Functional Calculus on Lie Groups and Wave Propagation

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1. Introduction

Consider a "sum of squares" operator

$$L = -\sum_{j=1}^{k} X_j^2 + X_{k+1} \tag{1.1}$$

on a smooth manifold M of dimension d, where $X_1, \ldots, X_{k+1}$ are smooth, real vector fields on M satisfying the following bracket condition: $X_1, \ldots, X_{k+1}$, together with the iterated commutators $[X_{j_1}, [X_{j_2}, [\ldots [X_{j_{\ell-1}}, X_{j_\ell}] \ldots]]]$, span the tangent space of M at every point of M. If $k = d$, then L might for example be a Laplace-Beltrami operator. If $k < d$, then L is not elliptic, but, according to a celebrated theorem of L. Hörmander [14], it is still hypoelliptic. Operators of this type arize in various contexts, for instance in higher dimensional complex analysis (see e.g. [32]). Assume in addition that L is essentially selfadjoint on $C_0^\infty(M)$ with respect to some volume element dx. Then the closure of L, again denoted by L, admits a spectral resolution $L = \int_0^\infty \lambda \, dE_\lambda$ on $L^2(M)$, and any function $m \in L^\infty(\mathbb{R}^+)$ gives rise to an L^2-bounded operator

$$m(L) := \int_0^\infty m(\lambda) \, dE_\lambda.$$

An important question is then under which additional conditions on the spectral multiplier m the operator $m(L)$ extends from $L^2 \cap L^p(M)$ to an L^p-bounded operator, for a given $p \neq 2$. If so, m is called an *L^p-multiplier for L*, and we write $m \in \mathcal{M}^p(L)$.

Since, without additional properties of M and L, there is little hope in finding answers to this questions, we shall assume that M is a connected Lie group G, with right-invariant Haar measure dx, and that $X_1, \ldots, X_k$ are left-invariant vector fields which generate the Lie algebra $\mathfrak{g}$ of G. Moreover, for simplicity, we shall assume $X_{k+1} = 0$, so that L is a so-called *sub-Laplacian.* The choice of a right-invariant Haar measure and left-invariant vector fields ensures that the formal transposed ${}^tX$ of $X \in \mathfrak{g}$ is given by $-X$, so that, by a straight-forward extension of a well-known theorem by E. Nelson and F. Stinespring, L is selfadjoint.

The objective of the talk will be to survey some of the relevant developments concerning this question, and moreover to link it to questions concerning estimates for the associated *wave equation*, more precisely the following Cauchy-problem:

$$\begin{aligned} &(\frac{\partial^2}{\partial t^2} + L)\, u(x,t) = 0 \quad \text{on} \quad G \times \mathbb{R}, \\ &u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = 0, \end{aligned} \tag{1.2}$$

whose solution is given by $u(\cdot,t) = \cos(t\sqrt{L})f$.

The classical model case is the Laplacian $L = -\Delta = -\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$ on $\mathbb{R}^d$. Since, on $\mathbb{R}^d$, the spectral decomposition of the Laplacian is induced by the the Fourier transformation, $m(L)$ is here a *Fourier multiplier operator*

$$\widehat{m(L)f}(\xi) = \mu(\xi)\hat{f}(\xi) \ ,$$

with a radial Fourier multiplier $\mu(\xi) = m(|\xi|^2)$, and a sufficient condition for m to be an L^p-multiplier for $-\Delta$ follows from a well-known Fourier multiplier theorem going back to J. Marcinkiewicz, S. Mikhlin and L. Hörmander (see [18][13]):

Fix a non-trivial cut-off function $\chi \in C_0^\infty(\mathbb{R})$ supported in the interval $[1,2]$, and define for $\alpha > 0$

$$\|m\|_{sloc,\alpha} := \sup_{r>0} ||\chi\, m(r\cdot)||_{H^\alpha},$$

where $H^\alpha = H^\alpha(\mathbb{R})$ denotes the Sobolev-space of order α. Thus, $\|m\|_{sloc,\alpha} < \infty$, if m is locally in H^α, uniformly on every scale. Notice also that, up to equivalence, $\|\cdot\|_{sloc,\alpha}$ is independent of the choice of the cut-off function χ.

"MMH-Theorem". *Suppose that $\|m\|_{sloc,\alpha} < \infty$ for some $\alpha > d/2$. Then $m \in \mathcal{M}^p(-\Delta)$ for every $p \in]1,\infty[$. Moreover, $m(-\Delta)$ is of weak-type (1,1).*

This result is sharp with respect to the critical degree of smoothness $d/2$ for the multiplier for $p = 1$; for $1 < p < \infty$, less restrictive conditions follow by suitable interpolation with the trivial L^2-estimate (see [16], [27]).

The proof of this theorem is based on the following weighted L^2- estimate, which can be deduced from Plancherel's theorem: Let $K_m \in \mathcal{S}'(\mathbb{R}^d)$ be such that $\widehat{K_m} = \mu$, so that $m(L)f = f * K_m$. Then, for m supported in $]0,1]$,

$$\int_{\mathbb{R}^d} |(1+|x|)^\alpha\, K_m(x)|^2\, dx \le C||m||^2_{H^\alpha} \ . \tag{1.3}$$

Now, if G is an arbitrary Lie group, then it has been shown by Y. Guivarc'h and J. Jenkins (see e.g. [35]) that G is either of polynomial growth, or of exponential growth, in the following sense: Fix a compact neighborhood U of the identity element e in G. Then G has *polynomial growth*, if there exists a constant $c > 0$ such that $|U^n| \le cn^c$ for every $n \in \mathbb{N}$, where $|A|$ denotes the Haar measure of a Borel subset A of G. In that case, it is known that there is in fact an integer Q and $C > 0$ such that

$$C^{-1} n^Q \le |U^n| \le C n^Q \quad \text{for every } n \ge 1. \tag{1.4}$$

G is said to have *exponential growth,* if

$$|U^n| \geq Ce^{\kappa n} \quad \text{for every } n \geq 1, \tag{1.5}$$

for some $\kappa > 0, C > 0$. In this case, there does in fact also hold a similar estimate from above.

Clearly, Euclidean groups are of polynomial growth, and, more generally, the same is true for nilpotent groups.

From an analytic point of view, there is a strong difference between both types of groups: Whereas groups of polynomial growth are *spaces of homogeneous type* in the sense of R. Coifman and G. Weiss (compare [32]), so that standard methods from the Calderón-Zygmund theory of singular integrals do apply, this is not true of groups of exponential growth. I shall mainly concentrate on groups of polynomial growth, and only briefly report on some phenomena discovered in the recent study of a few examples of groups of exponential growth.

2. Polynomial volume growth

Beginning with some early work by A. Hulanicki and E.M. Stein, various analogous of the MMH-Theorem for groups of polynomial growth have been proved in the course of the past two decades. A main objective of these works by various authors, among them L. De Michele, G. Mauceri, J. Jenkins, M. Christ, S. Meda, A. Sikora and G. Alexopoulos (see e.g. [1], also for further references), was the quest for the sharp critical exponent of smoothness in the corresponding theorems on such groups, which is in fact not concluded yet.

Let us look at the important special case of a *stratified Lie group* G, whose Lie algebra $\mathfrak{g}$ admits a decomposition into subspaces

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_p,$$

such that $[\mathfrak{g}_i, \mathfrak{g}_k] \subset \mathfrak{g}_{i+k}$ for all i, k, and where $\mathfrak{g}_1$ generates $\mathfrak{g}$ as a Lie algebra. We then form L in (1.1) from a basis $X_1, \ldots, X_k$ of $\mathfrak{g}_1$, with $X_{k+1} = 0$. Such a group is clearly nilpotent and admits a one-parameter group of automorphisms $\{\delta_r\}_{r>0}$, called *dilations*, given by $\delta_r\Big|_{\mathfrak{g}_j} := r^j \mathrm{Id}_{\mathfrak{g}_j}$. Then L is homogeneous of degree 2 with respect to these dilations, and the bi-invariant Haar measure transforms under δ_r as follows:

$$\delta_r^*(dx) = r^Q dx, \tag{2.1}$$

where

$$Q := \sum_{j=1}^{p} j \dim \mathfrak{g}_j$$

is the so-called *homogeneous dimension* of G. It agrees with the growth exponent Q in (1.4). Notice that for groups which are nilpotent of step $p > 1$, the homogeneous dimension is greater than the Euclidean dimension $d = \dim G$; only for abelian groups, both are the same. The following theorem is due to M. Christ ([5], see also [17]):

Theorem 1. *If G is a stratified Lie group of homogeneous dimension Q, and if $\|m\|_{sloc,\alpha} < \infty$ for some $\alpha > Q/2$, then $m(L)$ is bounded on $L^p(G)$ for $1 < p < \infty$, and of weak type (1,1).*

If $m \in L^\infty(\mathbb{R}^+)$, then, by left–invariance and the Schwartz kernel theorem, it is easy to see that $m(L)$ is a convolution operator $m(L)f = f * K_m = \int_G f(y)K_m(y^{-1}\cdot)dy$, where a priori the convolution kernel K_m is a tempered distribution. We also write $K_m = m(L)\delta_e$. The main problem in proving Theorem 1 is to draw information on K_m, namely to show that K_m is a Calderón-Zygmund kernel, from relatively abstract information on the multiplier m. This is usually done by appealing to estimates for the *heat kernels* $p_t = e^{-tL}\delta_e,\ t > 0$, by some method of subordination. In fact, through work by D. Jerison and A. Sanchez-Calle for the case of stratified groups, and N. Varopoulos and his collaborators for more general Lie groups [35], one knows that p_t, say on a stratified group, satisfies estimates of the following form:

$$p_t(x) \le C_\varepsilon t^{-Q/2} e^{-\frac{d(x,e)^2}{4(1+\varepsilon)t}}\ , \tag{2.2}$$

which are essentially optimal. Here, d denotes the so-called *optimal control* or *Carnot–Carathéodory distance* associated to the Hörmander system of vector fields $X_1, \dots, X_k$ (see [35]), defined as follows: An absolutely continuous path $\gamma : [0,1] \to G$ is called *admissible*, if

$$\dot\gamma(t) = \sum_{j=1}^k a_j(t) X_j(\gamma(t)) \ \text{ for a.e. } \ t \in [0,1].$$

The *length* of γ is then given by $|\gamma| := \int_0^1 \left(\sum_j a_j(t)^2\right)^{1/2} dt$, and the associated distance function is defined by $d(x,y) := \inf\{|\gamma| : \gamma \text{ is admissible, and } \gamma(0) = x, \gamma(1) = y\}$, where $\inf \emptyset := \infty$. Hörmander's bracket condition ensures that $d(x,y) < \infty$ for every $x, y \in G$.

Observe that in (2.1) and (2.2), in comparizon to the classical case $G = \mathbb{R}^d$, the homogeneous dimension Q takes over the role of the Euclidian dimension d. Because of this fact, which is an outgrowth of the homogeneity of L with respect to the automorphic dilations, the condition $\alpha > Q/2$ in Theorem 1 appeared natural and was expected to be sharp. The following result, which was found in joint work with E.M. Stein [25], and independently also by W. Hebisch [11], came therefore as a surprise:

Fix $n \in \mathbb{N}$, and let $\mathbb{H}_n$ denote the *Heisenberg group* of Euclidean dimension $d = 2n+1$, for which the group law, expressed in coordinates $(x,y,u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, is

$$(x,y,u)\cdot(x',y',u') = (x + x',\ y + y',\ u + u' + \tfrac{1}{2}(x\cdot y' - x'\cdot y)),$$

where $x \cdot y$ denotes the Euclidean inner product. A basis of the Lie algebra of $\mathbb{H}_n$ is then given by the left-invariant vector fields

$$U = \frac{\partial}{\partial u}, \quad X_j = \frac{\partial}{\partial x_j} - \tfrac{1}{2} y_j \frac{\partial}{\partial u}, \quad Y_j = \frac{\partial}{\partial y_j} + \tfrac{1}{2} x_j \frac{\partial}{\partial u}, \quad j = 1, \dots, n.$$

The only nontrivial commutation relations among these elements are the Heisenberg relations $[X_j, Y_j] = U, \quad j = 1, \dots, n$. The corresponding sub-Laplacian $L := -\sum_{j=1}^{n} (X_j^2 + Y_j^2)$ is then homogeneous with respect to the automorphic dilations $\delta_r(x, u) := (rx, r^2 u)$, and the homogeneous dimension is $Q = 2n + 2$.

THEOREM 2. *For the sub-Laplacian L on $\mathbb{H}_n$, the statement in Theorem 1 remains valid under the weaker condition $\alpha > d/2$ instead of $\alpha > Q/2$.*

Even though the proofs in [11] and [25] are somewhat different in nature, both draw heavily on the fact that the Heisenberg group has a large group of symmetries. The approach in [25] rests on the following estimate which, surprisingly, is better then what the Euclidean analogue (1.3) would predict:

Let $m \in H^{3/2}$ be supported in the interval $]0, 1]$. Moreover let a "homogeneous norm" on $\mathbb{H}_n$ be given by $|(x, y, u)| := (|x|^4 + |y|^4 + u^2)^{1/4}$, so that in particular $|\delta_r g| = r|g|$ for every $g \in \mathbb{H}_n$ and $r > 0$. Then

$$\int_{\mathbb{H}_n} |(1 + |g|)^2 K_m(g)|^2 \, dg \le C ||m||^2_{H^{3/2}}.$$

For extensions of these results to groups of "Heisenberg type" and "Marcinkiewicz-type" multiplier theorems, see [21],[22].

It is an open questions whether Theorem 1 does hold under the weaker condition $\alpha > d/2$ for arbitrary groups of polynomial growth.

3. SUBORDINATION UNDER THE WAVE EQUATION AND THE CASE OF THE HEISENBERG GROUP

It does not seem possible to derive Theorem 2 from estimates for heat kernels alone. Some approaches to multiplier theorems on polynomially growing groups also make use of information on the associated wave equation (1.2), namely the finite propagation speed for these waves (see.eg. [1]), an idea apparently going back to M. Taylor (see e.g. [34]). However, also these approaches do not yield the sharp result in Theorem 2.

In this section we shall show how, on the other hand, stronger information on wave propagation, namely sharp Sobolev estimates for solutions to (1.2), might in fact lead to sharp multiplier theorems for such groups. For the case of the Heisenberg group, such estimates have been established very recently in joint work with E.M. Stein [26].

Consider the Cauchy problem (1.2). It is natural here to introduce Sobolev norms of the form

$$||f||_{L^p_\alpha} := ||(1 + L)^{\alpha/2} f||_{L^p}.$$

Estimates for $u(\cdot,t)$, for fixed time t, in terms of Sobolev norms of the initial datum f, then reduce essentially to corresponding estimates for the operator $e^{it\sqrt{L}}$. For $p=2$, this operator is unitary, hence bounded on $L^2(G)$, but for $p\neq 2$ this operator will lead to some loss of regularity.

For the classical case of the Laplacian on $\mathbb{R}^d$, such estimates have been established by A. Miyachi [19] and J. Peral [28]. Extensions to the setting of Fourier integral operators, and in particular to elliptic Laplacians, have been given by A. Seeger, C. Sogge and E.M. Stein [29].

However, for non-elliptic sub-Laplacians, the methods in the latter article, which rely on a representation of $e^{it\sqrt{L}}$ as a Fourier integral operator, break down – already the first step, namely to identify $\sqrt{L}$ as a pseudodifferential operator in a "good" symbol class, fails.

Nevertheless, making use of the detailed representation theory of $\mathbb{H}_n$, and in particular of some explicit formulas for certain projection operators due to R. Strichartz, the following analogue of Miyachi-Peral's result has been proved in [26]:

THEOREM 3. *Let L denote the sub-Laplacian on the Heisenberg group, and let $p\in[1,\infty]$. Then, for $\alpha>(d-1)\left|\frac{1}{p}-\frac{1}{2}\right|$, one has*

$$||e^{i\sqrt{L}}f||_{L^p}\leq C_{p,\alpha}||(1+L)^{\alpha/2}f||_{L^p}.$$

By a simple scaling argument, based on the homogeneity of L, one obtains the following estimate for arbitrary time t (we concentrate here on the most important case $p=1$):

$$||e^{it\sqrt{L}}f||_{L^1}\leq C(1+|t|)^{\alpha}||(1+L)^{\alpha/2}f||_{L^1},\quad \text{if } \alpha>(d-1)/2. \tag{3.1}$$

The multiplier theorem in Theorem 2 can be deduced from this result by means of the following principle, respectively variants of it:

SUBORDINATION PRINCIPLE. *Assume that L is a sub-Laplacian on a Lie group G satisfying (3.1) for some $\alpha>0$ and every $t\in\mathbb{R}$. Let $\beta>\alpha+1/2$. Then there is a constant $C>0$, such that for any multiplier $\varphi\in H^\beta(\mathbb{R})$ supported in $[1,2]$, the corresponding convolution kernel K_φ is in $L^1(G)$, and*

$$||K_\varphi||_{L^1(G)}\leq C||\varphi||_{H^\beta(\mathbb{R})}.$$

Proof. Observe first that $(1+L)^{-\varepsilon}\delta_e\in L^1(G)$ for any $\varepsilon>0$. This follows from the formula

$$(1+L)^{-\varepsilon}\delta_e=\frac{1}{\Gamma(\varepsilon)}\int_0^\infty t^{\varepsilon-1}e^{-t(1+L)}\delta_e\,dt=\frac{1}{\Gamma(\varepsilon)}\int_0^\infty t^{\varepsilon-1}e^{-t}p_t\,dt$$

and the fact that the heat kernel p_t is a probability measure on G.

Write

$$\varphi(\lambda)=\psi(\lambda)(1+\lambda^2)^{-\gamma},\quad \text{with } \gamma>\alpha/2,$$

and put $k := (1+L)^{-\gamma}\delta_e \in L^1(G)$.

Then $||\psi||_{H^\beta} \simeq ||\varphi||_{H^\beta}$, and

$$K_\varphi = \psi(\sqrt{L})((1+L)^{-\gamma}\delta_e) = \psi(\sqrt{L})k = \int_{-\infty}^{\infty} \hat{\psi}(t)e^{it\sqrt{L}}k\ dt.$$

Estimate (3.1) then implies

$$||K_\varphi||_{L^1} \leq \int_{-\infty}^{\infty} |\hat{\psi}(t)|(1+|t|)^\alpha||(1+L)^{\alpha/2}k||_{L^1}dt.$$

But, $(1+L)^{\alpha/2}k = (1+L)^{\alpha/2-\gamma}\delta_e \in L^1$, hence, by Hölder's inequality and Plancherel's theorem,

$$||K_\varphi||_{L^1} \leq C(\int_{-\infty}^{\infty} |\hat{\psi}(t)(1+|t|)^\beta|^2dt)^{1/2} \cdot (\int_{-\infty}^{\infty}(1+|t|)^{2(\alpha-\beta)}dt)^{1/2} \leq C'||\psi||_{H^\beta}.$$

Q.E.D.

In case of the Heisenberg group, it suffices to choose $\beta > \frac{d-1}{2} + \frac{1}{2} = \frac{d}{2}$ in this subordination principle. This is just the required regularity of the multiplier in Theorem 2, and one can in fact deduce Theorem 2 from Theorem 3 by a refinement of the above subordination principle and standard arguments from Calderón-Zygmund theory.

In view of the above considerations, it would be desirable to extend Theorem 3 to larger classes of polynomially growing groups. I do have some hope that such extensions may be achievable by linking the estimates more directly to the underlying geometry through methods from geometrical optics.

4. Exponential volume growth

Comparatively little is yet known in the case of groups with exponential volume growth, even if one deals with full Laplacians.

There are basically two, partially complementary, multiplier theorems of general nature available in this context, both requiring the multiplier to be holomorphic in some neighborhood of the L^2-spectrum of L for $p \neq 2$.

The first, applying to multipliers of so-called Laplace transform type, is due to E. M. Stein [31] and is based on the theory of heat diffusion semigroups and Littlewood-Paley-Stein theory. The second, initiated by M. Taylor (see e.g. [34]), applies to Laplace-Beltrami operators on Riemannian manifolds with "bounded geometry" and lower bound on the Ricci curvature, and makes use of the finite propagation speed of waves on these manifolds.

Let us say that a sub-Laplacian L is of *holomorphic L^p-type*, if there exist a point λ_0 in the L^2-spectrum and an open neighborhood U of λ_0 in $\mathbb{C}$, such that every multiplier $m \in M^p(L)$ extends holomorphically to U.

It is well-known that Riemannian symmetric spaces of the non-compact type are of holomorphic L^p-type for $p \neq 2$, see e.g. [7], [34].

In contrast, we say that L admits a *differentiable L^p-functional calculus*, if there is some integer $k \in \mathbb{N}$ such that $C_0^k(\mathbb{R}_+) \subset \mathcal{M}^p(L)$.

In 1991 W. Hebisch [10] showed that certain distinguished Laplacians L on a particular class of solvable Lie groups G with exponential volume growth, namely the "Iwasawa AN components" of complex semisimple Lie groups, do admit a differentiable L^1- functional calculus, and not only this: m lies in $\mathcal{M}^1(L)$ if and only if $m \in \mathcal{M}^1(-\Delta)$, where Δ denotes the Laplacian on the Euclidian space of the same dimension as G. For variants and extensions of these results, see e.g. [9], [12].

This surprising result does, however, not extend to arbitrary solvable Lie groups, as has recently been shown in joint work with M. Christ [6]. Consider the following group G_1, whose Lie algebra $\mathfrak{g}_1$ has a basis T, X, Y, U such that the only non-trivial commutation relations are

$$[T, X] = X, \ [T, Y] = -Y, \ [X, Y] = U,$$

and the associated Laplacian $L = -(T^2 + X^2 + Y^2 + U^2)$.

G_1 is a semidirect product of the Heisenberg group $\mathbb{H}_1$ with $\mathbb{R}$ (analogues do exist also for higher dimensional Heisenberg groups). Then L is of holomorphic L^p-type for every $p \neq 2$.

As has been proved by H. Leptin and D. Poguntke [15], G_1 is in fact the lowest dimensional solvable Lie group whose group algebra $L^1(G)$ is non-symmetric, and the existence of differentiable L^p- functional calculi for Laplacians on Lie groups seems to be related to the symmetry of the corresponding group algebras.

The few results known so far raise two major questions: Suppose G is a, say, solvable Lie group of exponential growth, and let L be a sub-Laplacian on G. Under which conditions is L of holomorphic L^p-type for $p \neq 2$, respectively, when does it admit differentiable L^p- functional calculi? In the latter case, do theorems of MMH-type hold? The last question would require a good understanding of the integral kernels $m(L)\delta_e$ "at infinity", and is still completely open.

5. Local smoothing for the wave equation

Let us turn back to the Laplacian $L = -\Delta$ on $\mathbb{R}^d$. Then $m(L)$ corresponds to the *radial* Fourier multiplier $\xi \mapsto m(|\xi|^2)$. For such radial multipliers and $1 < p < \infty$, $p \neq 2$, better L^p-estimates can be proved than those obtained from interpolating the MMH-estimate with the trivial L^2-estimate, by making use of the curvature of the sphere $|\xi| = 1$. Let us look at the important model case of the *Bochner-Riesz multipliers*

$$m_\alpha(\lambda) := (1 - \lambda)_+^\alpha.$$

By interpolation, the MMH-Theorem implies

$$m_\alpha \in \mathcal{M}^p(-\Delta), \text{ if } \alpha > (d-1)\,|\frac{1}{p} - \frac{1}{2}| \ . \tag{5.1}$$

However, the famous *Bochner-Riesz-conjecture* states that

$$m_\alpha \in \mathcal{M}^p(-\Delta), \text{ if } \alpha > \max\,(d\,|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}, 0). \tag{5.1}$$

Put $p_d := \frac{2d}{d-1}$. The conjecture reduces to proving that $m_\alpha \in \mathcal{M}^{p_d}(-\Delta)$ for every $\alpha > 0$, whereas (5.1) requires $\alpha > \frac{d-1}{2d}$ for the critical exponent $p = p_d$.

In two dimensions, the conjecture has been proved by L. Carleson and P. Sjölin [3] by means of a more general theorem on oscillatory integral operators (for a variant of their proof by L.Hörmander, and another approach due to C. Fefferman and A. Cordoba, see e.g. [32]). In higher dimensions, only partial results are known hitherto, see e.g. [2], [36].

The Bochner-Riesz-conjecture is again linked to the wave equation via the stronger *local smoothing conjecture*, due to C. Sogge, according to which the solution $u(x,t)$ to the Cauchy problem (1.2) for the wave equation satisfies space-time estimates

$$||u||_{L^{p_d}(\mathbb{R}^d\times[1,2])} \le C_\varepsilon ||(1-\Delta)^\varepsilon f||_{L^{p_d}(\mathbb{R}^d)}, \tag{5.3}$$

for all $\varepsilon > 0$, again with $p_d = 2d/(d-1)$.

This conjecture is still open even in two dimensions; for interesting partial results, see e.g. [30], [20].

It is known that the validity of the local smoothing conjecture would imply that of other outstanding conjectures in Fourier analysis, like the "restriction conjecture" or the "Kakeya conjecture".

A common theme underlying all these conjectures is the interplay between curvature properties (here, the curvature of the Euclidian sphere) and Fourier analysis. For a comprehensive account of the state of these conjectures and the correlations between them, see [33].

Let me finish by describing a recent joint result with A. Seeger [24] concerning the local smoothing conjecture.

Introduce polar coordinates $x = r\theta,\ r > 0,\ \theta \in S^{d-1}$, where S^{d-1} denotes the unit sphere in $\mathbb{R}^d$. Correspondingly, define mixed norms

$$||f||_{L^p(\mathbb{R}_+,L^2(S^{d-1}))} := \left(\int_0^\infty (\int_{S^{d-1}} |f(r\theta)|^2 d\theta\right)^{p/2} r^{d-1}dr)^{1/p}.$$

Moreover, denote by L_θ the Laplace-Beltrami-operator on the sphere S^{d-1}.

Theorem 4. *If u is the solution of the Cauchy problem (1.2), and if $2 \le p < p_d$, then*

$$||u||_{L^p(\mathbb{R}_+\times[1,2],L^2(S^{d-1}))} \le C_{p,\varepsilon}\, ||(1-L_\theta)^\varepsilon f||_{L^p(\mathbb{R}_+,L^2(S^{d-1}))},$$

for all $\varepsilon > 0$.

The mixed norm of u in this estimate has to be taken with respect to the variables $(r,t;\theta)$. Slightly sharper endpoint estimates will be contained in [24]. For the case of radial initial data f, endpoint results for $p = p_d$ had been obtained before in [23], [8].

The proof of Theorem 4 makes use of the development of $f(r\theta)$ with respect to θ into spherical harmonics and the corresponding Plancherel theorem on the sphere, and some explicit formulas for the integral kernel of $\cos t\sqrt{-\Delta}$ in polar coordinates, obtainable through the Hankel inversion formula (see [4]). The integral kernel is decomposed into suitable dyadic pieces which, after applying suitable coordinate changes and re-scalings, finally can be estimated by means of the following vector-valued variant of Carleson-Sjölin's theorem for oscillatory integral operators [24], whose proof does not simply follow from an extension of one of the existing proofs for the scalar valued case:

Vector-valued Carleson-Sjölin theorem. *Let $\phi \in C^\infty(\mathbb{R}^2 \times \mathbb{R})$ be a smooth, real phase function and $a \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R})$ be a compactly supported amplitude. Consider the oscillatory integral operator T_λ given by*

$$T_\lambda f(z) := \int e^{i\lambda\phi(z,y)} a(z,y) f(y)\, dy.$$

Suppose that the Carleson-Sjölin determinant $\det \begin{pmatrix} \phi''_{z_1 y} & \phi''_{z_2 y} \\ \phi'''_{z_1 yy} & \phi'''_{z_2 yy} \end{pmatrix}$ *does not vanish on the support of a (it is in this condition where some curvature condition is hidden). Assume that $2 \le p \le 4$, and put*

$$w_p(\lambda) := \tfrac{(\log(2+\lambda))^{1/2-1/p}}{(1+\lambda)^{1/2}}.$$

Then

$$\Big\| \big(\sum_j |T_{\lambda_j} f_j|^2 \big)^{1/2} \Big\|_{L^p(\mathbb{R}^2)} \le C \, \Big\| \Big(\sum_j w_p^2(|\lambda_j|) \left| f_j \right|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R})}$$

for every sequence of functions $f_j \in L^p(\mathbb{R})$ and every sequence of real numbers λ_j.

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