Reflection Principle in Higher Dimensions

Klas Diederich and Sergey Pinchuk

Abstract. The article discusses the use of the reflection principle in studying the following conjecture: Let $D, D' \subset \mathbb{C}^n$ be domains with smooth real-analytic boundaries and $f : D \to D'$ a proper holomorphic map. Then f extends holomorphically to a neighborhood of the closure of D.

1991 Mathematics Subject Classification: 32H40, 32H99, 32D15
Keywords and Phrases: Proper holomorphic maps, reflection principle, Segre variety, holomorphic correspondences

1 Boundary regularity of proper holomorphic maps

Let $D, D' \subset\subset \mathbb{C}^n$, $n \geq 2$, be domains and $f : D \to D'$ a proper holomorphic map. The following two questions are very natural to ask:

1) Suppose, the boundaries ∂D and $\partial D'$ are both $\mathcal{C}^\infty$-smooth. Does f always admit a $\mathcal{C}^\infty$ extension $\hat{f} : \overline{D} \to \overline{D'}$?

2) Suppose, the boundaries ∂D and $\partial D'$ are both $\mathcal{C}^\omega$-smooth. Does f always admit a holomorphic extension $\hat{f}$ to a neighborhood of $\overline{D}$?

Both questions in full generality are open. However, a lot has been found out about them since the early 70's. The emphasis of this article is on question 2). For a survey until 1989 see [13].

The modern development for question 1 started with the article by Ch. Fefferman [12], showing that there is a $\mathcal{C}^\infty$-extension of f, if ∂D and $\partial D'$ are both strictly pseudoconvex and f is biholomorphic. For question 2) the positive answer for strictly pseudoconvex domains was obtained by H. Lewy [15] and S. Pinchuk [16] independently (again f biholomorphic).

Concerning question 1, important further progress was made using methods by S. Webster, E. Ligocka and S. Bell. With them the positive answer was obtained in the case of pseudoconvex domains D, D' of finite type (see [7] and [3]). (A local version needed in section 3 is contained in [4].) After M. Christ discovered in [6], that on the so-called worm domains the $\overline{\partial}$-Neumann problem is not globally hypoelliptic, it has become clear, that these methods do not carry over to the general case of pseudoconvex domains.

Concerning question 2) again the case of pseudoconvex domains has been successfully treated independently in [2] and [8] (both articles also contain local versions). The case of question 2 for D and D' not necessarily pseudoconvex, has been positively solved for $n = 2$ in [10] building on previous work [11], [8] and [9].

The main methods used in treating question 2 (the real-analytic case) are variations of a reflection principle in several complex variables. There are two major forms, an analytic and a geometric one. Let us at first briefly look at the analytic variant. We choose real-analytic defining functions $\rho(z,\overline{z})$ and $\rho'(z',\overline{z'})$ for the domains D resp. D'. The properness of the map f implies, that we have $\rho'(f(z),\overline{f(z)}) \equiv 0$ on ∂D. In the case $n = 1$, by the implicit function theorem, this equation can be solved in the form $f(z) = \overline{\lambda'(f(z))}$ with a holomorphic function λ'. This gives the extension. In dimension $n > 1$, we need at least n independent equations giving the separation into holomorphic and antiholomorphic parts. Under suitable conditions on the boundaries, they can be obtained by applying tangental CR-operators to the equation $\rho'(f(z),\overline{f(z)}) \equiv 0$. In the strictly pseudoconvex case (see [15] and [16]) one differentiation is enough. However, for boundaries of finite type the number of differentiations is a-priori undetermined. Hence this method, in general, applies only if it is known in advance, that the map f extends in a $\mathcal{C}^\infty$ way up to ∂D.

The geometric version of the reflection principle uses the complexification of the defining functions and the so-called Segre varieties given by them. It will be explained in the next section. For $n = 1$ Segre varieties are just points such that this reflection principle is the well-known Schwarz principle. For $n = 2$ this version was successfully applied in [10] and the articles on which this was built. We point out, that [10] also includes many relevant results for arbitrary $n \geq 2$. A new general result is contained here in section 3.

2 Segre varieties and the geometric reflection principle

Let $D \subset\subset \mathbb{C}^n$ be a domain, such that ∂D is real-analytic smooth near $z^0 \in \partial D$. We may assume $z^0 = 0$. On a suitable open neighborhood W of 0 we can choose a real-analytic defining function $\rho(z,\overline{z})$ for D. After shrinking W the complexification $\rho(z,\overline{w})$ of ρ, which is holomorphic in z and antiholomorphic in w, is well-defined and has a power series convergent on $W \times W$. We now can associate to any point $w \in W$ its so-called "Segre variety" defined as

$$Q_w := \{ z \in W : \rho(z,\overline{w}) = 0\} \tag{2.1}$$

It is a closed complex submanifold of W not depending on the choice of the defining function ρ. It easily follows, that these Segre varieties are also invariant under biholomorphic changes of coordinate systems and, hence, under local biholomorphisms. The geometric reflection principle makes systematic use of these local invariants and their behavior (A complete list of basic properties needed can be found in Prop. 2.2 of [10]). We will now explain its main ideas and some more technical details needed in section 3 for the proof of Theorem 3.1.

For convenience we will use for $z \in \mathbb{C}^n$ the notation $z = ('z, z_n)$. We can choose so-called normal coordinates associated to ∂D at 0 (see [5]). With respect to them, ρ has the form

$$\rho(z,\overline{z}) = 2x_n + \sum_{j=0} \rho_j('z,'\overline{z})(2y_n)^j \tag{2.2}$$

with real-analytic functions ρ_j vanishing at 0 and without purely holomorphic or antiholomorphic terms. The complexification of ρ then can be written as

$$\rho(z,\overline{w}) = z_n + \overline{w}_n + \sum_{j=0}^{\infty} \rho_j('z,'\overline{w})(-i)^j(z_n - \overline{w}_n)^j \tag{2.3}$$

It follows, that one has

$$\rho(z,\overline{w}) = 0 \quad \Leftrightarrow \quad z_n + \overline{w}_n + \sum_{|k|>0} \overline{\lambda_k(w)}\,'z^k = 0 \tag{2.4}$$

where the summation is over multiindices $k = (k_1, \ldots, k_{n-1})$ with $k_j \geq 0$ and each λ_k is a holomorphic function on W. It follows from (2.4) for later use

$$\rho(z,\overline{w}) = (1 + \alpha(z,\overline{w}))\left(z_n + \overline{w}_n + \sum_k \overline{\lambda_k(w)}'z^k\right) \tag{2.5}$$

with a $\mathcal{C}^\omega-$function $\alpha(z,\overline{w})$, holomorphic in z, antiholomorphic in w, vanishing at 0.

For convenience we write $\lambda_0(w) := w_n$. The holomorphic map

$$W \ni w \longmapsto \hat{\lambda}(w) := \left(\lambda_k(w) : k \in \mathbb{N}_0^{n-1}\right)$$

is called the "Segre map". Because of the Noether property, there is an integer $L > 0$ associated to ∂D, such that the terms up to total order L in $\hat{\lambda}$ completely determine $\hat{\lambda}$. If L is chosen with this property we also call Segre map the part

$$W \ni w \longmapsto \lambda(w) := (\lambda_k(w) : |k| \leq L) \in \mathbb{C}^N \tag{2.6}$$

It is important to observe, that the Segre map is often not injective. Therefore, the size of the complex-analytic sets

$$A_w := \{\, z : Q_z = Q_w \} \tag{2.7}$$

is decisive for the geometric reflection principle. We say

DEFINITION 2.1 *The domain D is called essentially finite at $0 \in \partial D$, if A_0 (and, hence, A_w for all w close to 0) is finite (see [11] and [1]).*

Real-analytic smooth hypersurfaces of finite type are always essentially finite. Furthermore, if ∂D is essentially finite at 0, the Segre map λ is finite and, hence, proper on W sufficiently small, . In this case, the set $\mathcal{S} := \lambda(W) \subset \mathbb{C}^N$ is closed complex analytic in a suitable neighborhood of $\lambda(0)$.

Let now $D, D' \subset\subset \mathbb{C}^n$ be real-analytic smooth domains. According to [14] they are of finite type. We will apply the above considerations both to D and D'. We introduce the notational convention, that the objects associated to D' will be denoted by the same letters as for D with a prime added (for instance, $Q'_{w'}$ is the Segre variety associated to $\partial D'$ at w').

Suppose now a proper holomorphic map $f : D \to D'$ is given. The program of using the Segre varieties for constructing a holomorphic extension of f to a neighborhood of $\overline{D}$ consists of the following two major steps:

1) Let $0 \in \partial D$ be an (arbitrary) point. Show, that there is a neighborhood W of 0, an open set $W' \subset \mathbb{C}^n$ and a proper holomorphic correspondence $F : W \to W'$ extending f from $W \cap D$ to W.

2) Let W be an open neighborhood of $0 \in \partial D$, $W' \subset \mathbb{C}^n$ open, and suppose, that a proper holomorphic correspondence $F : W \to W'$ extends f from $W \cap D$ to W. Show, that this implies the extendability of f as a holomorphic map to a neighborhood of 0.

This program has been carried out in full detail for $n = 2$ in [10]. However, many considerations of [10] are valid for general n and the step 2) for general n will be completed in this article in Theorem 3.1.

For the first step of the above-mentioned program we proceed essentially as follows: We choose the neighborhood W of 0 suitably and denote by W' a small open neighborhood of $\partial D'$. For $w' \in W'$, we denote by ${}^s w'$ the point on $Q'_{w'}$ on the complex normal through w' to $\partial D'$ and by ${}_{{}^s w'}Q'_{w'}$ the germ of $Q'_{w'}$ at ${}^s w'$. We put

$$V := \left\{ (w, w') \in \left(W \setminus \overline{D}\right) \times \left(W' \setminus \overline{D'}\right) : f(Q_w \cap D) \supset {}_{{}^s w'}Q'_{w'} \right\} \tag{2.8}$$

Notice, that, since the Segre map λ' is, in general, not injective, the set V will usually contain several points (w, w') lying over one point w.

After now showing at first by totally different techniques, that f extends as a holomorphic map to a neighborhood of a dense subset of ∂D, a long chain of steps distinguishing between boundary points of different CR-nature allows to show, that the set V can be extended across $\partial D \cap W$ in such a way, that a proper holomorphic correspondence $F : W \to W'$ is obtained extending f.

In step 2) an extending proper holomorphic correspondence $F : W \to W'$ is given. It induces a continuous extension of f to $W \cap \overline{D}$. Again one uses the fact, that f extends holomorphically across a dense subset of ∂D to deduce from the invariance of the Segre varieties under biholomorphisms the following much stronger invariance property with respect to F (Corollary 4.2 and 5.5 of [10]):

Theorem 2.2 *If neighborhoods W of 0 and W' of $0' := f(0)$ are chosen suitably, then there is a bijective holomorphic map $\varphi : \mathcal{S} \to \mathcal{S}'$, such that the diagram*

$$\begin{array}{ccc} \mathcal{S} & \overrightarrow{\varphi} & \mathcal{S}' \\ \lambda \uparrow & & \uparrow \lambda' \\ W & \overrightarrow{\hat{F}} & W' \end{array}$$

is commutative. (Here we denoted by $\hat{F}$ the set-valued map induced by F.)

We mention the following immediate consequence needed in the proof of Theorem 3.1. It concerns the functions λ_k from (2.4):

Lemma 2.3 *Under the hypothesis of Theorem (2.2) and with W, W' chosen as there, for every multiindex $k = (k_1, \ldots, k_{n-1})$, $|k| > 0$, the set $\lambda'_k(\hat{F}(z))$ consists of a unique complex number for every $z \in W$ and this defines a holomorphic function on W.*

3 Extending correspondences are maps in all dimensions

We will show in this section:

Theorem 3.1 *Let $D, D' \subset\subset \mathbb{C}^n$ be domains, $n \geq 2$. Suppose that $z^0 \in \partial D$ and $z'^0 \in \partial D'$ have open neighborhoods W resp. W' such that $\partial D \cap W$ and $\partial D' \cap W'$ are smooth real-analytic, essentially finite hypersurfaces and let $f : D \to D'$ be a proper holomorphic map. Furthermore, suppose, that the given map f extends as a proper holomorphic correspondence F to a neighborhood of z^0 such that $\hat{F}(z^0) = z'^0$. Then the map f extends holomorphically to a neighborhood of z^0.*

Proof: We may assume, that $z^0 = 0 = z'^0$, that the given correspondence F extending f is defined over W and the coordinates z, z' have been chosen to be normal at 0. Hence, a suitable defining function $\rho \in \mathcal{C}^\omega(W)$ can be written as in (2.2), similarly for D' near 0. We apply all notions of section 2. After rescaling the coordinates, we have polydiscs $U \subset W$ and $U' \subset W'$ around 0 of radius 2, such that $\hat{F}(U) \subset U'$ and the following property holds:

All functions $\rho(z, w)$, $\rho_j('z, 'w)$, $\lambda_k(w)$, $\sum_k \lambda_k(w)'z^k$ and the corresponding functions for the image are holomorphic in polydiscs around 0 of radius 2 in the corresponding dimensions. In particular, the series $\sum_k |\lambda_k(w)|$, $\sum_k |\frac{\partial \lambda_k}{\partial w_n}(w)|$ and the corresponding series for the image converge uniformly on compact subsets of U (resp. U'). Because of the normality of the coordinates we also have $\lambda'_k(0) = 0$ and $\frac{\partial \lambda'_k}{\partial w'_n}(0) = 0$ for all k. Therefore we have

$$\sum_k |\lambda'_k(0)| = 0 \quad \text{and} \quad \sum_k \left| \frac{\partial \lambda'_k}{\partial w'_n}(0) \right| = 0 \tag{3.1}$$

Since, as explained in [10], Prop.7.2, $f_n(z) = z_n h(z)$ on U with h holomorphic and $h(0) \neq 0$, we can make a biholomorphic coordinate change by replacing z_n by $z_n h(z)$. However, we have to be aware of the fact that the new coordinates are no longer normal for ∂D at 0.

Now the series $\sum_k \lambda'_k(w')\overline{\lambda'_k(\overline{\zeta}')}$ converges on $U' \times U'$ and represents a holomorphic function there. Putting $\zeta' := \overline{w}'$ and $w' \in \hat{F}(z)$, we get because of Lemma 2.3 $\sum_k |\lambda'_k(w')|^2 \in \mathcal{C}^\omega(U')$ and $\sum_k |\lambda'_k(\hat{F}(z))|^2 \in \mathcal{C}^\omega(U)$.

Since $\partial D'$ is supposed to be essentially finite, we may assume, that for all $a' \in U'$ the set

$$A'_a := \{ ('w', w'_n) \in U' : w'_n = a'_n, \lambda'_k(w') = \lambda'_k(a') \; \forall |k| > 0\} \tag{3.2}$$

is finite.

We now introduce the following two decisive auxiliary open sets depending on a sufficiently large number $M \gg 1$ and $\varepsilon \in (-\frac{1}{M}, 0]$:

$$\mathcal{D}'(M,\varepsilon) := \left\{ w' \in U' : 2\operatorname{Re} w'_n + M|w'_n|^2 + M\sum_k |\lambda'_k(w')|^2 < \varepsilon \right\} \tag{3.3}$$

$$\mathcal{D}(M,\varepsilon) := \left\{ z \in U : 2x_n + M|z_n|^2 + M\sum_k \left|\lambda'_k\left(\hat{F}(z)\right)\right|^2 < \varepsilon \right\} \tag{3.4}$$

We have

Lemma 3.2 *The open sets $\mathcal{D}'(M,\varepsilon)$ and $\mathcal{D}(M,\varepsilon)$ are pseudoconvex and their boundaries are of finite type at all points in U resp. U' where they are smooth.*

Proof: Both open sets are obviously inside the polydisk U resp. U' as sublevelsets of plurisubharmonic functions (for (3.4) we know from Lemma 2.3, that the $\lambda'_k(\hat{F}(z))$ are holomorphic functions on U). Hence they are pseudoconvex.

Next we observe, that, the defining function for $\mathcal{D}'(M,\varepsilon)$ from (3.3) can be rewritten

$$\rho'_{M,\varepsilon}(w', \overline{w}') := M\left|w'_n + \frac{1}{M}\right|^2 + M\sum_k |\lambda'_k(w')|^2 - \varepsilon - \frac{1}{M} \tag{3.5}$$

If $\partial\mathcal{D}'(M,\varepsilon)$ is smooth near a point $w'^0 \in U'$ and $h : \Delta \to \partial\mathcal{D}'(M,\varepsilon)$ is a holomorphic map with $h(0) = w'^0$ and $h(\Delta) \subset \partial\mathcal{D}'(M,\varepsilon)$, then because of (3.5), h_n and $\lambda'_k \circ h(t)$ have to be constant for all k. Since A'_a is finite, h itself has to be constant showing that $\partial\mathcal{D}'(M,\varepsilon)$ is of finite type at w'^0. The reasoning for $\mathcal{D}(M,\varepsilon)$ goes the same way. □

In general, it is not true that $\mathcal{D}'(M,\varepsilon) \subset D'$ (resp. $\mathcal{D}(M,\varepsilon) \subset D$). However, we have the following crucial

Lemma 3.3 *If $M \gg 1$ is sufficiently large, then one has for any $\varepsilon \in (-\frac{1}{M}, 0]$*
a) the non-smooth part of $\partial\mathcal{D}'(M,\varepsilon)$ is contained in D';
b) the non-smooth part of $\partial\mathcal{D}(M,\varepsilon)$ is contained in D.

Proof: We show at first a). Since $\varepsilon \le 0$, $w' \in \partial\mathcal{D}'(M,\varepsilon)$ implies because of (3.5)

$$M\left|w'_n + \frac{1}{M}\right|^2 + M\sum_k |\lambda'_k(w')|^2 \le \frac{1}{M} \tag{3.6}$$

Hence, the next three estimates follow directly

$$-\frac{2}{M} \le \operatorname{Re} w'_n \le 0, \quad |w'_n|^2 \le \frac{4}{M^2}, \quad \sum_k |\lambda'_k(w')|^2 \le \frac{1}{M^2} \tag{3.7}$$

Since 0 is the only solution of the system

$$w'_n = 0,\ \lambda'_k(w') = 0\ \forall |k| > 0$$

shrink to the origin for $M \to \infty$. In particular, necessarily also $w' \to 0$ as $M \to \infty$.

Let now $\partial \mathcal{D}'(M,\varepsilon)$ be non-smooth at w'. Then grad $\rho'_{M,\varepsilon}(w',\overline{w}') = 0$. Hence

$$\frac{\partial \rho'_{M,\varepsilon}}{\partial w'_n}(w',\overline{w}') = 0$$

implying

$$\frac{1}{M} + \operatorname{Re} w'_n + \operatorname{Re} \sum_k \frac{\partial \lambda'_k}{\partial w'_n}(w')\overline{\lambda_k(w')} = 0 \tag{3.8}$$

By (3.7) we have $|\lambda'_k(w')| \le \frac{1}{M}$ and, therefore,

$$\left| \sum_k \frac{\partial \lambda'_k}{\partial w'_n}(w')\overline{\lambda'_k(w')} \right| \le \frac{1}{M} \sum_k \left| \frac{\partial \lambda'_k}{\partial w'_n}(w') \right|$$

Because of (3.1) the sum on the right side is $o(1)$ for $w' \to 0$ uniformly in ε (this uniformity in $\varepsilon \in (-\frac{1}{M}, 0]$ holds in all the following estimates). Hence we get

$$\sum_k \frac{\partial \lambda'_k}{\partial w'_n}(w')\overline{\lambda'_k(w')} = o\left(\frac{1}{M}\right) \text{ for } M \to \infty \tag{3.9}$$

Together with (3.8) we get

$$\operatorname{Re} w'_n = -\frac{1}{M} + o\left(\frac{1}{M}\right) \tag{3.10}$$

Using again $|\lambda'_k(w')| \le \frac{1}{M}$ we also obtain

$$\sum_k \overline{\lambda'_k(w')}' w'^k = o\left(\frac{1}{M}\right) \tag{3.11}$$

Putting (3.10) and (3.11) into (2.5), we deduce

$$\rho'(w',\overline{w}') = (1 - \alpha'(w',\overline{w}'))\left(2 \operatorname{Re} w'_n + \sum_k \overline{\lambda'_k(w')}' w'^k\right) = -\frac{2}{M} + o\left(\frac{1}{M}\right) < 0$$

for large M uniformly in ε. Hence $w' \in D'$ finishing part a).

For showing b) we consider the defining function

$$\rho_{M,\varepsilon}(z,\overline{z}) := 2 \operatorname{Re} z_n + M|z_n|^2 + M \sum_k \left| \lambda'_k\Big(\hat{F}(z)\Big) \right|^2 - \varepsilon \tag{3.12}$$

of $\mathcal{D}(M,\varepsilon)$ and keep in mind that $f_n(z_n) = z_n$. Let now $z \in \partial\mathcal{D}(M,\varepsilon)$ be a non-smooth boundary point. In complete analogy to a) we get

$$2\,\mathrm{Re}\, z_n + M|z_n|^2 + M\sum_k \left|\lambda'_k\Big(\hat{F}(z)\Big)\right|^2 \le 0 \tag{3.13}$$

and the three inequalities

$$-\frac{2}{M} \le x_n \le 0, \quad |z_n|^2 \le \frac{4}{M^2}, \quad \sum_k \left|\lambda'_k\Big(\hat{F}(z)\Big)\right|^2 \le \frac{1}{M^2} \tag{3.14}$$

and since, again, $\hat{F}(z) \to \{0\}$ as $z \to 0$, we have

$$\sum_k \overline{\lambda'_k\Big(\hat{F}(z)\Big)}\Big[{}'\hat{F}(z)\Big]^k = o\Big(\frac{1}{M}\Big) \tag{3.15}$$

However, since $\frac{\partial\lambda'_k(\hat{F}(z))}{\partial z_n}$ does not necessarily vanish at 0, the analogue of (3.10) might not hold. But there exists at least a $c > 0$ such that for large $M \gg 1$

$$x_n \le -\frac{c}{M} \tag{3.16}$$

Namely, if (at least on a suitable subsequence) $x_n = o(\frac{1}{M})$, then we get from (3.13) $|\lambda'_k(\hat{F}(z))| = o(\frac{1}{M})$ and, therefore,

$$\sum_k \frac{\partial\lambda'_k\Big(\hat{F}(z)\Big)}{\partial z_n}\overline{\lambda'_k\Big(\hat{F}(z)\Big)} = o\Big(\frac{1}{M}\Big)$$

This, however, is a contradiction to

$$\frac{1}{M} + x_n + \mathrm{Re}\sum_k \frac{\partial\lambda'_k}{\partial z_n}\Big(\hat{F}(z)\Big)\overline{\lambda'_k\Big(\hat{F}(z)\Big)} = 0 \tag{3.17}$$

which holds in analogy to (3.8). This shows (3.16).

In order to show that $z \in D$ we will use the multivalued "function" $\rho'(\hat{F}(z), \overline{\hat{F}(z)})$, observing at first, that according to Prop. 7.1 from [10], for any fixed z, all its values have the same sign, namely, D always goes to D' under $\hat{F}(z)$ and the exterior goes to the exterior.

Because of (2.5) we have

$$\rho'\Big(\hat{F}(z), \overline{\hat{F}(z)}\Big) = \Big(1 + \alpha'\Big(\hat{F}(z), \overline{\hat{F}(z)}\Big)\Big)\left(2x_n + \sum_k \overline{\lambda'_k\Big(\hat{F}(z)\Big)}\Big[{}'\hat{F}(z)\Big]^k\right)$$

with $\alpha'(0,0) = 0$. Hence, we obtain from (3.15) and (3.16)

$$\rho'\Big(\hat{F}(z), \overline{\hat{F}(z)}\Big) \le -\frac{2c}{M} + o\Big(\frac{1}{M}\Big) < 0 \tag{3.18}$$

Therefore, $z \in D$. □

The next essential step in proving Theorem 3.1 is to show

Lemma 3.4 *If $M \gg 1$ is chosen as in Lemma 3.3, then f extends holomorphically to a proper map $\hat{f} : \mathcal{D}(M,0) \to \mathcal{D}'(M,0)$.*

Proof: For such M and ε close to $-\frac{1}{M}$, $\mathcal{D}(M,\varepsilon)$ is a small neighborhood of the set

$$A := \left\{ z_n = -\frac{1}{M}, \lambda'_k\left(\hat{F}(z)\right) = 0 \; \forall |k| > 0 \right\}$$

Because of (2.5) we have for any $z \in A$ (notice, that $z_n = -\frac{1}{M}$)

$$\rho'\left(\hat{F}(z), \overline{\hat{F}(z)}\right) = \left(1 + \alpha'\left(\hat{F}(z), \overline{\hat{F}(z)}\right)\right) \cdot 2x_n < 0$$

Hence $\mathcal{D}(M,\varepsilon) \subset D$ and $\mathcal{D}'(M,\varepsilon) \subset D'$ if in addition $\varepsilon \in (-\frac{1}{M}, 0]$ is close to $-\frac{1}{M}$. Therefore, by the definition of $\mathcal{D}(M,\varepsilon)$, $f : \mathcal{D}(M,\varepsilon) \to \mathcal{D}'(M,\varepsilon)$ is proper holomorphic.

Now let $\varepsilon \in (-\frac{1}{M}, 0]$ be maximal such that f extends to a proper holomorphic map $\hat{f} : \mathcal{D}(M,\varepsilon) \to \mathcal{D}'(M,\varepsilon)$. Notice at first, that this map is extended as a proper holomorphic correspondence to a neighborhood of $\overline{\mathcal{D}(M,\varepsilon)} \cap U$ by F. Therefore, $\hat{f}$ is continuous up to the boundary. Because of Lemma 3.3 and known results about holomorphic extension as mentioned in section 2 this map extends as a proper holomorphic map to $\mathcal{D}(M,\tilde{\varepsilon})$ with $\tilde{\varepsilon} > \varepsilon$ unless $\varepsilon = 0$. □

End of the proof of Theorem 3.1: By applying the same arguments as at the end of the last proof and using that $0 \in \partial\mathcal{D}(M,0)$, we see, that $\hat{f}$ extends holomorphically to a neighborhood of 0. □

References

1. Baouendi, M. S., Jacobowitz, H., Treves, F.: *On the analyticity of CR mappings*, Ann. Math. 122 (1985), 365–400.
2. Baouendi, M. S., Rothschild, L. P.: *Germs of CR maps between real analytic hyperfaces*, Invent. Math. 93 (1988), 481–500.
3. Bell, S., Catlin, D.: *Boundary regularity of proper holomorphic mappings*, Duke Math. J. 49 (1982), 385–396.
4. Bell, S., Catlin, D.: *Regularity of CR mappings*, Math. Z. 199 (1988), 357–368.
5. Chern, S. Y., Moser, J.: *Real hypersurfaces in complex manifolds*, Acta Math. 133 (1974), 219–271.
6. Christ, M.: *Regularity properties of the $\overline{\partial}_b$ equation on weakly pseudoconvex CR manifolds of dimension 3*, J. Amer. Math. Soc. 1 (1988), 587–646.
7. Diederich, K., Fornæss, J. E.: *Boundary regularity of proper holomorphic mappings*, Inventiones Math. 67 (1982), 363–384.
8. Diederich, K., Fornæss, J. E.: *Proper holomorphic mappings between real-analytic pseudoconvex domains in* $\mathbb{C}^n$, Math. Ann. 282 (1988), 681–700.

9. Diederich, K., Fornæss, J. E., Ye, Z.: *Biholomorphisms in dimension 2*, J. Geom. Analysis 4 (1994), 539–552.

10. Diederich, K., Pinchuk, S. I.: *Proper holomorphic maps in dimension 2 extend*, Indiana Univ. Math. J. 44 (1995), 1089–1126.

11. Diederich, K., Webster, S.: *A reflection principle for degenerate real hypersurfaces*, Duke Math. J. 47 (1980), 835–845.

12. Fefferman, C.: *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Inventiones Math. 26 (1974), 1–65.

13. Forstnerič, F.: *A survey on proper holomorphic mappings*, Proceedings of the Special Year in SCV's at the Mittag-Leffler Institute (Princeton, N. J.) (Fornæss, J. E., ed.), Math. Notes, vol. 38, Princeton University Press.

14. Lempert, L.: *On the boundary behavior of holomorphic mappings*, Contributions to Several Complex Variables (in honour of Wilhelm Stoll) (Braunschweig - Wiesbaden) (Howard, A., Wong, P.M., eds.), Vieweg and Sons, pp. 193–215.

15. Lewy, H.: *On the boundary behavior of holomorphic mappings*, Acad. Naz. Linc. 35 (1977), 1–8.

16. Pinchuk, S.: *On the analytic continuation of holomorphic mappings*, Math. USSR-Sb. 27 (1975), 375–392.

Klas Diederich
Mathematik, Univ. Wuppertal
Gausstr. 20
D-42097 Wuppertal, GERMANY
klas@math.uni-wuppertal.de

Sergey Pinchuk
Department of Mathematics
Indiana University
Bloomington, IN 47405, USA
pinchuk@indiana.edu