Unexpected Solutions of First and Second Order Partial Differential Equations

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Abstract. This note discusses a general approach to construct Lipschitz solutions of $Du \in K$, where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ and where K is a given set of $m \times n$ matrices. The approach is an extension of Gromov's method of convex integration. One application concerns variational problems that arise in models of microstructure in solid-solid phase transitions. Another application is the systematic construction of singular solutions of elliptic systems. In particular, there exists a 2×2 (variational) second order strongly elliptic system $\operatorname{div} \sigma(Du) = 0$ that admits a Lipschitz solution which is nowhere C^1.

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1 Introduction and examples

In this note we discuss a general method to construct solutions to a large class of nonlinear first and second order partial differential equations. The method makes strong use of work by Gromov (who substantially generalized earlier results of Nash and Kuiper) and is especially suitable for nonconvex problems where standard compactness arguments fail. One application concerns the (unexpected) existence of solutions in mathematical models of solid-solid phase transformations (see Example c) below). Another application is the recent resolution of the regularity question for weak solutions to the Euler-Lagrange equations of multiple integrals.

Theorem 1.1 *There exists a smooth, strongly elliptic* 2×2 *system*

$$-\operatorname{div} \sigma(Dv) = 0, \quad v : \mathbb{R}^2 \to \mathbb{R}^2 \tag{1.1}$$

that admits

(i) nontrivial Lipschitz solutions with compact support;

(ii) Lipschitz solutions that are nowhere C^1.

Moreover σ *can be chosen such that (1.1) is the Euler-Lagrange equation of a variational integral* $\int f(Dv)\,dx$, *where* f *is smooth and uniformly quasiconvex in the sense of Morrey.*

The existence of irregular solutions of the Euler-Langrange equations (1.1) is in sharp contrast with the (partial) regularity theory for minimizers of quasiconvex integrals developed by Evans [Ev 86], Acerbi-Fusco, Giaquinta-Modica, Fusco-Hutchinson and many others (due to space constraints I keep references to the minimum; the slightly enlarged version [MS 98] contains a more detailed list of references).

This raises the question which structure conditions on σ are needed to ensure a good regularity theory for quasilinear systems (Tartar has raised this issue in connection with the closely related question of compactness and stability of solutions, see e.g. [Ta 79], p. 160) For a scalar equation the De Giorgi-Moser-Nash Theorem shows that ellipticity is the natural condition. For systems, there is a large literature for monotone σ (see [Gi 83] for a summary, further references and a sketch of the history) and many results can be extended to so-called quasimonotone σ, but these conditions are too restrictive for applications e.g. to nonlinear elasticity. Similar issues arise for problems in nondivergence form as recent counterexamples by Nadirashvili ([Na 97]) show. In the theory of harmonic maps there are also striking differences between minimizers, weak solutions of the Euler-Langrange equations and the intermediate class of so-called stationary harmonic maps (see [He 97], [Si 96] for recent overviews).

We remark in passing that our counterexamples to regularity are quite different from the famous examples of De Giorgi, Bombieri-De Giorgi-Giusti and many subsequent works. The latter are based on finding equations that admit certain point singularities like $x/|x|$ (or certain cones), while our approach uses the fact that the equation is compatible with certain large oscillations of Du (small oscillations must be smooth by ellipticity). The construction of counterexamples is thus reduced to certain algebraic calculations in the space of matrices (see Section 4 below). Here our point of view is strongly influenced by Tartar's work [Ta 79], [Ta 98].

Before we return to the case of 2×2 systems let us review the general setting and some illustrative examples. Given a subset of the $m \times n$ matrices $M^{m \times n}$, a (bounded) domain $\Omega \subset \mathbb{R}^n$ and a map $u_0 : \Omega \to \mathbb{R}^m$ we seek to find Lipschitz maps $u : \Omega \to \mathbb{R}^m$ that satisfy

$$\begin{aligned} Du(x) &\in K \quad \text{for a.e.} \quad x \in \Omega, && (1.2) \\ u &= u_0 \quad \text{on} \quad \partial\Omega. && (1.3) \end{aligned}$$

Generalizations to problems of the form $F(x, u(x), Du(x)) = 0$ a.e., to maps between manifolds and to higher order derivatives are possible. In order to avoid technicalities as much as possible I focus on (1.2) and (1.3) in the following. This setting already includes a number of interesting examples.

Example a) (Scalar u, Hamilton-Jacobi equations) Let $m = 1$. It follows from Theorem 2.4 below that (1.2), (1.3) has a solution if u_0 is C^1 (or piecewise C^1 and continuous) and

$$Du_0 \in \text{int conv} K.$$

For affine u_0 the condition $Du_0 \in \overline{\text{conv}}K$ is clearly necessary. On the other hand, the examples $K = \{a, b\}$ or $K = \{(-1,-1),(-1,1),(1,-1),(1,1)\}$ show that the condition $Du_0 \in \overline{\text{conv}}K$ is in general not sufficient, even if u_0 is affine. If K is the level set of a convex coercive function there is also a good theory of viscosity solutions as developed by Kružkov, Crandall-Lions and many others. In general, Theorem 2.4 below yields existence of solutions in many cases where no viscosity solution exists, but the solutions have much weaker properties (no uniqueness, no comparison principle), a more detailed comparision appears in recent work of Cardaliaguet-Dacorogna-Gangbo-Georgy.

b) (Isometries) If $K = O(n)$ or

$$K = O(n, m) = \{F \in M^{m\times n} :\ F^T F = id_{\mathbb{R}^n}\}$$

then (1.2), (1.3) admit a solution if u_0 is a 'short' map, i.e.

$$Du_0 \in \text{int conv}K = \{F \in M^{m\times n} : \lambda_{max}(F^T F) < 1\},$$

see [Gr 86], Chapter 2.4.11, p. 216. In fact for $m > n$ (and $u_0 \in C^1$) one can obtain C^1 solutions, see [Gr 86], Chapter 2.4.9, Thm. (A), p. 203 .

c) (Two-well problem) In the study of phase transitions in crystals ([BJ 87], [CK 88]; see [Mu 98] for further references) the set

$$K = SO(2)A \cup SO(2)B \subset M^{2\times 2},$$

with A, B symmetric, positive definite, $\det A = \det B = 1$ arises. Theorem 3.2. below shows that solutions exist if $u_0 \in C^{1,\alpha}$(for $\quad 0 < \alpha < 1$) and

$$Du_0 \in \text{int}\,\overline{\text{conv}}K \cap \{\det = 1\}.$$

d) $m \times 2$ (Elliptic systems) Let $\sigma : M^{m\times 2} \to M^{m\times 2}$ be a C^1 map and consider the second order system

$$-\text{div}\,\sigma(Dv) = 0 \quad \text{in}\,\Omega, \tag{1.4}$$

i.e. $-\sum_{\alpha=1}^{2} \partial_\alpha \sigma_{i\alpha}(Dv) = 0$, for $i = 1, \cdots m$. If Ω is simply connected then (1.4) can be expressed as

$$\sigma(Dv)J = Dw, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and if we let $u = \begin{pmatrix} v \\ w \end{pmatrix}$, then (1.4) can be rewritten as

$$Du \in K, \quad K = \left\{ \begin{pmatrix} F \\ G \end{pmatrix} \in M^{2m\times 2} : \sigma(F)J = G \right\}.$$

e) (Four-point configuration). The following example played an important rôle in clarifying different convexity notions in the calculus of variations and was discovered independently (in different contexts) by several authors ([AH 86], [CT 93], [Ta 93]). It will be crucial in the construction of nontrivial solutions to 2×2 elliptic systems. Let (see Figure 1 in section 4)

$$K = \{A_1, A_2, A_3, A_4\},\ -A_1 = A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix},\ -A_2 = A_4 = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}.$$

One easily checks that all solutions of $Du \in K$ are trivial. Corollary 4.1 below shows that there is a large number of nontrivial maps whose gradient stays in an arbitrarily small neighbourhood of K.

2 Convex integration

The first striking results on solutions of relations like (1.2) appeared in the fundamental work of Nash [Na 54] and Kuiper [Ku 55] on isometric immersions. Specifically, Kuiper showed that for any $\varepsilon > 0$ there exist an isometric C^1 immersion $u : S^2 \to \mathbb{R}^3$ that maps S^2 in a ball of radius ε, while a classical theorem of Hilbert states that C^2 isometric immersions are rigid motions (Borisov studied rigidity and non-rigidity in $C^{1,\alpha}$). Extending these ideas Gromov [Gr 86] developed a general method, called 'convex integration' to treat (1.2) and much more general partial differential relations (Spring's recent book [Sp 98] gives a detailed exposition). The main emphasis in [Gr 86] is on the construction of C^1 solutions. In the context of equidimensional isometric immersions Gromov also studies the Lipschitz case in detail and later states a general result for Lipschitz solutions, see Chapter 2.4.11, p. 218. The setting is that of jet bundles and thus the result covers in particular systems of the form $F(x, u(x), \ldots, D^{(m)}u(x)) = 0$ a.e. in Ω subject to $D^{(l)}u = v^{(l)}$ on $\partial\Omega, 0 \le l \le m-1$.

A short self-contained proof for the special case (1.2), (1.3) appeared in [MS 96]. Following work of Cellina on ordinary differential inclusions, a slightly different approach based on Baire's theorem was pursued in a series of papers by Dacorogna and Marcellini, beginning with [DM 97], [DM 98]. As we shall see, Gromov's setting (or that of Dacorogna and Marcellini) suffices to discuss Examples a) and b), while for c)–e) additional ideas are needed.

The basic idea of convex integration is that nontrivial solutions of (1.2), (1.3) exist if Du_0 takes values in (the interior of) a suitable convex hull, called the P-convex hull. For sets $K \subset M^{m\times n}$ the notion of P-convexity reduces to what is called lamination convexity in [MS 96] ([MP 98] use the term set-theoretic rank-1 convexity). A set $E \subset M^{m\times n}$ is lamination convex if for all matrices $A, B \in E$ that satisfy $\mathrm{rk}(B - A) = 1$, the whole segment $[A, B]$ is contained in E. The lamination convex hull E^{lc} is the smallest lamination convex set containing E. The relevance of rank-1 matrices stems from the fact that they arise exactly as gradients of maps $x \mapsto u(x \cdot n)$ which only depends on one variable. These maps (and slight modifications thereof; see Lemma 2.2 below) are the basic building blocks in Gromov's construction. In the scalar case $m = 1$ lamination convexity of course reduces to ordinary convexity.

The construction of solutions now proceeds in two steps. First one considers open sets $U \subset M^{m\times n}$, and this case is easily reduced to an open neighbourhood of two matrices A, B with $\operatorname{rk}(B-A) = 1$. Secondly one passes to general sets K by approximating them from the inside by open sets contained in K^{lc}. In the following we say that a map $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is piecewise linear if it is continuous, if there exist finitely or countably many disjoint sets Ω_i whose union has full measure such that $u_{|\Omega_i}$ is affine and if Du is (essentially) bounded.

LEMMA 2.1 *Suppose that $U \subset M^{m\times n}$ is open and bounded and that $u_0 : \Omega \to \mathbb{R}^m$ is piecewise linear and satisfies*

$$Du_0 \in U^{lc} \ \ a.e. \tag{2.1}$$

Then there exists, for any $\delta > 0$, a piecewise linear u such that

$$Du \in U \ \ a.e., \quad u = u_0 \ \ on\ \partial\Omega\,, \tag{2.2}$$

$$\sup|u - u_0| < \delta\,. \tag{2.3}$$

For the proof it clearly suffices to consider the case $u_0(x) = Fx$. Using the fact that U^{lc} can be defined inductively by successive addition of rank-1 segments one easily reduces the proof of Lemma 2.1 to the following special case (see [MS 96]).

LEMMA 2.2 *Suppose that $\operatorname{rk}(B-A) = 1$, i.e. $B - A = a \otimes n$, and let $F = \lambda A + (1-\lambda)B$. If U is an open neighbourhood of $\{A, B\}$ then there exists a piecewise linear u such that*

$$Du \in U \ \ a.e., \quad u = Fx \ \ on\ \partial\Omega\,, \tag{2.4}$$

$$\sup|u(x) - Fx| < \delta\,. \tag{2.5}$$

To construct u, assume without loss of generality $F = 0, n = e_n$ and consider the set $M = (-1,1)^{n-1} \times \varepsilon(-\lambda, 1-\lambda)$ and the one-dimensional map $v(x) = ah(x_n)$, where $h'(x_n) = 1-\lambda$ for $x_n < 0$, $h'(x_n) = -\lambda$ for $x_n > 0, h(0) = \varepsilon\lambda(1-\lambda)$. Then $Dv \in \{A, B\}$ and $h > 0$ in A. The function $u(x) = ag(x)$, with $g(x) = h(x) - \varepsilon\sum_{i=1}^{n-1}|x_i|$ has the desired properties on the diamond-shaped set $\tilde{M} = M \cap \{g > 0\}$. For general sets $\Omega \subset \mathbb{R}^n$ one can use Vitali's theorem to exhaust Ω by disjoint scaled copies of $\tilde{M}$. Choosing the scaled copies sufficiently small one obtains (2.5).

This finishes the argument for open sets U. For general sets K one needs an suitable approximation by open sets.

DEFINITION 2.3 (GROMOV) *A sequence of open sets $U_i \subset M^{m\times n}$ is an in-approximation of a set $E \subset M^{m\times n}$ if*

(i) the U_i are uniformly bounded;

(ii) $U_i \subset U_{i+1}^{lc}$;

(iii) $U_i \to E$ in the following sense: if $F_i \in U_i$ and $F_i \to F$ then $F \in E$.

Example For $m = 1$ the shells $U_i = \{x : 1 - 2^{-i+2} < |x| < 1\}$ are an in-approximation of S^{n-1} and $U_1 = B^n = \text{int conv} S^{n-1}$.

Theorem 2.4 ([Gr 86, p. 218; [MS 96]) *Suppose that $\{U_i\}$ is an in-approximation of $K \subset M^{m\times n}$ and that $u_0 : \Omega \to \mathbb{R}^m$ is C^1 (or piecewise C^1) and satisfies*

$$Du_0 \in U_1 \text{ a.e.}$$

Then there exists $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ such that

$$Du \in K \text{ a.e.}, \quad u = u_0 \text{ on } \partial\Omega$$

For the proof one uses Lemma 2.1 to inductively construct approximations $u^{(i)}$ with $Du^{(i)} \in U_i$. The key point is to assure that the $Du^{(i)}$ converge strongly. At first glance it is surprising that this can be achieved since the construction in Lemma 2.1 yields solutions with highly oscillatory gradients. Nonetheless by a judicious choice of the C^0 error δ in Lemma 2.1 one can ensure that the oscillations added in each iteration step are essentially independent of the previous ones and only effect a set of small measure. This construction, which is reminiscent of the construction of continuous, nowhere differentiable functions is one of the key ideas of convex integration (in [DM 97] it is replaced by a very elegant, but slightly less flexible, Baire category argument); see [MS 96] for the details.

3 Constraints and sets without rank-1 connections

The theory explained so far applies to Example a) and b) but not to c) - e). As regards c), the constraint $\det F = 1$ is stable under lamination convexity since $F \mapsto \det F$ is affine in rank-1 directions. Hence the set K in c) does not admit an in-approximation by open sets. The set K in e) contains no rank-1 connections and hence $K^{lc} = K$, and similarly U^{lc} contains only points near K for small neighbourhoods U of K. As regards d), Ball [Ba 80] showed that for strongly elliptic systems that arise as Euler-Lagrange equations (i.e. $\sigma = Df, f : M^{m\times n} \to \mathbb{R}$) again $K^{lc} = K$. It turns out, however, that the previous results can be extended to a slightly larger hull than K^{lc}, namely the rank-1 convex hull K^{rc} (called the functional rank-1 convex hull in [MP 98]), and that this hull can be nontrivial in Examples d) and e).

We say that a function $f : E \to \mathbb{R}$ is rank-one convex on a set E if it is convex on each rank-one line $t \mapsto F + ta\otimes n$. For a compact set K we define the rank-one convex hull relative to E by

$$K^{rc,E} = \left\{ F \in M^{m\times n} \, : \, f(F) \le \inf_K f, \, \forall f : E \to \mathbb{R} \text{ rank-1 convex} \right\},$$

i.e. $K^{rc,E}$ consists of those points that cannot be separated from K by rank-1 convex functions. For a (relatively) open set U the set $U^{rc,E}$ is defined as the union of all $K^{rc,E}$ where $K \subset U$ is compact. If $E = M^{m\times n}$ we simply write K^{rc}

and U^{rc}. The main result is the following variant of Lemma 2.1. Given an $r \times r$ minor M ($r \geq 2$) and a real number $t \neq 0$ we let

$$\Sigma = \left\{ F \in M^{m\times n} : M(F) = t \right\}.$$

Lemma 3.1 *(i) Let $U \subset M^{m\times n}$ be open, let $F \in U^{rc}$ and let $\varepsilon > 0$. Then there exists a piecewise linear map $u : \Omega \to \mathbb{R}^m$ that satisfies*

$$\begin{aligned} Du &\in U^{rc} \text{ a.e.}, \quad u(x) = Fx \text{ on } \partial\Omega, \\ \text{meas}\,\{Du &\notin U\} < \varepsilon|\Omega|. \end{aligned}$$

(ii) If U is relatively open in Σ and $F \in U^{rc,\Sigma}$, then u can be chosen such that in addition $Du \in U^{rc,\Sigma} \subset \Sigma$ a.e.

By a simple iteration one obtains the counterpart of Lemma 2.1 with U^{lc} replaced by U^{rc} (or $U^{rc,E}$ if a constraint is imposed). The proof of part(i) uses three facts. First, for a compact set K, the rank-1 convex hull K^{rc} consists of the barycentres of a certain class $\mathcal{M}^{rc}(K)$ ('laminates') of probability measures supported on K. Precisely, a probability measure belongs to $\mathcal{M}^{rc}(K)$ if and only if $\langle \nu, f\rangle \geq f(\langle \nu, id\rangle)$ for all rank-1 convex f. Secondly, we use a result of Pedregal [Pe 93] that laminates can be approximated (in the weak$*$ topology of measures) by simpler measures, the so-called laminates of finite order, that are supported on U^{rc}, where U is a (small) neighbourhood of K. The class $\mathcal{L}(U^{rc})$ of laminates of finite order is defined inductively as follows: all Dirac masses δ_F with $F \in U^{rc}$ belong to $\mathcal{L}(U^{rc})$. If $\sum_{i=1}^k \lambda_i \delta_{F_i}$ belongs to $\mathcal{L}(U^{rc})$ and if $F_k = \mu A + (1-\mu)B$, $A, B \in U^{rc}$, $\text{rk}(B-A) = 1$, $\mu \in (0,1)$, then $\sum_{i=1}^{k-1} \lambda_i \delta_{F_i} + \lambda_k \mu \delta_A + \lambda_k (1-\mu)\delta_B$ belongs to $\mathcal{L}(U^{rc})$. Third, we inductively use Lemma 2.2 to associate to each $\nu \in \mathcal{L}(U^{rc})$ a map $u : \Omega \to \mathbb{R}^m$ such that $Du \in U^{rc}$ and $|\text{meas}\,\{Du = F_i\} - \lambda_i \text{meas}\,\Omega| < 2^{-i}\varepsilon$. Then u has the desired properties.

To treat the case with constraint one first has to extend Pedregal's result to this situation. Secondly one has to prove a version of Lemma 2.2 which includes the constraint $Du \in \Sigma$ (one can relax (2.4) to the conditions $Du \in U^{rc}$ and meas $\{Du \notin U\} < \varepsilon$.) To this end one first obtains a C^∞ approximation by smoothing and considering a flow of a suitable (divergence-free) vectorfield. Then one constructs a piecewise linear approximation using, among other facts, a result of Dacorogna and Moser on the solvability of $\det Du = f$ in $C^{k,\alpha}$ spaces.

By iteration and the same approximation argument as in the proof of Theorem 2.4 one finally obtains the following result. We say that $\{U_i\}$ is an *rc*-in-approximation of K if the conditions in Definition 2.3 hold with U_i^{lc} replaced by U_i^{rc}.

Theorem 3.2 *(i) Suppose that $K \in M^{m\times n}$ admits an rc-in-approximation by open sets U_i. Suppose that $u_0 \in C^1(\Omega;\mathbb{R}^m)$ (or u_0 piecewise C^1) and*

$$Du_0 \in U_1^{rc}.$$

Then there exists a map u that satisfies

$$Du \in K \text{ a.e. }, \quad u = u_0 \text{ on } \partial\Omega.$$

(ii) If $K \subset \Sigma$ and $u_0 \in C^{2,\alpha}(\Omega;\mathbb{R}^m)$ then the same assertion holds if the U_i are only relatively open in Σ. If $m = n = r$ then the condition $u_0 \in C^{1,\alpha}(\Omega;\mathbb{R}^m)$ suffices.

Remark. 1. If K is open (or relatively open in Σ) one can take the trivial in-approximation $U_i \equiv K$.
2. As in Gromov's work one can achieve the boundary condition in the stronger sense of fine approximation: for each function $\eta \in C^0(\Omega)$ with $\eta > 0$ there exists a solution that satisfies $|u - u_0| < \eta$. In particular, if $u_0 = 0$ one can find solutions with $Du = 0$ on $\partial\Omega$.

4 Applications

Example a) Lamination convexity reduces to ordinary convexity and an in-approximation with $U_1 \supset \operatorname{int}\operatorname{conv} K$ is given by $U_i = \{F \in \operatorname{conv} K : 0 < \operatorname{dist}(F, K) < 2^{-i+2}\operatorname{diam} K\}$. Hence the assertions in Section 1 follow from Theorem 2.4.

Example b) One easily checks that

$$K^{lc} = K^{rc} = \operatorname{conv} K = \left\{F \in M^{m\times n} : \lambda_{max}(F^T F) \le 1\right\}.$$

It follows that

$$U_i = \left\{F \in M^{m\times n} : 1 - 2^{-i+2} < \lambda_{max}(F^T F) < 1\right\}$$

provides an in-approximation with $U_1 = \operatorname{int}\operatorname{conv} K$, and Theorem 2.4 applies.

Example c) Let $\Sigma = \left\{F \in M^{2\times 2} : \det F = 1\right\}$. By a result of Šverák [Sv 93]

$$K^{lc} = K^{rc} = \Sigma \cap \operatorname{conv} K.$$

Thus $U_i = \Sigma \cap \left\{F \in \operatorname{int}\operatorname{conv} K : 0 < \operatorname{dist}(F, K) < 2^{-i+2}\operatorname{diam} K\right\}$ provides an in-approximation (relative to Σ) with $U_1 \supset \Sigma \cap \operatorname{int}\operatorname{conv} K$.

Example e) Let

$$J_1 = -J_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_2 = -J_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $K^{lc} = K$ but $K^{rc} = [-1, 1]^2 \cup \bigcup_{i=1}^4 [J_i, A_{i+1}]$ (see Figure 1).

As an immediate consequence of Theorem 3.2 we obtain:

Corollary 4.1 *Let $U \supset K$ be open, and suppose that $F \in U^{rc} \supset K^{rc}$. Then there exist $u : \Omega \to \mathbb{R}^2$ such that*

$$Du \in U \ \text{a.e.}, \quad u = Fx \ \text{on } \partial\Omega.$$

As regards Example d) we recall the result mentioned in the introduction.

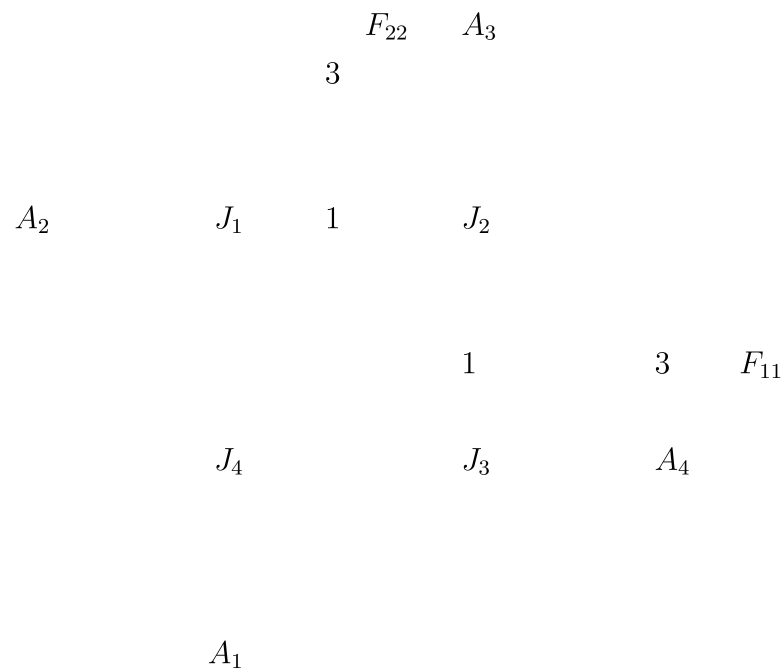

Figure 1: The set $\{A_1, A_2, A_3, A_4\}$ is lamination convex but the rank-one convex hull contains the shaded square and the line segments $[J_i, A_{i+1}]$.

THEOREM 4.2 *There exists a smooth strongly elliptic* 2×2 *system*

$$-\mathrm{div}\sigma(Dv) = 0, \quad v : \mathbb{R}^2 \to \mathbb{R}^2 \tag{4.1}$$

that admits

(i) nontrivial Lipschitz solutions with compact support;

(ii) Lipschitz solutions that are nowhere C^1*.*

Moreover σ *can be chosen such that (4.1) is the Euler-Lagrange equation of a variational integral* $\int f(Dv)\,dx$*, where* f *is smooth and uniformly quasiconvex in the sense of Morrey.*

Sketch of proof. Our interest lies mainly in the variational case but the main idea can already be seen in the simpler non-variational situation. The key idea is to embed the four-point configuration in Figure 1 in the set

$$K = \left\{ \begin{pmatrix} F \\ G \end{pmatrix} : F, G \in M^{2\times 2},\ \sigma(F)J = G \right\}.$$

This turns out to be surprisingly simple. Consider first the restriction of σ to diagonal matrices and let

$$\sigma_{11}(F_{11}, F_{22}) = F_{11} - g(F_{22}), \quad \sigma_{22}(F_{11}, F_{22}) = F_{22} - h(F_{11}).$$

Strong ellipticity on diagonal matrices reduces to the conditions

$$\frac{\partial \sigma_{11}}{\partial F_{11}} \geq c > 0, \quad \frac{\partial \sigma_{22}}{\partial F_{22}} \geq c > 0,$$

and is clearly satisfied. Moreover g and h can be chosen such that the set $\{\sigma_{11} = \sigma_{22} = 0\}$ includes the points A_1, A_2, A_3, A_4 in Figure 1 and $(0,0)$. If we extend σ to nondiagonal matrices by $\sigma_{12} = kF_{12}$, $\sigma_{21} = kF_{21}$ then σ is elliptic for sufficiently large k, and a careful analysis shows that K^{rc} typically contains a neighbourhood U of $0 \in M^{4\times 2}$, and K admits an rc-in-approximation $\{U_i\}$ with $U_1 = U$.

Let Ω be a smooth and bounded domain in $\mathbb{R}^n$. By Theorem 3.2 there exist a solution

$$Du \in K \text{ a.e.}, \quad u = 0 \text{ on } \partial\Omega.$$

Writing $u = \begin{pmatrix} v \\ w \end{pmatrix}$ we obtain

$$-\mathrm{div}\sigma(Dv) = 0 \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega.$$

Since $Dw = \sigma(Dv)J$, the trace theorem yields $\sigma(Dv)n = 0$ on $\partial\Omega$.

Now extend v by zero to $\mathbb{R}^n$. Since $\sigma(0) = 0$ the map v is a solution of (4.1) with compact support. Regarding (ii) one can use (i) and scaling to construct solutions that can only be regular on a set of arbitrarily small measure. To obtain the full strength of (ii) one has to slightly modify the construction in the proof of Theorem 2.4.

5 Some open problems

A necessary condition for the solvability of

$$Du \subset K \text{ a.e. in } \Omega, \quad u(x) = Fx \text{ on } \partial\Omega$$

is that F belongs to the so-called quasiconvex hull K^{qc} of K which in general is bigger than the rank-1 convex hull K^{rc} (see [Sv 95] or [Mu 98] for definitions and further references). This raises the following questions

- Does Theorem 3.2 hold if one replaces U_i^{lc} by U_i^{qc} in the definition of in-approximation?
- Can one compute (or estimate) K^{qc} for the set K in Example d)?
- Can one find manageable conditions on σ that guarantee $K^{qc} = K$?

Even checking whether $K^{rc} = K$ is in general not obvious. The following Theorem gives a recent example.

Theorem 5.1 *Let $f(F) = \det F$, for $F \in M^{2\times 2}$, and let $\sigma(F) = Df(F) = \det F \operatorname{cof} F$. Then the set*

$$K = \left\{ \begin{pmatrix} F \\ G \end{pmatrix} : \sigma(F)J = G \right\}$$

satisfies $K^{rc} = K$.

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