From Finite- to Infinite-Dimensional Phenomena in Geometric Functional Analysis on Local and Asymptotic Levels

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The geometry and linear-metric structure of high dimensional convex bodies make an essential contribution to the understanding of the geometry, structure and some purely infinite-dimensional properties of Banach spaces. An asymptotic approach that studies finite-dimensional geometric properties "stabilized at infinity" makes it possible to identify regularities behind an apparent lack of structure. Recently, a deeper understanding of the infinite nature of Banach spaces has opened possibilities to study some previously intractable linear-topological problems by refined essentially finite-dimensional methods. By putting together certain sophisticated finite-dimensional random constructions we can create new phenomena of infinite flavour in arbitrary Banach spaces.

1. Finite-dimensional phenomena We start the discussion of "random quotients" of finite-dimensional normed spaces. Properties of such spaces reveal a striking interplay between high dimensional geometry and the linear structure of normed spaces. We shall also briefly mention some related properties of Gaussian matrices. Consider the following theorem (the terminology is explained below).

Theorem 1 *For $0 < \alpha < 1$ and $K \geq 1$ there exists $f(\alpha, K) > 0$ such that for every $n \geq 1$, whenever X is an n-dimensional normed space all of whose $[\alpha n]$-dimensional subspaces are K-isomorphic, then X is $f(\alpha, K)$-isomorphic to ℓ_2^n.*

This result is an isomorphic finite-dimensional version of two questions from Banach's book ([Ba32]): regarding an n-dimensional symmetric convex body all of whose k-dimensional sections are affinely equivalent, and the homogeneous Banach space problem. For the former question see Gromov's work [Gr67]; the solution to the latter was obtained by Gowers [G94a], in conjunction with [KT95], and will be discussed later.

Theorem 1 was proved by Bourgain in [B87] for sufficiently small α, and in [MT88] for all α, yielding the function $f(\alpha, K)$ less than $cK^{3/2}$ for $0 < \alpha < 2/3$, and cK^2, for $2/3 \leq \alpha < 1$, where $c = c(\alpha)$. The general line of an argument (the same in both papers) depends on two separate parts of the theory. The first part studies Euclidean sections of convex bodies, yielding upper estimates for the distance of such sections to ellipsoids. It was initiated in the late 60s and has

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been well developed throughout the intervening period, resulting in the discovery of many deep results relating a variety of geometric characteristics (see Milman's surveys [Mi86b] and [Mi96]). The other part investigates lower estimates; it was initiated by the Gluskin result (below); its development in a general form started with the present theorem. For lower estimates it is natural to work with quotient spaces (corresponding to projections of convex bodies); then the results for subspaces (corresponding to sections of convex bodies) follow by duality. Volume-type characteristics that appear in both parts are different, though, and it requires additional ingenuous arguments to put them together.

We need some notation. For convenience, we describe the real case only, the complex case follows by standard modifications. On $\mathbb{R}^n$ we consider the natural Euclidean norm $\|\cdot\|_2$, and by B_2^n we denote the closed Euclidean unit ball. By ℓ_1^n we denote $\mathbb{R}^n$ with the unit ball $B_1^n = \{x = (a_i) \mid \sum |a_i| \le 1\}$. Any convex compact centrally symmetric body $B \subset \mathbb{R}^n$ determines the normed space E for which B is the unit ball, and any n-dimensional normed space has (many) representations of such a form. The polar body B° is the unit ball in the dual space E^*. By $\{e_i\}$ we denote the unit vector basis of $\mathbb{R}^n$. By $\mathrm{vol}_n(\cdot)$ we denote the n-dimensional Lebesgue measure. If X_1, X_2 are isomorphic Banach spaces, the Banach–Mazur distance is defined by $d(X_1, X_2) = \inf \|T\| \, \|T^{-1}\|$, with the infimum running over all isomorphisms T from X_1 onto X_2; if $d(X_1, X_2) \le d$, we say that the spaces are d-isomorphic. For $B \subset \mathbb{R}^n$ we let

$$v_k(B) = \sup_{F,\, \dim F = k} \left(\mathrm{vol}_k\, P_F B \big/ \mathrm{vol}_k\, B_2^k \right)^{1/k}, \qquad \text{for } 1 \le k \le n,$$

where P_F is the orthogonal projection on a subspace $F \subset \mathbb{R}^n$.

Let $E = (\mathbb{R}^n, B_E)$; by properly identifying E with $\mathbb{R}^n$ we may further assume that B_2^n is the ellipsoid of minimal volume containing B_E. The simplest form of a lower estimate used in the proof in Theorem 1 says ([MT88]): *Let* $0 < \alpha < 1$. *There exist* $m = [\alpha n]$*-dimensional quotients* F_1, F_2 *of* E *such that*

$$d(F_1, F_2) \ge c(\alpha) v_{[m/4]}(B_E)^{-2}. \tag{1}$$

(In fact, these quotients are "random", in sense to be explained shortly.)

Estimates as in (1) are a conceptualization of the discovery of Gluskin in [Gl81a] (see also [Gl86]), who determined an asymptotic growth of the diameter of the Minkowski compactum of all m-dimensional normed spaces, by showing that: *There exists* $c > 0$ *such that for every* $m \ge 1$ *there exist ("random")* m*-dimensional quotients of* ℓ_1^{3m}, F_1 *and* F_2 *such that* $d(F_1, F_2) \ge cm$. (By the classical John theorem, $d(F_1, F_2) \le m$, for all m-dimensional normed spaces.)

Gluskin's new point of view triggered extensive investigations of a natural class of "random"quotients of ℓ_1^n, by a number of researchers (including Mankiewicz and Szarek among others). Further studies showed that the resulting bodies are "rigid", in a sense that the underlying normed spaces admit few well bounded linear operators. This is nicely expressed using the notion of mixing operators ([S86], [M88]). An operator $T \in L(\mathbb{R}^m)$ is called k-mixing, where $1 \le k \le m/2$, if there exists a subspace $H \subset \mathbb{R}^m$ with $\dim H \ge k$ such that $\|P_{H^\perp} T x\|_2 \ge \|x\|_2$, for every $x \in H$ (where $P_{H^\perp}$ is the orthogonal projection from $\mathbb{R}^m$ onto $H^\perp$). Note that for every projection P with $k = \mathrm{rank}\, P \le m/2$, $2P$ is k-mixing. We shall concentrate on quotients of proportional dimension.

Theorem 2 ([S83], [S86]) *There is $c>0$ such that for every integer $m \geq 1$ there is an m-dimensional quotient of ℓ_1^{2m}, F, such that every projection P on F with $m/4 \leq \operatorname{rank} P \leq 3m/4$, satisfies $\|P : F \to F\| \geq c\sqrt{m}$. More generally, for every $[m/4]$-mixing operator T on $\mathbb{R}^m$, $\|T : F \to F\| \geq c\sqrt{m}$.*

The first statement ([S83]) settled the so-called finite-dimensional basis problem (solved independently in [Gl81b]), showing an example of a sequence of finite-dimensional spaces F_n with $\operatorname{bc}(F_n) \to \infty$. Let us recall the fundamental classical definition. A sequence $\{x_i\}$ in a Banach space X is a Schauder basis, if every $x \in X$ admits the unique representation as a convergent series $x = \sum_i a_i x_i$. In such a situation, for $k = 1, 2, \ldots$, define the projections $P_k : X \to X$ by $P_k(x) = \sum_{i=1}^k a_i x_i$, for $x \in X$. Then $\operatorname{bc}(\{x_i\}) = \sup_k \|P_k\| < \infty$. If a Banach space X has a Schauder basis, the basis constant of X is defined as $\operatorname{bc}(X) = \inf \operatorname{bc}(\{x_i\})$, where the infimum is taken over all bases $\{x_i\}$ in X. So clearly, $\operatorname{bc}(F) \geq c\sqrt{m}$, for F as in the theorem.

An important aspect of these constructions is their random character. For problems requiring technically involved geometric phenomena the Gaussian setting is the easiest to use. Let $\gamma_1, \gamma_2, \ldots$ be independent real valued Gaussian variables with distribution $N(0,1)$. Let $m \geq 1$. Set $g = m^{-1/2} \sum_{i=1}^m \gamma_i e_i \in \mathbb{R}^m$. Let $n > m$ and $k = n - m$. Let $g_1, \ldots, g_k$ be independent $\mathbb{R}^m$-valued variables with the same distribution as g. Consider a *Gaussian projection* $Q_\omega : \mathbb{R}^n \to \mathbb{R}^m$ defined on the unit vector basis in $\mathbb{R}^n$ by $Q_\omega(e_i) = e_i$ for $1 \leq i \leq m$ and $Q_\omega(e_i) = g_{i-m}(\omega)$ for $m < i \leq n$. Given a normed space $E = (\mathbb{R}^n, B_E)$, by an m-dimensional *Gaussian quotient* of E we understand the space $F_\omega = (\mathbb{R}^m, B_{F_\omega})$ where $B_{F_\omega} = Q_\omega(B_E)$.

Of course this approach is related to Gaussian matrices. Convex geometric analysis discovered, often for its own needs, some deep results about such matrices. Since they may be of importance for many other areas of mathematics, we shall briefly digress to comment upon them.

Let $G = G_n(\omega)$ be an $n \times n$ matrix with independent Gaussian $N(0, 1/n)$ entries. Let $\{s_k(G)\}_{k \geq 1}$ be the sequence of singular (s-)numbers of G (i.e., the eigenvalues of $(G^*G)^{1/2}$ arranged in the non-increasing order, counting multiplicities). Their distribution is described by the classical Wigner Semi-circle Law [W55], which however has a qualitative character only. A quantitative distributional inequality was proved by Szarek in [S90]: *For $d \leq n/2$, $\mathcal{P}\{c_1 d/n \leq s_{n-d}(G) \leq c_2 d/n\} \geq 1 - C\exp(-cd^2)$, where $c_1, c_2, c, C > 0$ are absolute constants.* For further refinements and references see [S91]. In the other direction, Gordon studied (cf. e.g., [Go88], [Go92]) the majorization of Gaussian processes, in particular the maximum and the minimum of $\|G(x)\|_2$ over all $x \in B$, for an arbitrary symmetric convex body $B \subset \mathbb{R}^n$. Here G is possibly a rectangular $m \times n$ Gaussian matrix. For example, if $m = \alpha n < n$, this easily implies sharp estimates for the norms of G and of G^{-1} depending on α (established earlier e.g., via complicated combinatorial arguments [Ge80]), and many other geometric applications.

The above quotients of ℓ_1^n can be taken as Gaussian quotients; and the sets of (pairs of) ω's for which the lower estimates do not hold, have the measure exponentially small in n. In the last decade many sophisticated properties of random Gaussian quotients F of ℓ_1^n have been established, connected with factorizations of operators and the distance to the cube ([S90]), actions of compact groups of operators ([M88], [M98]) and others.

When these constructions are considered in the framework of *arbitrary* normed spaces, as for example in Theorem 1, their random character becomes even more crucial. Randomness is the main reason for the connection of linear structure and volumes. Also, families of random projections (or sections) of high-dimensional convex bodies display a curious dichotomous behaviour: they are either nearly Euclidean or else, they have an unusually rigid structure as discussed above. Thus the rigidity becomes a "random alternative" to being Euclidean.

Theorem 3 ([MT88], [MT94]) *Let $n \geq 1$ and let E' be a $2n$-dimensional normed space. There exists a quotient space E of E' with $\dim E = n$, and a Euclidean norm on E such that identifying E with $\mathbb{R}^n$ (and the Euclidean norm with $\|\cdot\|_2$), condition (1) is satisfied for a random pair of m-dimensional Gaussian quotients of E ($m = [\alpha n]$). Furthermore, letting $m = [99n/100]$, for a random m-dimensional Gaussian quotient F of E the estimate $\|T : F \to F\| \geq c v_{[m/100]}(B_E)^{-1}$ is valid for all $[m/10]$-mixing operators $T \in L(\mathbb{R}^m)$, with an absolute constant $c > 0$. Hence $\mathrm{bc}(F) \geq c v_{[m/100]}(B_E)^{-1}$ as well.*

For technical reasons, m has to be sufficiently close to n, but its specific value is of no importance. The estimates obtained are sharp: for $E = \ell_1^n$ we recover Gluskin's diameter result and Theorem 2. The preliminary step of passing from E' to E is designed to get the unit ball B_E in a "special position", i.e., having certain additional geometric properties with respect to the Euclidean structure in $\mathbb{R}^n$. This was achieved by using deep results from the convex geometric analysis: the inverse Brunn-Minkowski inequality ([Mi86a]) and the proportional Dvoretzky-Rogers Lemma ([BS88]).

It is also worthwhile to consider a more geometric approach to random families (of subspaces or quotients), through the orthogonal group. Let $\mathcal{G}_{n,m}$ denote the Grassmann manifold of all m-dimensional subspaces of $\mathbb{R}^n$, with the Haar measure. For $F \in \mathcal{G}_{n,m}$ let P_F be the orthogonal projection onto F. If $E = (\mathbb{R}^n, B_E)$ then F endowed with the unit ball $B_F = P_F(B_E)$ is a quotient space of E. We should mention, however, that the orthogonal approach is not equivalent to the Gaussian one; for example, it may be less sensitive to some involved structural properties of normed spaces.

Using more involved arguments it is possible to study invariants like these in Theorem 3 or others, for random quotients of the original space E, without passing to a special position. This reveals a striking threshold phenomenon, which, however, for some invariants can be quite indirect. For example, we have ([MT98b]): *Let E be an n-dimensional space identified with $\mathbb{R}^n$ in such a way that B_2^n is the ellipsoid of minimal volume containing B_E. There exists $1 \leq \varphi = \varphi_E$ such that:*

- *(i) "random" $(F_1, F_2) \in \mathcal{G}_{n,[n/2]} \times \mathcal{G}_{n,[n/2]}$ satisfies $d\big(P_{F_1}(B_E), P_{F_2}(B_E)\big) \geq \varphi$;*
- *(ii) "random" $F \in \mathcal{G}_{n,[n/8]}$ satisfies $(c/\sqrt{\varphi}) P_F(B_2^n) \subset P_F(B_E) \subset P_F(B_2^n)$, where $c > 0$ is an absolute constant.*

(Here "random" means "on a set of positive measure".) Intuitively, for any normed space E, identified with $\mathbb{R}^n$ as above, for any fixed K, the only way in which a random pair of $[n/2]$-dimensional quotients of E may be closer together than K is that random $[n/8]$-dimensional quotients of E are $C\sqrt{K}$-Euclidean. A kind of converse statement is trivially true: the distance between random $[n/8]$-dimensional quotients admits an upper bound by comparison with Euclidean space.

A detailed presentation of random quotients of finite-dimensional spaces, related infinite-dimensional constructions and an extensive bibliography, can be found in [MT98a].

2. INFINITE-DIMENSIONAL CONSTRUCTIONS A strong case for the emerging integration of finite-dimensional properties and the linear-topological structure of Banach spaces is made by the use of random quotient phenomena for constructions "inside" arbitrary Banach spaces. The first example combining finite-dimensional random quotients of ℓ_1^n into an infinite-dimensional space was given by Bourgain ([B86]), who constructed a real Banach space that admits two non-isomorphic complex structures. Then Szarek ([S87]) constructed a space without a sequence of uniformly bounded projections $\{P_n\}$ with $\sup_n \operatorname{rank}(P_n - P_{n-1}) < \infty$, hence without a Schauder basis.

At the root of these constructions lies a property still stronger than those discussed before: even adding to a quotient F the most regular space of all, does not remove an essential lack of well bounded operators. For example ([S86]): *A space F from Theorem 2 satisfies* $\operatorname{bc}(F \oplus_2 \ell_2) \geq cm^{1/4}$. This property can be formally deduced from a lower estimate for norms of all k-mixing operators on $\mathbb{R}^m$ ([MT94]), so an analogous fact holds in general too, by Theorem 3.

Before stating the next theorem recall that if X_n are Banach spaces, the ℓ_2-sum, $(\bigoplus X_n)_{\ell_2}$, is the Banach space of all sequences of vectors $z = (z_n)$, with $z_n \in X_n$ for all n, such that $\|z\|_{\oplus X_n} = (\sum \|z_i\|_{X_n}^2)^{1/2} < \infty$. If $X_n = X$ for all n, we write $\ell_2(X)$ instead of $(\bigoplus X)_{\ell_2}$.

The first construction of "gluing" together random quotients of finite-dimensional subspaces of an arbitrary Banach space X was done in [MT94] and it led to some interesting structural characterizations of Hilbert space. We give just one example.

THEOREM 4 [MT94] *Let X be a Banach space such that every subspace of every quotient of $\ell_2(X)$ has a Schauder basis. Then X is isomorphic to Hilbert space.*

Thus the theory has made a full circle, that started from Enflo's example of a Banach space without the approximation property ([E73]). Spaces Z without a Schauder basis can now be constructed in just three canonical operations, of the ℓ_2-sum and taking subspaces and quotients, starting from an *arbitrary* Banach space X not isomorphic to ℓ_2. Moreover, such spaces are of the form $Z = (\bigoplus Z_n)_{\ell_2}$, where Z_n are *finite-dimensional* quotients of subspaces of $\ell_2(X)$. It should be noted that the presence of the ℓ_2-sum is necessary for a characterization of Hilbert space. Johnson ([J79]) constructed a Banach space X not isomorphic to ℓ_2, all of whose quotients of subspaces have a basis.

It should be emphasised that the hypothesis of the theorem gives no *a priori* information on uniform boundedness of the basis constants involved. This produces a strong infinite-dimensional flavor, which could not exist in specific examples. It is surprising that this effect has been obtained by a fundamentally local (finite-dimensional) approach.

Another direction of the interplay between finite- and infinite-dimensional techniques is illustrated by the homogeneous Banach space problem ([Ba32]): *If an infinite-dimensional Banach space X is isomorphic to all of its infinite-dimensional closed subspaces, is X isomorphic to ℓ_2?* As already mentioned, the problem was solved in the positive by combining Gowers' dichotomy theorem [G94a] (Theorem 6 below) and a result from [KT95]. Its history has been explained in detail in [G94b] so we wish to limit ourselves to just a few comments on the local approach involved.

Before going on, we recall the classical definition that non-zero vectors $\{z_i\}$ in a Banach space are *unconditional* if there is C such that for any scalars $\{a_i\}$ and a sequence $\{\varepsilon_i\}$ of signs, one has $\|\sum \varepsilon_i a_i z_i\| \leq C\|\sum a_i z_i\|$.

The first obvious difficulty in attacking the homogeneous space problem is the lack of information on uniform boundedness of norms of the isomorphisms. (Even up to this day no direct proof is known that if X is homogeneous then X is *uniformly* isomorphic to all of its infinite-dimensional subspaces.) Luckily, Gowers' dichotomy theorem combined with properties of H.I. spaces (discussed in the next section), enables us to quickly overcome this difficulty and to conclude that a homogeneous space X must have an unconditional basis. Then the theorem is concluded by a result from [KT95]:

Theorem 5 *Let X be a Banach space with an unconditional basis. Then X contains either ℓ_2 or a subspace without an unconditional basis.*

There are two points worth making. Firstly, there exists a property of a space X, slightly weaker than having an unconditional basis, which is "local", that is, is determined by the behaviour of a certain numerical invariant on all finite-dimensional subspaces of X. Thus, once we find a sequence of finite-dimensional subspaces of X with this invariant tending to infinity, the closed span of these subspaces is a subspace without an unconditional basis. The second point is that under very mild geometric assumptions on X, the construction in Theorem 5 is combinatorial, and the resulting subspace preserves a lot of the structure of X. For example, it admits an unconditional decomposition into 2-dimensional subspaces (we omit a precise definition), and this implies that it has a Schauder basis.

3. Asymptotic infinite-dimensional geometry The asymptotic approach to geometric infinite-dimensional properties can be exemplified by the notion of an asymptotic structure, which depends on possibility of "stabilizing" finite-dimensional subspaces "at infinity" ([MiT93], [MMT95]; a forerunner of this notion was studied in [MiSh79]). We shall give just few examples to indicate the possibilities and directions of such results. The main point is that this is the most general asymptotic geometric notion that can be defined for an *arbitrary* Banach space. It clearly yields a richer theory than the classical notion of spreading models, based on Ramsey's combinatorial theorem (see e.g. in [BL84] for the definition).

For simplicity, we consider a Banach space X with a basis $\{x_i\}$. We need some notation. The set of all positive integers is denoted by $\mathbb{N}$. For $F, G \subset \mathbb{N}$ we write $F < G$ whenever $\max F < \min G$, or either F or G is empty. For a vector $z = \sum a_i x_i \in X$, the support of z is $\operatorname{supp}(z) = \{i \mid a_i \neq 0\}$. A block is a vector with a finite support; blocks are successive, $z_1 < z_2$, whenever $\operatorname{supp}(z_1) <$

supp (z_2). Sequences of vectors $\{e_i\}$ and $\{z_i\}$ are C-equivalent ($C \geq 1$) if for all sequences of scalars $\{a_i\}$ we have $(1/\sqrt{C})\|\sum a_i z_i\| \leq \|\sum a_i e_i\| \leq \sqrt{C}\|\sum a_i z_i\|$.

An n-dimensional normed space E with a basis $\{e_i\}$ is an asymptotic space of X (we write $E \in \{X\}_n$), if there exist successive blocks $z_1, \ldots, z_n$, as close to $\{e_i\}$ as we wish, and arbitrarily far and arbitrarily spread out with respect to the basis. Precisely, given $\varepsilon > 0$, for an arbitrarily large m_1 there is a block z_1 with $\{m_1\} <$ supp (z_1) such that for an arbitrarily large m_2 there is a block z_2 with $\{m_2\} <$ supp (z_2), etc., such that the blocks $\{z_1, \ldots, z_n\}$ obtained after n steps are successive and $(1+\varepsilon)$-equivalent to $\{e_i\}$. The asymptotic structure of X consists of all asymptotic spaces of X.

The concept of asymptotic structure in a natural way describes classes of Banach spaces rather than individual spaces. For example, a space X is called an *Asymptotic-*ℓ_p space, $1 \leq p \leq \infty$ (note the capital A) if there exists C such that for all n and $E \in \{X\}_n$, the basis in E is C-equivalent to the unit vector basis in ℓ_p^n. Thus an Asymptotic-ℓ_p space has the simplest possible asymptotic structure—recall that by Krivine's theorem ([K76]) for every X there is $1 \leq p \leq \infty$ such that $\ell_p^n \in \{X\}_n$ for every n. In fact, a block structure is not so very important in this definition: if the equivalence condition is relaxed to the condition that for all n, all $E \in \{X\}_n$ are C-isomorphic to ℓ_p^n, we still get the same class of Asymptotic-ℓ_p spaces, for $1 < p < \infty$ ([MMT95]).

It can be shown ([MMT95]) that: *If X is an Asymptotic-ℓ_p space ($1 < p < \infty$), there exists C satisfying the condition that for all n, representations of all $E \in \{X\}_n$ which are C-complemented by block projections can be found arbitrarily far and arbitrarily spread out. Conversely, the complementation condition implies that X is an Asymptotic-ℓ_p space for some* $1 \leq p \leq \infty$. (A block projection is a projection of a form $Px = \sum z_i^*(x) z_i$, where the sets supp $(z_i) \cup$ supp (z_i^*), for $i = 1, 2, \ldots$, are successive.) For classical spaces ℓ_p and c_0 the first statement is trivial; but in the asymptotic setting it requires a non-obvious stabilization step. The converse statement seems to have a truly asymptotic nature: the validity of its classical analogue requires strong additional assumptions ([LT71]).

A general stabilization argument shows ([KOS98], [MiT95]) that in every Asymptotic-ℓ_p space X even a higher level of structure can be automatically reached: X contains a subspace Y with a basis such that there exists C that for every n, any n normalized blocks of the basis with supports after n, are C-equivalent to the unit vector basis in ℓ_p^n. Such spaces are called *asymptotic-*ℓ_p, $1 \leq p \leq \infty$.

The first truly non-classical Banach space was discovered by Tsirelson [Ts74]. The implicit definition of its unit ball effectively *saturates* the space with a certain geometric property (i.e., each infinite-dimensional subspace has this property) which prevents the space and its dual from containing ℓ_p, for $1 \leq p < \infty$, or c_0. Saturation of spaces with desired (often complicated) properties is the fundamental ingredient of some spectacular developments of recent years. In the dual setting, put forward in [FJ74], the norm on space T is defined implicitly as the solution of an equation. T and T^* are an asymptotic-ℓ_1 and an asymptotic-ℓ_∞, respectively. A detailed study of these spaces and some of their variants appears in [CS89].

More generally, an investigation of the successive block structure of asymptotic-ℓ_1 spaces was done in [OTW97]. Among other results, natural geometric invariants have been introduced, localized to the Schreier families mentioned below, and certain regular behaviour was established. However, a non-block geometric structure of asymptotic-ℓ_1 spaces may be very diverse: for example a space may contain uniform copies of ℓ_∞^n for all n ([ADKM98]).

On the other hand, truly infinite-dimensional phenomena in general may not stabilize. This was first discovered as a conjunction of two results: a theorem by Milman [Mi69] and the above example by Tsirelson. However, this direction was not pursued for about 15 years. Only in the early 90s it became a central leitmotif in a series of breakthrough results by Gowers and Maurey [GM93], Odell and Schlumprecht [OS93] and Gowers [G94a] (see also the surveys [G94b] and [OS94]).

A passage between finite- and infinite-dimensional geometry may then be achieved by alternating localization and stabilization (as long as possible) of suitable invariants along hierarchies of families (with increasing complexity) of finite subsets of $\mathbb{N}$. This would result in saturating a space with combinatorial structures having required properties: each infinite-dimensional subspace would contain such a structure. An important, and in a sense universal, example of such a hierarchy, which unfortunately we have no place to describe, is given by Schreier families $\{\mathcal{S}_\alpha\}_{\alpha<\omega_1}$, introduced in [AA92]. (The concept of the asymptotic structure discussed above corresponds to family $\mathcal{S}_1$.)

Before we proceed, we need to briefly recall some of the phenomena involved.

In connection with the construction of a Banach space no subspace of which has an unconditional basis, a stronger property was identified in [GM93]: a space X is called *hereditarily indecomposable* (in short, H.I.) if no closed subspace Y of X can be written as a topological direct sum $W \oplus Z$, where W and Z are closed infinite-dimensional subspaces. The space constructed by Gowers and Maurey is H.I. The structure of the algebra $L(X)$ of bounded operators on an H.I. space X is particularly simple ([GM93]): *If X is H.I. and $T \in L(X)$ then $T = \lambda I + S$, where S is a strictly singular operator and λ is a scalar.* It is still an open question whether there exists a Banach space on which every bounded operator is a *compact* perturbation of a scalar, hence admits a non-trivial invariant subspace.

Another inspiring example was constructed by Argyros and Deliyanni [AD97]; their space is H.I. and asymptotic-ℓ_1: any n normalized blocks of the basis with supports after n are 2-equivalent to ℓ_1^n, the lack of stabilization, required in order that a space be H.I., depends on the Schreier families $\mathcal{S}_k$, when $k \to \infty$.

Recall the Gowers dichotomy theorem:

Theorem 6 ([G94a]) *Every Banach space contains a subspace that either has an unconditional basis or is hereditarily indecomposable.*

A Banach space X is called λ-*distortable* if there exists an equivalent norm $|\cdot|$ on X such that $\inf_{Y\subset X} \sup\{|x|/|y| \mid x, y \in Y, \|x\| = \|y\| = 1\} > \lambda$; and is *arbitrarily distortable* if it is λ-distortable for every $\lambda > 1$. For a detailed report on this notion, and in particular, on Schlumprecht's example [Sch91], we refer to [OS93] and [O98]. Here let us only recall the solution of the distortion problem:

Theorem 7 ([OS93]) *ℓ_p for $1 < p < \infty$ is arbitrarily distortable. Every Banach space contains ℓ_1 or c_0 or a λ-distortable subspace, for some $\lambda > 1$.*

A complete characterization of Banach spaces containing arbitrarily distortable subspaces is still unclear. Every Banach space either contains an arbitrarily distortable subspace or it contains a subspace of bounded distortion. This latter property means that there is $C < \infty$ such that any equivalent norm can be stabilized up to C, on a certain infinite-dimensional subspace of any given subspace Y. It was shown in [MiT93] that a space of bounded distortion contains an asymptotic-ℓ_p subspace, for some $1 \leq p \leq \infty$; and it was proved by Maurey ([Ma95]) that an asymptotic-ℓ_p space of type r for some $r > 1$, in which the basis is unconditional, is arbitrarily distortable. (A space has type r for some $r > 1$ if it does not contain copies of ℓ_1^n uniformly for all n.) Having the problem settled for a large class of spaces with an unconditional basis, Theorem 6 suggests that the next important case is that of hereditarily indecomposable spaces. It was widely expected that H.I. spaces should be arbitrarily distortable, and it is indeed so.

Theorem 8 ([T96]) *A Banach space X of bounded distortion contains a subspace with an unconditional basis. Consequently, any H.I. space is arbitrarily distortable.*

The main part of the argument uses the condition of bounded distortion to construct, for some fixed C, trees in X whose finite branches are built from C-unconditional sequences of successive blocks, and which have arbitrarily large countable ordinal index. An easy application of Kunen–Martin boundedness principle (see e.g., [D77]) shows the existence of a C-unconditional tree with an infinite branch, whose linear span will be the subspace with a C-unconditional basis.

As an immediate corollary we get that: *Every Banach space of type r for some $r > 1$ contains an arbitrarily distortable subspace.* This substantially limits the hypothetical possibility of the existence of a distortable space of bounded distortion. It would be very interesting if such a space existed, as it would demonstrate new geometric and combinatorial phenomena. The most prominent candidate is Tsirelson's space T (cf. e.g., [OTW97], [OT98]).

Returning to H.I. spaces, although their structure theory appears to have no bearing on spaces with an unconditional basis, a recent surprising and beautiful result of Argyros and Felouzis [AF98] shows that there is a direct connection between these two classes.

Theorem 9 ([AF98]) *Every Banach space either contains a subspace isomorphic to ℓ_1 or a subspace which is a quotient of an H.I. space. Furthermore, the class of Banach spaces which are quotients of H.I. spaces contains among others: spaces of type r for some $r > 1$ with an unconditional basis (in particular ℓ_p and L_p for $1 < p < \infty$), c_0, Tsirelson's space T and its dual.*

The proof of this result consists of two new essential ingredients. The first is an abstract interpolation scheme (originating in [DFJP74]) that yields a factorization of certain operators through H.I. spaces. The second is a geometric concept of thin sets, combined with an ingenious combinatorial construction of thin norming sets. The proof of the latter statement is geometric, while the former statement uses a rather complicated saturation argument.

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