Developments from Nonharmonic Fourier Series

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Abstract. We begin this survey by showing that Paley and Wiener's unconditional basis problem for nonharmonic Fourier series can be understood as a problem about weighted norm inequalities for Hilbert operators. Then we reformulate the basis problem in a more general setting, and discuss Beurling-type density theorems for sampling and interpolation. Next, we state some multiplier theorems, of a similar nature as the famous Beurling-Malliavin theorem, and sketch their role in the subject. Finally, we discuss extensions of nonharmonic Fourier series to weighted Paley-Wiener spaces, and indicate how these spaces are explored via de Branges' Hilbert spaces of entire functions.

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1. From Paley-Wiener to Hunt-Muckenhoupt-Wheeden

The theory of nonharmonic Fourier series begins with Paley and Wiener [18], who discovered that the trigonometric system $\{e^{ikx}\}$ remains an unconditional basis for $L^2(-\pi,\pi)$ when the integer frequencies k are replaced by "nonharmonic" frequencies λ_k satisfying $|\lambda_k - k| \le d$ for some $d < 1/\pi^2$. This result led to quite extensive activity around the problem of describing all unconditional bases of the form $\{e^{i\lambda_k x}\}$ for $L^2(-\pi,\pi)$. A decisive breaktrough was made by Pavlov [19], and a complete solution to the problem as just stated is now available [9,12,15].

We shall present below a survey of recent developments which are closely related to the problem of Paley and Wiener. Let us therefore begin by clarifying how the unconditional basis problem can be understood: It can be recast as a question concerning boundedness of Hilbert operators in certain weighted L^2 (or more generally L^p) spaces of functions and sequences, and thus leads us to the Hunt-Muckenhoupt-Wheeden theorem [7]. We will follow [12], in which this shift from Hilbert space geometry to weighted norm inequalities is made.

We restate the Paley-Wiener problem in terms of entire functions. Denote by PW^p $(0 < p \le \infty)$ the classical Paley-Wiener spaces, which consist of all entire functions of exponential type at most π whose restrictions to the real line are in L^p. We endow PW^p with the natural $L^p(\mathbb{R})$-norms, and note that they are Banach spaces when $1 \le p \le \infty$ and complete metric spaces when $0 < p < 1$. For $1 < p < \infty$, we say that a sequence of complex numbers $\Lambda = \{\lambda_k\}$, $\lambda_k = \xi_k + i\eta_k$ is a *complete interpolating sequence* for PW^p if the interpolation problem $f(\lambda_k) = a_k$ has a unique solution $f \in PW^p$ for every sequence $\{a_k\}$ satisfying

$$\sum_k |a_k|^p e^{-p\pi|\eta_k|}(1+|\eta_k|) < \infty.$$

Via the Paley-Wiener theorem, it is found that Λ is a complete interpolating sequence for PW^2 if and only if the system $\{e^{i\lambda_k x}\}$ is an unconditional basis for $L^2(-\pi,\pi)$.

Let us see how the Hilbert operator comes into play when we seek to describe complete interpolating sequences. Suppose Λ is a complete interpolating sequence for PW^p, $1<p<\infty$. Let us assume for simplicity that all the points of Λ lie in a horizontal strip, that $0\notin\Lambda$, and $\xi_k\le\xi_{k+1}$ for all k. It is easy to show that Λ has to be a separated sequence, i.e., $\inf_{j\ne k}|\lambda_j-\lambda_k|>0$, and also that it must be uniformly dense, i.e., that $\sup_k(\xi_{k+1}-\xi_k)<\infty$. In what follows, one should think of Λ roughly as an arithmetic progression.

If the function $f_0\in PW^p$ solves the interpolation problem $f_0(\lambda_k)=\delta_{0,k},\ k\in\mathbb{Z}$, then $f_0(\mu)\ne 0$ for $\mu\in\mathbb{C}\setminus\Lambda$, since otherwise the function $(z-\lambda_0)(z-\mu)^{-1}f_0(z)$ belongs to PW^p and vanishes on Λ, contradicting the uniqueness of the solution of the interpolation problem. It is a short step from this observation to conclude that the limit

$$S(z)=\lim_{R\to\infty}\prod_{|\lambda_k|<R}(1-z/\lambda_k) \tag{1}$$

exists and defines an entire function of exponential type π. This function is called the *generating function* of the sequence Λ. It follows that if $a=\{a_j\}$ is a sequence such that $a_j=0$ except for finitely many j's, the unique solution of the interpolation problem $f(\lambda_j)=a_j$, has the form

$$f(z)=\sum_j\frac{a_j}{S'(\lambda_j)}\frac{S(z)}{(z-\lambda_j)}.$$

Now if $\Gamma=\{\gamma_j\}$ is any other separated and uniformly dense sequence lying in a horizontal strip, a classical inequality of Plancherel and Pólya [11, pp. 50–51] shows that

$$\sum_j|f(\gamma_j)|^p\lesssim\int_{\mathbb{R}}|f(x)|^p dx.$$

(We write $g\lesssim h$ whenever there is a positive constant C such that $g\le Ch$, and $g\simeq h$ if both $g\lesssim h$ and $h\lesssim g$.) Because the solution of the interpolation problem is unique, the open mapping theorem implies that $\sum|f(\lambda_j)|^p\simeq\int|f(x)|^p dx$, and so

$$\sum_j|f(\gamma_j)|^p\lesssim\sum_j|a_j|^p. \tag{2}$$

We claim that this inequality is just a weighted norm inequality for a discrete Hilbert operator. To see this, let ℓ^p_w be the space of all sequences $b=\{b_k\}$ satisfying $\|b\|^p_{w,p}:=\sum|b_k|^p w_k<\infty$ for some positive weight sequence $w=\{w_j\}$. If we put $u=\{|S'(\lambda_j)|^p\}$ and $v=\{|S(\gamma_j)|^p\}$, (2) says that the Hilbert operator $H_{\Lambda,\Gamma}:\ell^p_u\to\ell^p_v$ defined as

$$(\mathcal{H}_{\Lambda,\Gamma}b)_j=\sum_k\frac{b_k}{\gamma_j-\lambda_k},$$

is a bounded operator.

So far we have not assumed anything about Γ, except that it is separated and uniformly dense. We may in fact tailor it specifically to Λ in such a way that the weights u and v become identical, apart from a multiplicative constant. To see how this can be done, set $\varepsilon = \inf_{j\neq k} |\lambda_j - \lambda_k|/3$, and observe that since S has no zeros in the disk $|z-\lambda_j| \leq \varepsilon$, we can find a point γ_j with $|\gamma_j - \lambda_j| = \varepsilon$ and

$$|S(\gamma_j)| = \varepsilon |S'(\lambda_j)|.$$

We are now in a familiar situation, and obtain in accordance with the celebrated Hunt-Muckenhoupt-Wheeden theorem [7] that the weight $w = \{|S'(\lambda_j)|^p\}$ must satisfy a discrete Muckenhoupt (A_p) condition:

$$\sup_{k\in\mathbb{Z},n>0} \left(\frac{1}{n}\sum_{j=k+1}^{k+n} w_j\right)\left(\frac{1}{n}\sum_{j=k+1}^{k+n} w_j^{-\frac{1}{p-1}}\right)^{p-1} < \infty. \tag{3}$$

The analogy is clear: The classical continuous (A_p) condition for a positive weight $v(x) > 0,\ x \in \mathbb{R}$ is

$$\sup_I \left\{\left(\frac{1}{|I|}\int_I v dx\right)\left(\frac{1}{|I|}\int_I v^{-\frac{1}{p-1}} dx\right)^{p-1}\right\} < \infty, \tag{4}$$

where I ranges over all intervals in $\mathbb{R}$, and the Hunt-Muckenhoupt-Wheeden theorem [7] says that (3) is necessary and sufficient for boundedness of the classical Hilbert operator on the weighted space of functions $L^p(\mathbb{R}; vdt)$. It is clear that (3) is essentially a special case of (4). In fact, in our case, we may use either of the conditions, because it may be proved that (3) with $w = \{|S'(\lambda_j)|^p\}$ is equivalent to (4) with $v = |S(x)/\mathrm{dist}(x,\Lambda)|^p$.

The above reasoning has provided an essential piece of evidence for the main theorem of [12], which we will now state. We remove the assumption that Λ be located in a horizontal strip. It is then convenient to introduce the distance function

$$\delta(z,\zeta) = \frac{|z-\zeta|}{1+|z-\bar{\zeta}|},$$

which expresses that we deal with Euclidean geometry close to the real axis and hyperbolic geometry far away from the real axis. We say that Λ is *δ-separated* if $\inf_{j\neq k} \delta(\lambda_j,\lambda_k) > 0$. Moreover, Λ is said to satisfy the *two-sided Carleson condition* if for any square Q of side-length $l(Q)$ and with one of its sides sitting on the real axis, we have

$$\sum_{\lambda_k \in Q\cap\Lambda} |\Im\lambda_k| \leq Cl(Q),$$

with C independent Q.

The main theorem of [12] is:

Theorem 1. *A sequence $\Lambda = \{\lambda_k\}$ of complex numbers is a complete interpolating sequence for PW^p $(1 < p < \infty)$ if and only if the following three conditions hold.*

(i) *The sequence Λ is δ-separated and satisfies the two-sided Carleson condition.*

(ii) *The limit $S(z)$ in* (1) *exists and represents an entire function of exponential type π.*

(iii) *The weight $(|S(x)|/\mathrm{dist}(x,\Lambda))^p$ $(x \in \mathbb{R})$ satisfies the (A_p) condition* (4).

This theorem should be read in the following way: Condition (i) is a separation condition in which the Carleson condition is present because we solve in particular an interpolation problem in H^p; (ii) is mainly a density condition, as it gives the type of S; (iii) is a condition on the "balance" of the sequence. It is in fact a working condition, if one makes use of the equivalence between the (A_2) and Helson-Szegö conditions. For instance, the so-called Kadets 1/4 theorem, which says that $|\lambda_k - k| \le d < 1/4$ is the best possible inequality in the Paley-Wiener condition, is a direct consequence (see [9]). A similar perturbation result can be proved for PW^p, as shown in [12].

2. Beurling-type density theorems for sampling and interpolation

Stated as an interpolation problem for entire functions, the Paley-Wiener basis problem makes sense for a large class of holomorphic spaces. In this section, we shall extend the setting, and then consider the complementary situation that complete interpolating sequences are nonexistent. Building on a basic contribution by Beurling [2], who considered a problem of balayage of Fourier-Stieltjes transforms and a corresponding interpolation problem, we reformulate the Paley-Wiener problem by seeking to describe separately so-called sampling and interpolating sequences. Again, problems of this type can be traced back to classical work on nonharmonic Fourier series [5,8]; for the modern state of research on such nonharmonic Fourier series, see [21].

Assume we are given a weighted L^p space of holomorphic functions defined on some domain Ω in the complex plane. We denote this space by $\mathcal{B}$ and assume that the functional of point evaluation $f \mapsto f(z)$ is bounded for each $z \in \Omega$. The norm of this functional is called the *majorant* of $\mathcal{B}$, and it is denoted by $M(z)$. If $p = 2$, then $\mathcal{B}$ is a Hilbert space and $M(z) = \sqrt{K(z,z)}$, where $K(z,\zeta)$ is the reproducing kernel of the space. We say that a sequence of distinct points $\Lambda = \{\lambda_k\}$ in Ω is a *sampling sequence* for $\mathcal{B}$ if $\|f\|_{\mathcal{B}} \simeq \|\{f(\lambda_k)/M(\lambda_k)\}\|_{\ell^p}$ for $f \in \mathcal{B}$. We say that Λ is an *interpolating sequence* for $\mathcal{B}$ if the interpolation problem $f(\lambda_k) = a_k$ has a solution $f \in \mathcal{B}$ whenever $\{a_k/M(\lambda_k)\} \in \ell^p$. Finally, we say that Λ is a *complete interpolating sequence* for $\mathcal{B}$ if it is both sampling and interpolating. It is not difficult to check (using the open mapping theorem) that this definition is in line with the one given in the previous section.

Saying that a complete interpolating sequence is both a sampling and an interpolating sequence is a way of expressing that it exists as a compromise between two competing density conditions: A sampling sequence should be uniformly "dense", while an interpolating sequence should be uniformly "sparse". However, the reasoning of the previous section shows that there is more to it than only competing density conditions: Existence of complete interpolating sequences is tied to norm inequalities for Hilbert operators between weighted L^p spaces. This means that we can expect to find such sequences only when $1 < p < \infty$ and in spaces with a special underlying geometry.

In this section, we shall present an aspect of the following striking dichotomy: *Geometric density conditions characterize sampling and interpolating sequences if and only if there are no complete interpolating sequences.* Of course, we are not able to claim that the truth of this statement is universal, but it covers at least three wide classes of model spaces: weighted Paley-Wiener spaces PW^p_ψ (to be considered in Section 4), weighted Fock spaces F^p_ψ, and weighted Bergman spaces A^p_ψ (to be defined shortly). In all three cases, the growth of functions is controled by e^ψ, where ψ is a subharmonic function whose Laplacian has an appropriate behavior compared to the underlying geometry: in the Paley-Wiener case, $\Delta\psi$ is supported by the real line and the Riesz measure of ψ is $\mu(x)dx$, with $\mu(x) \simeq 1$; in the Fock case, $\Delta\psi(z) \simeq 1$ for all $z \in \mathbb{C}$; in the Bergman case, $\Delta\psi(z) \simeq (1-|z|^2)^{-2}$ for all z in the unit disk $\mathbb{D}$. A common feature is that density conditions for sampling and interpolation are expressed in terms of $\Delta\psi$. We note that *it is only for Paley-Wiener spaces that we have weighted norm inequalities for the Hilbert operators attached to the possible complete interpolating sequences.*

We comment first on the Fock and Bergman cases, which are similar. We will only present results for Bergman spaces; the Fock case has been treated in the recent paper [17]. The results to be presented here for Bergman spaces are new, and we shall sketch proofs which are quite different from those of [17]. We call these results *Beurling-type density theorems*, because results of this type were first presented by Beurling and because certain parts of Beurling's analysis seem indispensable in whatever setting we consider.

We need to give a precise definition of the weighted Bergman spaces A^p_ψ. Suppose a subharmonic function ψ on the unit disk is given, whose Laplacian satisfies $\Delta\psi(z) \simeq (1-|z|^2)^{-2}$ for all $z \in \mathbb{D}$. Let dm denote Lebesgue area measure on $\mathbb{C}$. Define

$$\|f\|^p_{\psi,p} = \int_{\mathbb{D}} |f(z)|^p e^{-p\psi(z)} (1-|z|^2)^{-1} dm(z)$$

for $p < \infty$, and $\|f\|_{\psi,\infty} = \sup_z |f(z)|e^{-\psi(z)}$. We denote by A^p_ψ $(0 < p \le \infty)$ the set of all functions f analytic in $\mathbb{D}$ such that $\|f\|_{\psi,p} < \infty$. A prime example is obtained by setting $\psi(z) = -\beta\log(1-|z|^2)$, with $\beta > 0$.

Now set

$$\rho(z,\zeta) = \left|\frac{z-\zeta}{1-\bar{\zeta}z}\right|,$$

which is the pseudohyperbolic distance between z and ζ. We say that a sequence $\Lambda = \{\lambda_j\}$ is *ρ-separated* if $\inf_{j\ne k} \rho(\lambda_j,\lambda_k) > 0$. For a fixed ρ-separated sequence Λ, we denote by $n(z,r)$ the number of points $\lambda_k \in \Lambda$ which satisfy $\rho(z,\lambda_k) < r$, and set correspondingly

$$a_\psi(z,r) = \int_{\rho(z,\zeta)<r} \Delta\psi(\zeta) dm(\zeta).$$

The *lower uniform density of* Γ *with respect to* ψ is defined as

$$D_\psi^-(\Lambda) = \liminf_{r\to 1^-} \inf_{z\in\mathbb{D}} \frac{\int_0^r n(z,t)dt}{\int_0^r a_\psi(z,t)dt},$$

and the *upper uniform density of* Γ *with respect to* ψ is

$$D_\psi^+(\Lambda) = \limsup_{r\to 1^-} \sup_{z\in\mathbb{D}} \frac{\int_0^r n(z,t)dt}{\int_0^r a_\psi(z,t)dt}.$$

We have then the following two Beurling-type density theorems.

THEOREM 2. *A sequence* Λ *is sampling for* A_ψ^p *if and only if it contains a* ρ*-separated subsequence* Λ' *satisfying* $D_\psi^-(\Lambda') > 1/\pi$ *and in addition, when* $0 < p < \infty$*, it is a finite union of* ρ*-separated sequences.*

THEOREM 3. *A sequence* Λ *is interpolating for* A_ψ^p *if and only if it is* ρ*-separated and satisfies* $D_\psi^+(\Lambda) < 1/\pi$*.*

For $\psi(z) = -\beta\log(1-|z|^2)$ these are the main results of [20]. In the next section, we will sketch how the general case follows from these special results, via a certain multiplier theorem. Here we restrict ourselves to making two remarks concerning the proof for $\psi(z) = -\beta\log(1-|z|^2)$; in this case, with a slight abuse of notation, we set $A_\psi^p = A_\beta^p$ and $\|\cdot\|_{\psi,p} = \|\cdot\|_{\beta,p}$.

First, we would like to point out what is the core of Beurling's approach as it appears when transferred to $\mathbb{D}$. Namely, A_β^p enjoys the following group invariance: If τ is a Möbius self-map of $\mathbb{D}$, the operator T_τ defined by

$$(T_\tau f)(z) = (\tau'(z))^{\beta+1/p} f(\tau(z))$$

acts isometrically on A_β^p. This implies that sampling and interpolating sequences are Möbius invariant, and in fact, by a normal family argument, any compact-wise limit of a sequence $\tau_n\Lambda$, where τ_n are Möbius self-maps of $\mathbb{D}$, is sampling/interpolating if Λ is sampling/interpolating. An analysis of such compact-wise limits plays an essential role in Beurling's scheme. This part of Beurling's proof is of a general nature and is applicable whenever we have a suitable group invariance; we refer to [16] for a discussion of how the notion of "group invariance" can be extended to spaces with general weights.

Our second remark concerns the proof of the sufficiency of the density condition for interpolation. In [22], this was done by first relating the upper uniform density to a density used by Korenblum for describing the zeros of functions in A_β^∞, and then use this relation to construct a linear operator of interpolation. A less intricate and more direct proof, using Hörmander-type L^2 estimates for $\bar\partial$, has later been given by Berndtsson and Ortega-Cerdà [1]. This approach works also for F_ψ^p.

We end this section with a few words about the original interpolation problem considered by Beurling [2], to illustrate that "Beurling-type" density conditions may be rather subtle. Beurling considered interpolating values only along the real axis, in which case uniform densities of real sequences yield a complete description. If we permit complex sequences Λ, we are led to combine techniques from entire functions and Hardy spaces in a nontrivial manner, and to solve simultaneously an interpolation problem in H^∞.

Suppose Λ is δ-separated, and let h be a positive number. Denote by $n_h^+(r)$ the maximum number of points from Λ to be found in a rectangle of the form $\{z = x + iy : \ t < x < t + r, \ |y| < h\}$, where t is any real number. The upper uniform density of Λ is defined to be

$$D^+(\Lambda) = \lim_{h\to\infty} \lim_{r\to\infty} \frac{n_h^+(r)}{r}.$$

We have then the following "mixed" Beurling-type and Carleson theorem.

THEOREM 4. *A sequence Λ is interpolating for PW^∞ if and only if it is δ-separated, satisfies the two-sided Carleson condition, and $D^+(\Lambda) < \tau/\pi$.*

This result is proved in [17]. A key ingredient in the proof will be presented in the next section. There is of course a similar result for the sampling problem, but it is more elementary. The result holds also when PW^∞ is replaced by PW^p, $p < 1$, which is an easier case than $p = \infty$.

3. THE ROLE OF MULTIPLIERS

The most distinguished example of a *multiplier theorem* is the following deep result of Beurling and Malliavin [3,10]: *If f is an entire function of exponential type with bounded logarithmic integral,*

$$\int_{\mathbb{R}} \frac{\log^+ |f(x)|}{1 + x^2} dx < \infty,$$

then, for every $\varepsilon > 0$ there exists an entire function g of exponential type ε with both $|g|$ and $|fg|$ bounded on the real axis.

In this section, we discuss how certain more modest multiplier theorems fit into our theory. As for the Beurling-Malliavin theorem, proofs are based on atomizing Riesz measures of certain subharmonic functions, but the details are quite straightforward in our case. However, it should be noted that we obtain more precise estimates on what corresponds to the product $|fg|$ above. This is why these multiplier theorems have an interesting role to play in our subject.

For the Paley-Wiener case, we have the following mulitplier theorem:

THEOREM 5. *Suppose Λ is a δ-separated sequence and ω is a subharmonic function of the form*

$$\psi(z) = \int_{-\infty}^{\infty} [\log |1 - z/t| + (1 - \chi_{[-1,1]}(t))\Re z/t]\mu(t)dt, \tag{5}$$

where $\mu(t) \simeq 1$. Then there exists an entire function g with δ-separated zero sequence $Z(g)$ lying in a horizontal strip, with $\delta(\Lambda, Z(g)) > 0$, and such that $|g(z)|e^{-\psi(z)} \simeq \delta(z, \Gamma)$.

The corresponding result for the disk is:

THEOREM 6. *Suppose Λ is a ρ-separated sequence in $\mathbb{D}$, and let ϕ be subharmonic in $\mathbb{D}$ so that its Laplacian $\Delta\phi$ satisfies $\Delta\phi(z) \simeq (1-|z|^2)^{-2}$ for all $z \in \mathbb{D}$. Then there exists a function g analytic in $\mathbb{D}$, with ρ-separated zero sequence $Z(g)$ and $\rho(Z(g), \Lambda) > 0$, and such that $|g(z)| \simeq \rho(z, Z(g))e^{\phi(z)}$.*

There is also an analogous result for the Fock case [14]. The two theorems above are in fact inspired by that result. A proof of Theorem 5 can be found in [16], while Theorem 6 is a slight variant of Theorem 2 of [22].

Theorem 5 is a key ingredient in the proof of Theorem 4. It is used both to transform Beurling's interpolation problem into an H^∞ problem, and to "correct" H^∞ solutions to produce solutions which are entire functions. We give only a hint how the first transformation is done. If we assume Λ satisfies the conditions of Theorem 4 and set $\varepsilon = 1-D^+(\Lambda)$, then Theorem 5 yields the existence of a function h vanishing on Λ, and satisfying the estimate $|h(z)| \simeq e^{\pi(1-\varepsilon/2)|\Im z|}\delta(z, Z(h))$, where $Z(h)$ is the zero sequence of h. (Incidentally, this argument shows that every interpolating sequence for PW^∞ is contained in a sequence which is a complete interpolating sequence for each of the spaces PW^p, $1 < p < \infty$. It is a striking fact that, on the other hand, there exists an interpolating sequence for PW^2 which is not a subsequence of any complete interpolating sequence for PW^2, as shown in [21].)

Next, we sketch how Theorem 6 can be used to prove Theorems 2 and 3 from the case of regular weights $-\beta\log(1-|z|^2)$. To this end, we begin by showing that A^p_ψ can be embedded into A^p_β for a sufficently large β: Choose β so large that $\phi(z) = \beta\log(1/(1-|z|^2)) - \psi(z)$ is a subharmonic function satisfying the condition of Theorem 6. Taking g to be the function of Theorem 6, it is clear that $f \in A^p_\psi$ if and only if $fg \in A^p_\beta$, and that $\|f\|_{\psi,p} \simeq \|f\|_{\beta,p}$. In other words, *we may associate A^p_ψ with the closed subspace of A^p_β which consists of functions vanishing on $Z(g)$.*

We now take Λ to be the ρ-separated sequence of Theorem 6, and claim that then *Λ is sampling/interpolating for A^p_ψ if and only if $Z(g) \cup \Lambda$ is sampling/interpolating for A^p_β.* The sufficiency of the condition $Z(g) \cup \Lambda$ being sampling/interpolating is trivial in view of the observation we just made, while the necessity can be obtained from the fact that $Z(g)$ is interpolating for A^p_β, as follows from Theorem 3 in the case of regular weights. Now Theorems 2 and 3 follow from the regular case by a simple rewriting of the density conditions.

For other applications of Theorem 6, see [6,22].

4. FROM DE BRANGES TO WEIGHTED PALEY-WIENER SPACES

Suppose ψ is a subharmonic function in $\mathbb{C}$ of the form (5) with $\mu(t) \simeq 1$. Set $w = e^{-\psi}$, and define

$$\|f\|^p_{w,p} = \int_{\mathbb{R}} |f(t)w(t)|^p\,dt$$

for $p < \infty$, and $\|f\|_{w,\infty} = \sup_z |f(z)|e^{-\psi(z)}$. We denote by PW^p_ψ $(0 < p \le \infty)$ the set of all entire functions f such that $\|f\|_{w,p} < \infty$ and $\log|f(z)| \le C_\varepsilon + \psi(z) + \varepsilon|z|$ for all $\varepsilon > 0$. The Phragmén-Lindelöf principle ensures that these spaces are complete with respect to their norms.

Following the reasoning at the end of Section 3, we may extend Theorem 1 and Theorem 4 to cover these weighted Paley-Wiener spaces. Thus our choice of weights is natural if we wish to see how far the basic results of nonharmonic Fourier series can be extended. But weighted Paley-Wiener spaces are interesting for other reasons. One particularly interesting point is the connection to de Branges' Hilbert spaces of entire functions [4], and that this link can be used to explore the nature of weighted Paley-Wiener spaces. We shall briefly indicate how this may work. The presentation is based on [14], where a complete treatment can be found. We stick from now on to the Hilbert space case $p = 2$.

A natural question is: Why is our choice of weights $e^{-\psi}$ reasonable? It is quite easy to see that our condition on the weight implies $M(x)w(x) \simeq 1$. By means of de Branges' theory, we can prove that this relation, which is a regularity condition on w, in fact characterizes weighted Paley-Wiener spaces. To be more precise, suppose $\mathcal{H}$ is a Hilbert space of entire functions whose norm is given by $\|\cdot\|_{w,2}$, where w is a positive weight function. We assume the functional of point evaluation is bounded for each $z \in \mathbb{C}$, and further that $\mathcal{H}$ is closed under the operations $f(z) \mapsto f(z)(z-\overline{\zeta})/(z-\zeta)$ (provided $f(\zeta)=0$) and $f(z) \mapsto f^*(z)$, where $f^*(z) = \overline{f(\overline{z})}$. If $M(x)w(x) \simeq 1$, we say that w is a *majorant weight*. Then the following holds:

Theorem 7. *A positive function w is a majorant weight for some space $\mathcal{H}$ if and only if there exists a function $\mu(x) \simeq 1$ and a real entire function g such that*

$$\log w(x) + g(x) + \int_{-\infty}^{\infty} [\log|1-x/t| + (1-\chi_{[-1,1]}(t))x/t]\mu(t)dt \in L^\infty. \tag{6}$$

The function g represents an inessential part of the weight, because a replacement of w by we^{-g} corresponds to multiplying all functions in $\mathcal{H}$ by a factor $\exp g/2$. Then assuming $g \equiv 0$, it is plain from de Branges' theory that $M(x)w(x) \simeq 1$ and the form of w together force $\mathcal{H}$ to be a weighted Paley-Wiener space.

It is interesting to note that if w has a bounded logaritmic integral, the condition (6) of Theorem 7 says that we have a representation $\log w = u + v$, with $u \in L^\infty$ and $(\tilde{v})' \in L^\infty$, where $\tilde{v}$ denotes the Hilbert transform of v.

The proof of Theorem 7 is based on converting the problem in the following way. By de Branges' theory, $\mathcal{H}$ coincides with a de Branges space $H(E)$; here E is an entire function without zeros in the upper half-plane, $|E(z)| \geq |E(\bar{z})|$ for all $\Im z > 0$, and $f \in H(E)$ if and only if f/E and f^*/E both belong to H^2 of the upper half-plane. That $\mathcal{H} = H(E)$ means in particular that $\|f/E\|_2 = \|f\|_{w,2}$ for all $f \in \mathcal{H}$. With this relation established, the proof becomes a problem of exploring the distribution of the zeros of E. To give a hint about the nature of the problem, we mention the following result. Let $\Lambda = \{\xi_k - i\eta_k\}$ denote the zero sequence of E, and suppose $\xi_k \leq \xi_{k+1}$ for all k. Then: *$H(E)$ equals (up to norm equivalence) a weighted Paley-Wiener space if and only if Λ is uniformly dense and a finite union of separated sequences, the sequence $\Lambda \cap \{z : \Im z > -\varepsilon\}$ is separated for some $\varepsilon > 0$, and $\{\eta_k\}$ is a discrete (A_2) weight.* The analogue of Theorem 1 for PW_ψ^2 is used to prove the last part of this statement.

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