Rectifiability, Analytic Capacity, and Singular Integrals

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Abstract. This is a survey of some interplay between geometric measure theory (rectifiability), complex analysis (analytic capacity) and harmonic analysis (singular integrals). Vaguely, it deals with the following three principles:

1. The analytic capacity of a 1-dimensional compact subset of the complex plane $\mathbf{C}$ is zero if and only if E is purely unrectifiable.
2. The analytic capacity of a 1-dimensional compact subset E of $\mathbf{C}$ is positive if and only if the Cauchy singular integral operator is L^2-bounded on a large part of E.
3. Singular integrals behave nicely on an m-dimensional subset E of $\mathbf{R}^n$ if and only if E is in some sense rectifiable.

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1. Analytic capacity; finite length. First a general remark: since the list of complete references would be very long I have omitted many which the reader can find in [C1], [D1], [G] or [M2]. The analytic capacity of a compact subset E of $\mathbf{C}$ was defined by Ahlfors in 1947 as

$$\gamma(E) = \sup_f \lim_{z\to\infty} |zf(z)|$$

where the supremum is taken over all analytic functions $f\colon \mathbf{C}\setminus E \to \mathbf{C}$ such that $|f(z)| \le 1$ and $f(\infty) = 0$. Ahlfors showed that (see [G]) $\gamma(E) = 0$ if and only if E is removable for bounded analytic functions. That is, whenever U is an open set containing E, any bounded analytic function in $U \setminus E$ has an analytic extension to U. Or equivalently, the only bounded analytic functions in $\mathbf{C}\setminus E$ are constants. This is all very easy (but Ahlfors proved deep results about the existence of an extremal and its properties) and this characterization of removability is quite complex analytic. One would wish to find a geometric characterization. This is often called the Painlevé problem, since Painlevé started to study it about 100 years ago.

There are two very easy results, see [G] or [M2]: If the 1-dimensional Hausdorff measure $\mathcal{H}^1(E) = 0$, then $\gamma(E) = 0$. If the Hausdorff dimension of E $\dim E > 1$, then $\gamma(E) > 0$. Thus the following recent theorem of David [D2] leaves the question open only for sets E with $\mathcal{H}^1(E) = \infty$ and $\dim E = 1$ (but there is quite a variety of them).

1.1. Theorem. *Let $E \subset \mathbf{C}$ be compact with $\mathcal{H}^1(E) < \infty$. Then $\gamma(E) = 0$ if and only if $\mathcal{H}^1(E \cap \Gamma) = 0$ for every rectifiable curve Γ.*

Sets E such that $\mathcal{H}^1(E \cap \Gamma) = 0$ for every rectifiable curve Γ are called purely unrectifiable according to Federer's terminology. Besicovitch studied their properties extensively in the 20's and 30's and called them irregular. They and their rectifiable (regular) counterparts and higher dimensional generalizations are quite basic in geometric measure theory.

I discuss the proof of Theorem 1.1, which also brings forth clearly the role of singular integrals. Suppose first that E is not purely unrectifiable. Then it meets some rectifiable curve in positive length. Some rather easy arguments show that it meets also some Lipschitz graph Γ with small Lipschitz constant in positive length. Calderòn showed in 1977 that the Cauchy singular integral operator C_Γ,

$$C_\Gamma g(z) = \lim_{\varepsilon \to 0} \int_{\Gamma \setminus B(z,\varepsilon)} \frac{g(\zeta)}{\zeta - z} \, d\mathcal{H}^1 \zeta,$$

is bounded in $L^2(\Gamma)$ for such a Γ. (Later Coifman, McIntosh and Meyer showed that this is true for all Lipschitz graphs.) By that time it was already known, see [C1], for example, that then there is some bounded non-negative function h on Γ such that $f = C_\Gamma h$ is bounded in $\mathbf{C} \setminus E$. Thus $\gamma(E) > 0$.

The last step is based on a duality argument using the Hahn–Banach theorem and no constructive method of finding a non-constant bounded analytic function in $\mathbf{C} \setminus E$ is known even if Γ is C^1. If it is $C^{1+\varepsilon}$, then such a method exists, see [G].

Suppose then that $\gamma(E) > 0$. We should find a rectifiable curve Γ such that $\mathcal{H}^1(E \cap \Gamma) > 0$. First, there is a non-constant bounded analytic function f in $\mathbf{C} \setminus E$ vanishing at infinity, and an easy argument, see, e.g., [M2], using the Cauchy integral formula yields a bounded Borel function $\varphi \colon E \to \mathbf{C}$ such that $f = C_E \varphi$.

Let us assume that $\mathcal{H}^1\big(E \cap B(z,r)\big) \le Cr$ for $z \in \mathbf{C}$ and $r > 0$; this is not a really serious restriction. Then it is still easy to see that even the maximal function $C_E^* \varphi$,

$$C_E^* \varphi(z) = \sup_{\varepsilon > 0} \left| \int_{E \setminus B(z,\varepsilon)} \frac{\varphi(\zeta)}{\zeta - z} \, d\mathcal{H}^1 \zeta \right|$$

is bounded in $\mathbf{C}$. Suppose we would be lucky enough to find φ so that it is also non-negative. Set $\mu = \varphi \mathcal{H}^1 | E$. Using Fubini's theorem as Melnikov and Verdera did in [MV] we get for all $\varepsilon > 0$,

$$\begin{aligned}\infty > C &\geq \int \left| \int_{\mathbf{C}\setminus B(z,\varepsilon)} \frac{1}{\zeta - z}\, d\mu\zeta \right|^2 d\mu z \\ &= \iiint\limits_{A_\varepsilon} \frac{1}{(z_1 - z_3)\overline{(z_2 - z_3)}}\, d\mu z_1\, d\mu z_2\, d\mu z_3 + O(1) \\ &= \frac{1}{6} \iiint\limits_{A_\varepsilon} \sum_\sigma \frac{1}{(z_{\sigma(1)} - z_{\sigma(3)})\overline{(z_{\sigma(2)} - z_{\sigma(3)})}}\, d\mu z_1\, d\mu z_2\, d\mu z_3 + O(1).\end{aligned}$$

Here σ runs through all six permutations of $\{1,2,3\}$ and $A_\varepsilon = \{(z_1, z_2, z_3) : |z_i - z_j| > \varepsilon \text{ for } i \neq j\}$. To get that the error term is bounded is an easy estimate using $\mu\big(B(z,r)\big) \leq Cr$ for all z, r. Now a remarkable identity found by Melnikov in [M] says that

$$\sum_\sigma \frac{1}{(z_{\sigma(1)} - z_{\sigma(3)})\overline{(z_{\sigma(2)} - z_{\sigma(3)})}} = c(z_1, z_2, z_3)^2$$

where $c(z_1, z_2, z_3)$ is the reciprocal of the radius of the circle passing through $z_1, z_2, z_3 \in \mathbf{C}$. This is 0 if and only if these points are collinear. The number $c(z_1, z_2, z_3)$ is called the Menger curvature of the triple (z_1, z_2, z_3). Menger introduced it in the early 30's to define the curvature for continua in compact, convex metric spaces, see [K]. Using the above formulas and letting $\varepsilon \to 0$, we obtain

$$c^2(\mu) \equiv \iiint c(x,y,z)^2\, d\mu x\, d\mu y\, d\mu z < \infty.$$

So now we have some geometric information about μ, and looking at it more closely we find that "most" (in μ-sense) triples of points which lie close to each other must be nearly collinear. This gives good hopes for a construction of rectifiable curves which carry positive μ measure.

The following theorem was first proved by David and then Legér [L] gave a different proof which also allows a higher dimensional version. Note that the situation is somewhat similar to that in Jones's traveling salesman result in [J].

1.2. THEOREM. *If $\mu = \varphi\mathcal{H}^1|E$, $\varphi \in L^\infty(E)$, $\varphi \geq 0$, $\mathcal{H}^1(E) < \infty$ and $c^2(\mu) < \infty$, then there are rectifiable curves Γ_i such that*

$$\mu\Big(\mathbf{C} \setminus \bigcup_{i=1}^{\infty} \Gamma_i\Big) = 0.$$

We have then that $\mathcal{H}^1(E \cap \Gamma_i) > 0$ for some i, and so E is not purely unrectifiable.

This would end the proof of Theorem 1.1 except that we have made the unjustified assumption that $\varphi \geq 0$. Note that in the above argument to get $c^2(\mu) < \infty$ we did not need the uniform boundedness of the Cauchy transform; we only needed the boundedness in L^2. Hence Theorem 1.1 follows if we can show:

1.3. Theorem. *If there is a non-zero $\varphi \in L^\infty(E)$ such that $C_E^*\varphi$ is bounded, then there is $F \subset E$ such that $\mathcal{H}^1(F) > 0$ and the truncated operators $C_{F,\varepsilon}$;*

$$C_{F,\varepsilon}g(z) = \int_{F\setminus B(z,\varepsilon)} \frac{g(\zeta)}{\zeta - z}\, d\mathcal{H}^1\zeta,$$

are uniformly bounded in $L^2(\mathcal{H}^1|F)$.

Then we can use the constant function 1 to get $c^2(\mathcal{H}^1|F) < \infty$.

Theorem 1.3 follows from [DM] and [D2]. First φ was transformed to an accretive function ψ (i.e., $\operatorname{Re}\psi \geq \delta > 0$) with L^2-estimates for the Cauchy transform in [DM] with a construction relying on ideas of Christ from [C2], where Theorem 1.3 was proved for AD-regular sets (see Section 3). Then David proved a general $T(b)$-theorem in [D2] yielding Theorem 1.3. A little later Nazarov, Treil and Volberg gave in [NTV3] a different simpler proof for Theorem 1.3 also obtaining a general $T(b)$-theorem.

The problem of removable sets for Lipschitz harmonic functions is very much like that for bounded analytic functions. Theorem 1.1 is valid also in this case, but we don't know if the two classes of removable sets are exactly the same. The reason for this similarity is that rather than studying bounded harmonic functions we are studying harmonic functions with bounded gradient and the gradient of the fundamental solution $c\log|z|$ is essentially the Cauchy kernel.

This problem is interesting also in $\mathbf{R}^n$. There are several partial results, see [MP], but nothing like the analog of Theorem 1.1, even for $(n-1)$-dimensional AD-regular sets. Now the kernel is $|x|^{-n}x$, $x \in \mathbf{R}^n$, but we don't know anything useful to replace Melnikov's identity with.

2. Analytic capacity; infinite length. If $\mathcal{H}^1(E) < \infty$, then by a result of Besicovitch E is purely unrectifiable if and only if

$$\mathcal{H}^1\big(p_\theta(E)\big) = 0 \quad \text{for almost all } \theta \in [0,\pi), \tag{2.1}$$

where p_θ is the orthogonal projection onto the line making angle θ with the real axis. Vitushkin conjectured in the 60's that (2.1) would be equivalent to $\gamma(E) = 0$ for all compact sets $E \subset \mathbf{C}$. Thus Theorem 1.1 says that he was right when $\mathcal{H}^1(E) < \infty$. The general conjecture was shown to be false in [M1] where it was shown that (2.1) is not conformally invariant, but this did not say which of the two implications is false. In [JM] Jones and Murai gave a concrete example where (2.1) holds but $\gamma(E) > 0$. It is not known if $\gamma(E) = 0$ implies (2.1). Now Melnikov has a new conjecture:

2.2. Conjecture. *For any compact $E \subset \mathbf{C}$, $\gamma(E) > 0$ if and only if there is a (non-negative) Radon measure μ on E such that $\mu(E) > 0$, $\mu\big(B(z,r)\big) \leq r$ for all $z \in \mathbf{C}$, $r > 0$, and $c^2(\mu) < \infty$.*

Melnikov proved in [M] that the "if part" of this conjecture is true. In fact, he proved the quantitative estimate

$$\gamma(E) \geq C\frac{\mu(\mathbf{C})^{3/2}}{\big(\mu(\mathbf{C}) + c^2(\mu)\big)^{1/2}},$$

if μ is as in Conjecture 2.2.

Using this result of Melnikov, Joyce and Mörtes have given in [JoM] another example with simpler arguments where (2.1) holds but $\gamma(E) > 0$.

Conjecture 2.2 is not known even for non-degenerate continua; then the analytic capacity is positive. Another test case, which is open, is given by the Cantor sets of Garnett in [G, p. 87]. There is a mistake in [G] and the characterization for $\gamma(E) > 0$ given in Theorem 2.2 is not correct, see [M3] and [E] for this and some related results.

3. Singular integrals on regular sets. A compact subset E of $\mathbf{C}$ is called AD-regular (Ahlfors–David) if there exists a positive number C such that

$$r/C \leq \mathcal{H}^1\big(E \cap B(z,r)\big) \leq Cr \quad \text{for } z \in E,\ 0 < r < 1.$$

The following theorem was proved in [MMV] using the above relations between the Cauchy kernel and Menger curvature. This also gave Theorem 1.1 for AD-regular sets. Some generalizations, but still partial results of Theorem 1.1, were given by Lin [Li] (doubling condition) and Pajot [P] (positive lower density).

3.1. Theorem. *Let $E \subset \mathbf{C}$ be AD-regular. The truncated operators $C_{E,\varepsilon}$ are uniformly bounded in $L^2(\mathcal{H}^1|E)$ if and only if E is uniformly rectifiable, that is, there is an AD-regular curve containing E.*

The uniform L^2-boundedness of $C_{E,\varepsilon}$ is equivalent to the boundedness of the principal value operator C_E, if we know that the principal values exist almost everywhere for a dense set of functions. But we don't know this a priori, hence the above formulation. This is also equivalent to the boundedness of the maximal operator C_E^*.

It is obvious what the AD-regularity means for m-dimensional subsets of $\mathbf{R}^n$; we just replace r by r^m and $\mathcal{H}^1$ by the m-dimensional Hausdorff measure $\mathcal{H}^m$. It is less obvious what the uniform rectifiability should mean if $m > 1$, but David and Semmes have shown that there exist several natural equivalent definitions and they have developed an extensive theory of such sets, see [DS]. They have also studied singular integrals on them and shown that they are bounded in $L^2(\mathcal{H}^m|E)$ for a large class of Calderón–Zygmund kernels. The converse is also valid, i.e., L^2-boundedness implies uniform rectifiability, if one assumes the L^2-boundedness for the operators related to all kernels of the type $\varphi(|x|)\,|x|^{-m-1}x$, $x \in \mathbf{R}^n$, where φ is a smooth non-negative function. However, it is not known if the converse is valid if one only uses one single kernel, for example, the Riesz kernel $K_m(x) = |x|^{-m-1}x$. The problem is again that we don't have anything like the curvature identity. Farag [F] has looked at different ways of forming sums of permutations starting from K_m, but all of them take both positive and negative values and are thus difficult to use.

We can also ask if results like Theorem 3.1 hold for other kernels in the plane. The same method works for the real and imaginary parts of the Cauchy kernel, for example, but I don't know any other essentially different kernel for which this, or some other, method would work. Joyce has looked at the kernels $|z|^{-2k}\,z^{2k-1}$, $k = 1, 2, \ldots$, and again found for the sum of permutations both positive and negative values, when $k > 1$.

One can also study m-regular sets E for non-integral m, but Vihtilä showed in [Vi] that then the singular integral operator related to K_m is never bounded in $L^2(\mathcal{H}^m|E)$.

4. EXISTENCE OF PRINCIPAL VALUES. In the previous section we saw that the L^2-boundedness of the singular integral operators is often equivalent to uniform rectifiability. For non-uniform rectifiability there are characterizations with the existence of principal values. A subset E of $\mathbf{R}^n$ is called m-rectifiable if there are C^1 (or, equivalently, Lipschitz) m-dimensional surfaces S_i such that

$$\mathcal{H}^m\left(E \setminus \bigcup_{i=1}^{\infty} S_i\right) = 0.$$

4.1. THEOREM. *Let $E \subset \mathbf{C}$ be $\mathcal{H}^1$ measurable with $\mathcal{H}^1(E) < \infty$. Then E is 1-rectifiable if and only if*

$$\lim_{\varepsilon \to 0} \int_{E \setminus B(z,\varepsilon)} \frac{1}{\zeta - z}\, d\mathcal{H}^1\zeta$$

exists for $\mathcal{H}^1$ almost all $z \in E$.

The fact that the existence of principal values implies rectifiability was proved by Tolsa in [T3] using results of Nazarov, Treil and Volberg from [NTV3] and the curvature method. Hence this is restricted to the Cauchy kernel and 1-dimensional sets. It is not known if the analogue of Theorem 4.1 holds for m-dimensional sets if $m \geq 2$. The existence of principal values was proved in [MM]. Verdera gave a different proof in [V] which also works in general dimensions (see also [M2]). With an extra condition on positive lower density we have the following, see [MPr] or [M2].

4.2. THEOREM. *Let $E \subset \mathbf{R}^n$ be $\mathcal{H}^m$ measurable with $\mathcal{H}^m(E) < \infty$. Then E is m-rectifiable if and only if for $\mathcal{H}^m$ almost all $x \in E$,*

$$\liminf_{r \to 0} r^{-m}\mathcal{H}^m\big(E \cap B(x,r)\big) > 0$$

and

$$\lim_{\varepsilon \to 0} \int_{E \setminus B(x,\varepsilon)} \frac{x - y}{|x-y|^{m+1}}\, d\mathcal{H}^m y$$

exists.

Huovinen proved in [H] a result analogous to Theorem 4.2 for some other kernels in $\mathbf{C}$.

Tolsa has given in [T2] a complete geometric characterization, involving curvature, of those Radon measures μ on $\mathbf{C}$ for which

$$C_\nu(z) = \lim_{\varepsilon \to 0} \int_{\mathbf{C} \setminus B(z,\varepsilon)} \frac{1}{\zeta - z}\, d\nu\zeta$$

exists for μ almost all z for all Radon measures ν in $\mathbf{C}$.

There is another very nice result in [T2]: if the Cauchy operator $g \mapsto (1/z) * (g d\mu)$ is bounded in $L^2(\mu)$ (meaning again the uniform boundedness of the truncated operators), then the principal values $C_\mu(z)$ exist for μ almost all $z \in \mathbf{C}$. This is again known (essentially) only for the Cauchy kernel, since the proof uses curvature.

5. Calderón–Zygmund theory in non-homogeneous spaces. We have already mentioned several times the works of Nazarov, Treil and Volberg [NTV1–3] and Tolsa [T1–3]. Their starting point was the following question. Let μ be a Radon measure in $\mathbf{R}^n$. If μ is doubling; $\mu\big(B(x,2r)\big) \leq C\mu\big(B(x,r)\big)$ for all $x \in \operatorname{spt}\mu$, $r > 0$ (or, in other words, $(\operatorname{spt}\mu, \mu)$ is a space of homogeneous type), most of the Calderón–Zygmund theory of singular integrals is valid. Surprisingly, the works mentioned above show that almost always the doubling condition is not needed at all. Tolsa uses the curvature method, and this is again restricted to the Cauchy kernel. Nazarov, Treil and Volberg have developed a beautiful method using random lattices of dyadic cubes and showing that with a large probability such a lattice is in a good position in order that useful estimates can be established. This works for general Calderón–Zygmund kernels in $\mathbf{R}^n$. Then one obtains in great generality such basic results as the equivalence of the L^2-boundedness to the L^p-boundedness for $1 < p < \infty$ and to the weak L^1-boundedness, Cotlar's inequality, $T(1)$- and $T(b)$-theorems.

The $T(b)$-theorem for singular integral operators T such as the Cauchy operator says that if there exists $b \in L^\infty(\mu)$ such that $\operatorname{Re} b \geq \delta > 0$ (this can be replaced with weaker conditions) and $T(b) \in \mathrm{BMO}(\mu)$, then T is bounded in $L^2(\mu)$. The first such theorem without any doubling condition was proved by David in [D2]; this was the last missing piece in the proof of Theorem 1.1. There is some difference with David's $T(b)$-theorem and that of Nazarov, Treil and Volberg since David is defining BMO with generalized "dyadic cubes" which depend on the measure μ.

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