longrightarrow \{\text{H-C-modules}\}$$

by

$$\text{(1.2b)} \qquad \mathcal{F} \quad \longmapsto \quad \mathrm{H}^k(X, \mathcal{F} \otimes_{\mathbb{C}} \mathcal{O}_X^{\mathrm{alg}}(\lambda))\,.$$

Here, just as in our previous discussion, the sheaf $\mathcal{F}$ is properly viewed as an element in the $K$-equivariant derived category of $\mathbb{C}$-sheaves with twist $-\lambda$. The symbol $\mathcal{O}_X^{\mathrm{alg}}(\lambda)$ stands for the twisted sheaf of complex algebraic functions on $X$. It is a subsheaf of $\mathcal{O}_X^{\mathrm{an}}(\lambda)$. On the other hand, in [MUV] we construct an equivalence of categories

$$\text{(1.3)} \qquad \{G_{\mathbb{R}}\text{-equivariant sheaves on } X\} \xrightarrow{\ \Gamma\ } \{K\text{-equivariant sheaves on } X\}\,.$$

[2]Under this identification the value $\lambda = 0$ corresponds to the element "$\rho$ = half the sum of positive roots" in $\mathfrak{t}^*$.

This equivalence is constructed via an averaging procedure. Loosely speaking, we average a $G_{\mathbb{R}}$-sheaf over the orbits of $K/K_{\mathbb{R}}$. The Harish-Chandra module of the representation associated to the $G_{\mathbb{R}}$-sheaf $\mathcal{F}$ is gotten by applying the Beilinson-Bernstein functor (1.2) to the sheaf $\Gamma\mathcal{F}$. The commutative diagram below summarizes our discussion:

$$(1.4)\qquad \begin{array}{ccc} \{G_{\mathbb{R}}\text{-representations}\} & \longrightarrow & \{\text{H-C-modules}\} \\ {\scriptstyle (1.1)}\uparrow & & \uparrow{\scriptstyle (1.2)} \\ \{G_{\mathbb{R}}\text{-equivariant sheaves on } X\} & \xrightarrow{\Gamma} & \{K\text{-equivariant sheaves on } X\}\,. \end{array}$$

We have have not specified the degree $k$ that we should use for the cohomology groups in formulas (1.1b) and (1.2b). If we restrict $\lambda$ to lie in the dominant cone then there is a natural choice of a subcategory of complexes of $G_{\mathbb{R}}$-sheaves and a subcategory of complexes of $K$-sheaves such that the functors (1.1b) and (1.2b) are nonzero on these subcategories only for the value $k = 0$. Furthermore, restricted to these subcategories the functors (1.1b) and (1.2b) are equivalences, provided that the parameter $\lambda$ is regular[3]. On the $K$-side the subcategory has a useful characterization. It consists of $K$-equivariant perverse sheaves. On the $G_{\mathbb{R}}$-side no direct characterization is known.

The motivation for the original work of Beilinson-Bernstein was to understand how standard representations decompose into irreducibles (Kazhdan-Lusztig conjectures). Via the functor (1.2) this problem translates into the problem of understanding how standard perverse sheaves decompose into irreducible perverse sheaves. This problem, in turn, can be solved by using the theory of mixed sheaves. For a survey, see Beilinson's talk at the Warsaw congress [B]. In the same vain other questions in representation theory can be translated to questions about the geometry of (closures of) $K$-orbits. Because $K$ is a complex algebraic group, we are in the context of (complex) algebraic geometry. The situation on the $G_{\mathbb{R}}$-side is different. The $G_{\mathbb{R}}$-orbits are only semi-algebraic sets and hence appear to be more difficult to work with. In the rest of this paper we use topological techniques that allow one to work on the $G_{\mathbb{R}}$-side. Although the categories of $G_{\mathbb{R}}$-equivariant sheaves and $K$-equivariant sheaves are equivalent certain things appear to be easier to extract from one side than the other. For example, it appears impossible at this time to give a proof of the Kazhdan-Lusztig conjectures on the $G_{\mathbb{R}}$-side. On the other hand, the character formula that we explain in the next section crucially depends on the $G_{\mathbb{R}}$-side.

## 3. A geometric character formula.

In this section we will explain a character formula which can be viewed as a generalization of Kirillov's "universal character formula", valid for all admissible representations. Recall that to any representation we can associate its character which is a conjugation invariant, locally $L^1$-function on the group $G_{\mathbb{R}}$. The function on $\mathfrak{g}_{\mathbb{R}}$ gotten by pulling back the character under the exponential map (and multiplied by the square root of the Jacobian) is called the Lie algebra character.

[3]If we identify $\mathrm{H}^2(X, \mathbb{C})$ with a Cartan $\mathfrak{t}$ this amounts to the regularity of $\lambda + \rho$.

Let us recall the formula proposed by Kirillov:

(2.1)

The Lie algebra character of the representation
associated to the coadjoint orbit $\mathcal{O}_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{R}}^*$
is the Fourier transform of the canonical measure on $\mathcal{O}_{\mathbb{R}}$.

As a coadjoint orbit, $\mathcal{O}_{\mathbb{R}}$ has a canonical symplectic form and hence a canonical measure. In [R1] Rossmann gave a proof of Kirillov's formula for tempered representations, i.e., for the irreducible unitary representations that "appear" in the regular representation $L^2(G_{\mathbb{R}})$.

In [R2], Rossmann made the following proposal to obtain a Kirillov type character formula in general. Let us fix the parameter $\lambda \in \mathrm{H}^2(X,\mathbb{C})$ and let us consider $G_{\mathbb{R}}$-representations associated to this parameter. Recall that the dual $\mathfrak{t}^*$ of any Cartan $\mathfrak{t} \subset \mathfrak{g}$ can be identified with $\mathrm{H}^2(X,\mathbb{C})$ (see footnote 1). Hence, the element $\lambda$ specifies a coadjoint $G$-orbit $\Omega_\lambda \subset \mathfrak{g}^*$. If $\lambda$ is regular, as we will assume from now on, there is an isomorphism $\mu_\lambda : T^*X \to \Omega_\lambda$, due to Rossmann, which he calls the twisted moment map. To have some feel for this map, we describe it loosely. First of all, it is the twisted version of the moment map $\mu : T^*X \to \mathfrak{g}^*$ for the $G$-action. The moment map $\mu$ has its image in the nilpotent cone $\mathcal{N}^*$ in $\mathfrak{g}^* \cong \mathfrak{g}$. Note that, under the identification $\mathfrak{g}^* \cong \mathfrak{g}$, the cotangent space $T_x^*X$ is identified with $\mathfrak{n}_x$, where $\mathfrak{n}_x$ is the nilpotent radical of the Borel subalgebra corresponding to $x$. With these identifications the map $\mu$ is the identity on $T_x^*X$. To describe $\mu_\lambda$, let $U_{\mathbb{R}}$ be the compact form of $G$ which is "compatible" with $K_{\mathbb{R}}$ and $G_{\mathbb{R}}$. Because $U_{\mathbb{R}}$ acts transitively on $X$, the flag manifold $X$ can be identified with a canonical $U_{\mathbb{R}}$-orbit inside $\Omega_\lambda$. The map $\mu_\lambda$ is obtained by translating the moment map $\mu$ by the $U_{\mathbb{R}}$-embedding of $X$ in $\Omega_\lambda$. On the zero section of $T^*X$ the map $\mu_\lambda$ reduces to the $U_{\mathbb{R}}$-embedding of $X$ in $\Omega_\lambda$. The twisted moment map is $U_{\mathbb{R}}$-equivariant and only real algebraic, not complex algebraic.

Let us consider the complex vector space spanned by the Lie algebra characters of all the representations associated to the parameter $\lambda$. This is the space of invariant eigendistributions (associated to the parameter $\lambda$, which we have assumed to be regular). Rossmann shows that any invariant eigendistribution on $\mathfrak{g}_{\mathbb{R}}$ can be uniquely written in the following form. We set

$$T^*_{G_{\mathbb{R}}}X \;=\; \bigcup_{S \text{ a } G_{\mathbb{R}}\text{-orbit}} T^*_S X \;\subset\; T^*X\,.$$

Here $T^*_S X$ denotes the conormal bundle of the orbit $S$ in $X$; by definition $T^*_S X$ is a subspace of $T^*X$. If we let $n$ denote the complex dimension of $X$, then the space $T^*_{G_{\mathbb{R}}}X$ has real dimension $2n$. Let us denote by $\mathrm{H}^{inf}_{2n}(T^*_{G_{\mathbb{R}}}X,\mathbb{C})$ the space of $2n$-cycles with closed (possibly infinite dimensional) support in $T^*_{G_{\mathbb{R}}}X$ with coefficients in $\mathbb{C}$. Rossmann shows that for any invariant eigendistribution $\theta$ on $\mathfrak{g}_{\mathbb{R}}$ associated to $\lambda$ there exists a unique cycle $C \in \mathrm{H}^{inf}_{2n}(T^*_{G_{\mathbb{R}}}X,\mathbb{C})$ such that

$$\theta(\phi) \;=\; \frac{1}{(2\pi i)^n}\int_{\mu_\lambda(C)} \widehat{\phi}\,\sigma_\lambda^n\,.$$

Here $\phi$ is any smooth compactly supported function on $\mathfrak{g}_{\mathbb{R}}$ and $\sigma_\lambda$ is the canonical complex algebraic symplectic form on $\Omega_\lambda$. In other words, we can view the construction of the character as a map

$$\{G_{\mathbb{R}}\text{-representations}\} \longrightarrow \mathrm{H}^{inf}_{2n}(T^*_{G_{\mathbb{R}}}X, \mathbb{C}). \tag{2.2}$$

To understand the map (2.2) geometrically, the right hand side immediately suggests that we should parametrize the $G_{\mathbb{R}}$-representations, by $G_{\mathbb{R}}$-sheaves (rather than by $K$-sheaves). Then, as is shown in [SV2], the map (2.2) coincides with the characteristic cycle construction of Kashiwara

$$\mathrm{CC} : \{G_{\mathbb{R}}\text{-sheaves on } X\} \longrightarrow \mathrm{H}^{inf}_{2n}(T^*_{G_{\mathbb{R}}}X, \mathbb{Z}). \tag{2.3}$$

We discuss this construction briefly in the next section. Note that (2.3) shows, in particular, that the map (2.2) factors through $\mathrm{H}^{inf}_{2n}(T^*_{G_{\mathbb{R}}}X, \mathbb{Z})$. To summarize:

Theorem. *The Lie algebra character of the representation associated to $G_{\mathbb{R}}$-sheaf $\mathcal{F}$ is given by*

$$\theta(\mathcal{F})(\phi) = \frac{1}{(2\pi i)^n} \int_{\mu_\lambda(\mathrm{CC}(\mathcal{F}))} \widehat{\phi}\, \sigma_\lambda^n\,, \qquad (\phi \in C_c^\infty(\mathfrak{g}_{\mathbb{R}}))\,.$$

When $\mathcal{F}$ gives rise to a discrete series representation or, more generally, to a tempered representation, our formula reduces to the original formula of Rossmann: one shows that the cycle $\mu_\lambda(\mathrm{CC}(\mathcal{F}))$ is homologous to the appropriate coadjoint orbit.

*Remark.* As we explained in the first section, we can, completely equivalently, parametrize representations either by $K$-sheaves or by $G_{\mathbb{R}}$-sheaves. The $K$-side seems, at least at the first sight, more appealing and simpler as it allows one to work entirely in the realm of complex algebraic geometry. However, as the theorem above shows, from the point of view of understanding characters of representations the $G_{\mathbb{R}}$-side seems indispensable.

## 4. The Characteristic Cycle Construction.

Let $X$ be a real algebraic manifold of dimension $n$ which we assume, for simplicity, to be oriented. We consider constructible sheaves on $X$, i.e., sheaves of $\mathbb{C}$-vector spaces with the following property: there exists a (semi-)algebraic decomposition of $X$ such that the sheaf restricted to any constituent of the decomposition is constant of finite rank. As before we consider complexes of constructible sheaves and we should be working in the context of derived categories. Given a (complex of) constructible sheaves $\mathcal{F}$ on $X$, Kashiwara in [K1] shows how to associate to it a Lagrangian, $\mathbb{R}^+$-invariant cycle $\mathrm{CC}(\mathcal{F})$ in $T^*X$. Recall that $T^*X$ has a canonical symplectic structure and that conormal bundles of smooth submanifolds are prototypes of Lagrangian, $\mathbb{R}^+$-invariant submanifolds of $T^*X$. The construction of $\mathrm{CC}(\mathcal{F})$ is Morse-theoretic. The cycle $\mathrm{CC}(\mathcal{F})$ measures how the local Euler characteristic (=the Euler characteristic of the stalks) of $\mathcal{F}$ changes as we move to a particular direction from a point on $X$. From this description it is apparent that CC satisfies the following properties:

(a) $\mathrm{CC}(\mathbb{C}_X) = [X]$,
(b) CC is additive in short exact sequences,
(c) CC is locally defined on $X$.

Here the symbol $[X]$ stands for the zero section viewed as a cycle on $T^*X$ with its given orientation. The index theorem of Kashiwara [K1] states that the global Euler characteristic of $\mathcal{F}$ coincides with the intersection product of the zero section $[X]$ and $\mathrm{CC}(\mathcal{F})$. By property a) above, this amounts to a generalization to sheaves of the classical index theorem of Hopf: the Euler characteristic of a compact manifold $X$ is given by the self intersection number of the zero section in $T^*X$. Kashiwara's index theorem can be generalized to the relative case: for a proper map $f : X \to Y$ and a sheaf $\mathcal{F}$ on $X$ we can describe the characteristic cycle of the push-forward of $\mathcal{F}$ in terms of $\mathrm{CC}(\mathcal{F})$ and an intersection product [KSa].

To be able to calculate the the effect of CC under all the operations on sheaves it is necessary and sufficient to have a formula for the characteristic cycle of a pushforward under an open embedding. As this is our most important tool, we will give the statement. To this end, let $j : U \hookrightarrow X$ be an open embedding and let $f$ be a defining equation for the boundary of $U$. Then, for a sheaf $\mathcal{F}$ on $U$, we have

$$\text{(d)} \qquad \mathrm{CC}(Rj_*\mathcal{F}) \;=\; \lim_{s\to 0^+} \left( \mathrm{CC}(\mathcal{F}) + s\frac{df}{f} \right) .$$

This formula is proved in [SV1]. It is modeled after a similar formula proved by Ginzburg in the complex analytic case. The properties (a)-(d) completely determine the operation CC, i.e., they could be taken as axioms. The construction CC amounts to a (weak) but very workable form of microlocalization.

## 5. Nilpotent invariants.

In this section, as an application of our techniques, we will identify two rather different invariants of representations. Both of these invariants involve nilpotent orbits. Invariants that involve nilpotent orbits are particularly interesting because, as was explained in §1, it is generally believed that unitary representations are best parametrized using such data. One of the invariants, due to Vogan, is purely algebraic and the other, due to Barbasch-Vogan [BV], is analytic. The statement that these invariants coincide has become known as the Barbasch-Vogan conjecture.

Let us consider an irreducible representation $V$ of $G_{\mathbb{R}}$. The analytic invariant is defined as follows. Let $\theta$ denote the Lie algebra character of the representation $V$. Take the Fourier transform of the leading term of the asymptotic expansion of $\theta$ at the origin. Barbasch and Vogan show that this Fourier transform is a $\mathbb{C}$-linear combination of canonical measures on nilpotent coadjoint orbits in $i\mathfrak{g}_{\mathbb{R}}{}^*$. In other words, this Fourier transform can be written as

$$\mathrm{WF}(V) \;=\; \sum a_j[\mathcal{O}_j^{\mathbb{R}}]\,,$$

where the $\mathcal{O}_j^{\mathbb{R}}$ are $G_{\mathbb{R}}$-orbits in $i\mathfrak{g}_{\mathbb{R}}^* \cap \mathcal{N}^*$ and $a_j \in \mathbb{C}$. Recall that $\mathcal{N}^*$ denotes the "nilpotent cone" in $\mathfrak{g}^* \cong \mathfrak{g}$. The cycle $\mathrm{WF}(V)$ is called the wave front cycle of $V$.

The algebraic invariant is defined via the Harish-Chandra module $M$ of $V$. We choose a $K$-invariant good filtration $M_j$ of $M$ with respect to the canonical filtration of the universal enveloping algebra $U(\mathfrak{g})$. The associated graded $\operatorname{gr}(M)$ is a module over the symmetric algebra $S(\mathfrak{g})$. As such, it determines a well defined algebraic cycle on $\mathfrak{g}^*$. The support of this cycle coincides with the support of the module $\operatorname{gr}(M)$. Vogan [V] shows that the algebraic cycle is $K$-invariant and is supported on $\mathfrak{p}^* \cap \mathcal{N}^*$. The space $\mathfrak{p}$ is given by the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Hence we have associated to $V$ a cycle

$$\operatorname{Ass}(V) \;=\; \sum b_j [\mathcal{O}_j^K]\,,$$

where the $\mathcal{O}_j^K$ stand for $K$-orbits in $\mathfrak{p}^* \cap \mathcal{N}^*$ and $b_j$ are non-negative integers. The cycle $\operatorname{Ass}(V)$ is called the associated cycle of $V$. In [Se] Sekiguchi constructs a bijection between $G_\mathbb{R}$-orbits on $i\mathfrak{g}_\mathbb{R}^* \cap \mathcal{N}^*$ and $K$-orbits on $\mathfrak{p} \cap \mathcal{N}^*$. The following result is proved in [SV3]:

THEOREM. *The wave front cycle and the associated cycle coincide under the Kostant-Sekiguchi correspondence. In particular, the constants $a_j$ are non-negative integers.*

Let us briefly discuss the general structure of the argument. It can by summarized in the form of the following commutative diagram:

$$(5.1)\qquad \begin{array}{ccc}
\{G_\mathbb{R}\text{-representations}\} & \longrightarrow & \{\text{H-C-modules}\} \\
\wr\big\downarrow & & \big\downarrow\wr \\
\{G_\mathbb{R}\text{-equivariant sheaves on } X\} & \xrightarrow{\ \Gamma\ } & \{K\text{-equivariant sheaves on } X\} \\
\scriptstyle\mathrm{CC}\big\downarrow & & \big\downarrow\scriptstyle\mathrm{CC} \\
\{\text{Lagrangian cycles on } T^*_{G_\mathbb{R}}X\} & \xrightarrow{\ \Psi\ } & \{\text{Lagrangian cycles on } T^*_K X\} \\
\scriptstyle\mu_*\big\downarrow & & \big\downarrow\scriptstyle\mu_* \\
\{G_\mathbb{R}\text{-orbits in } \mathcal{N}^* \cap i\mathfrak{g}_\mathbb{R}^*\} & \xrightarrow{\ \psi\ } & \{K\text{-orbits in } \mathcal{N}^* \cap \mathfrak{p}^*\}\,.
\end{array}$$

The vertical arrows from the top to bottom can be identified with the wave front cycle and the associated cycle constructions, respectively. The crux of the argument is the explicit computation of the map $\Psi$ induced by $\Gamma$. This computation takes us outside of the realm of semi-algebraic and subanalytic sets. We make essential use of the geometric categories of [DM]. In particular, we work in the context of the geometric category associated to the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$. The last step is the identification of the map $\psi$, induced by $\Psi$, with the Kostant-Sekiguchi correspondence.

The fact that the vertical arrows in (5.1) amount to the invariants WF and Ass shows that they can be extracted from the appropriate characteristic cycles. Let us phrase this more precisely. Consider the diagram $X \xleftarrow{\pi} T^*X \xrightarrow{\mu} \mathcal{N}^*$ of spaces where $\pi$ is the projection and $\mu : T^*X \to \mathcal{N}^*$ is the moment map. If $\mathcal{F}$

is a $G_{\mathbb{R}}$-equivariant sheaf on $X$ then the corresponding wave front cycle is given by "microlocalizing" $\mathcal{F}$ via the CC construction to a cycle on $T^*X$ and then integrating this cycle[4] over the fibers of $\mu$. The analogous process on the $K$-side produces the associated cycle (this fact is due to J.-T. Chang). The characteristic cycles carry much more information than the wave front cycle and the associated cycle and it is conceivable that some of this extra information is crucial in understanding the unitary representations attached to nilpotent orbits. Furthermore, the construction CC is a bit too crude at least in one respect. The diagram (5.1) should be at least extended so that the objects in the last two rows are $G_{\mathbb{R}}$ and $K$-equivariant, respectively.

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[4]Before integrating the cycle, we multiply it with the cohomology class $e^{\lambda+\rho}$

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Kari Vilonen
Department of Mathematics
Brandeis University
Waltham, MA 02254
USA
vilonen@math.brandeis.edu