Rigidity and Renormalization in One Dimensional Dynamical Systems

W. de Melo[1]

Abstract. If two smooth unimodal maps or real analytic critical circle maps have the same bounded combinatorial type then there exists a $C^{1+\alpha}$ diffeomorphism conjugating the two maps along the corresponding critical orbits for some $\alpha > 0$. The proof is based on a detailed understanding of the orbit structure of an infinite dimensional dynamical system: the renormalization operator.

1. Introduction

A smooth discrete dynamical system is generated by a smooth transformation $f: M \to M$ of a compact manifold M called the phase space. Its dynamics involves an infinite number of maps, the iterates of f, defined inductively by $f^1 = f$, $f^n = f \circ f^{n-1}$. Accordingly, for points x in the phase space we have the notions of positive orbit, $\{x \in M: \ f^n(x); n \geq 0\}$, negative orbit, $\{y; f^m(y) = x, m \geq 0\}$ and the grand orbit, $\{y; f^m(y) = f^n(x), m, n \geq 0\}$. In the qualitative theory of dynamical systems, the natural equivalence relation to express the notion of "same dynamics" is conjugacy: f and g are conjugate if there exists a homeomorphism $h: M \to M$ such that $h \circ f = g \circ h$. Such a homeomorphism, called a conjugacy between f and g, maps orbits of f into orbits of g. If $0 \leq r \leq \infty$ then the space of C^r dynamical systems with the C^r topology is a Baire space and if $r < \infty$ it is even a Banach manifold. Similarly, if the parameter space, say P, is also a compact manifold then the space of C^r families of mappings $F: P \times M \to M$ is also a Baire space. Hence, we can talk about typical dynamical systems or typical parametrized families when they belong to a residual subset of the full space (in particular dense). In the case of a given specific parametrized family of dynamical systems, we have a different notion of typical: a property is Lebesgue typical if it is satisfied for maps corresponding to a full Lebesgue measure in the parameter space.

In real one-dimensional dynamical systems, the phase space is either a compact interval or the unit circle. In both cases we have an order structure and we say that two orbits have the same combinatorial type if the mapping that sends the $i-th$ element of one orbit into the $i-th$ element of the other orbit is order preserving. If this correspondence is smooth we say that the orbits have the same geometric type: indeed, a smooth map, being infinitesimally affine, preserves the small scale geometric properties of the orbits (for instance, the Hausdorff dimensions of the closures of the two orbits are the same).

Let us consider the following parametrized families of maps.

$$(1) \qquad q_a: [-1,1] \to [-1,1], \quad q_a(x) = -ax^2 + 1, \quad 0 < a \leq 2$$

[1]This work has been partially supported by the Pronex Project on Dynamical Systems

$$A_{a,b}\colon \mathbf{S}^1 \to \mathbf{S}^1; \qquad A_{a,b}(x) = x + a + \frac{b}{2\pi}\sin(2\pi x) \pmod{1}, \quad \text{where } 0 \le a \le 2 \quad \text{and } 0 \le b \tag{2}$$

All the maps in the family (1) have a unique critical point. If $b < 1$, the mapping $A_{a,b}$ is a circle diffeomorphism; if $b = 1$, it is a critical circle mapping: a smooth homeomorphism with a unique critical point of cubic type and, lastly, for $b > 1$ it is not invertible and the dynamics becomes "chaotic".

These maps have played a crucial role in the development of one-dimensional dynamics for two reasons. First, they exhibit all possible combinatorial behavior for a large class of maps, namely, the maps of the form $\phi \circ q_1 \circ \psi$, where ϕ and ψ are interval diffeomorphisms, which we call *fold maps* (or unimodal maps) and the circle maps of the form $\Phi \circ A_{0,1} \circ \Psi$, where Φ and Ψ are circle diffeomorphisms, and these are called *critical circle maps.* The second reason is that maps in these families exhibit a very complex dynamical behavior, varying wildly with the parameter and thus producing a rich bifurcation set. All these facts were already well established at the end of the 70's when new unexpected quantitative discoveries greatly enhanced the interest of mathematicians and physicists in the subject.

These discoveries came from two different sources. From pure mathematics through the fundamental work of M. Herman on the smoothness of conjugacies between circle diffeomorphisms in [H], [Yoa]. The other input came from physics. Inspired by the scaling laws observed in phase transition and the renormalization group ideas developed in statistical mechanics to explain these phenomena, [W], Feigenbaum [F] and independently Coullet-Tresser [CT], performing numerical experiments with parametrized families of interval maps similar to (1) above, detected similar scaling laws, both in the phase space and parameter space, that were universal in the sense that they were independent of the particular family under consideration. Furthermore, they conjectured that these quantitative properties might be explained using renormalization group ideas adapted to this setting. It would correspond to a dynamical system (the renormalization operator) acting in the space of one-dimensional dynamical systems and the scaling laws observed would be a consequence of the hyperbolicity of a fixed point of this operator. The renormalization operator, which will be defined in section 3, is just the first return map of the original dynamical system to a smaller interval around the critical point, rescaled to the original size. Hence, the iterates of the renormalization operator reveals the small scale geometric properties of the critical orbit. Similar experiments were performed for critical circle mappings in [FKS] and analogous conjectures were formulated [La2], [Ra].

After a computer assisted proof of these conjectures [La1], a great effort was made to provide a conceptual proof of these and some extended conjectures. The main contribution came from Sullivan [Su], who introduced in the theory several new ideas and tools from real and complex analysis. He was able to prove the existence of a Cantor set in the space of fold maps that is invariant under iteration of the renormalization operator and such that the restriction of the operator to this set is a homeomorphism topologically conjugate to a full shift of a finite number of symbols. The Feigenbaum-Coullet-Tresser's fixed point corresponds to one of the fixed point of the shift map. Furthermore, for each fold map f with bounded combinatorial type there exists a map g in the Cantor set, such that the iterates of the renormalization operator at f and g converges to each other. Also, the maps in the Cantor attractor are real analytic with very nice holomorphic extensions to the complex plane: they belong to a compact set in the space of quadratic-like maps in the sense of Douady and Hubbard, [DH]. After this, the powerful arsenal from conformal dynamics could be used and McMullen was able to prove the exponential convergence of iterates of the renormalization operator at quadratic like maps with the same bounded combinatorial type, thus establishing a strong rigidity result for such maps: their critical orbits have the same geometric type (see his paper in this volume and Theorem 3.5 in section 3). Finally, Lyubich in [Ly] proved the full hyperbolicity of the invariant Cantor set in the context of germs of quadratic-like maps. Once again Smale's horse-shoe shows up as a basic ingredient of dynamics!

To extend this result to the setting of smooth dynamical systems one has to overcome many technical difficulties most of them arising from the fact that although the space of dynamical systems is a nice Banach manifold the renormalization operator is not differentiable. The rigidity of the critical orbit of infinite renormalizable smooth maps of bounded type is related to the exponential convergence of iterates of the renormalization operator (Theorem 3.4 below) and an extension of the rigidity result for smooth mappings is discussed in [dMP] A partial result on the hyperbolicity in the setting of smooth maps was obtained in [D] and [FMP] to be discussed in the next section.

The results of Herman for circle diffeomorphisms can also be treated using renormalization ideas, as done in [SK], using only arguments from real analysis as in Herman's original proof.

We also point out that Martens in [M] proved the existence of the periodic points of the renormalization operator using only real analysis, extending the result to a broader class of maps having non-integer power law critical points.

2. Rigidity in phase space and parameter space.

A *fold map* (or unimodal map) is a smooth map f of a compact interval I that has a unique quadratic critical point $c_f \in I$, namely, $f = \phi \circ q \circ \psi$ where ϕ, ψ are $C^r, r \geq 2$, diffeomorphisms of compact intervals and $q(x) = x^2$.

The combinatorial type of any orbit of a fold map is determined by the combinatorial type of the critical orbit (see [MS], pp. 92). Therefore, we say that two fold maps f and g have the same *combinatorial type* if the mapping $f^i(c_f) \mapsto g^i(c_g)$, for $i \in \mathbf{N}$ is order preserving, where $f^1 = f$ and $f^i = f \circ f^{i-1}$ is the i-th iterate of f.

A fold map f is renormalizable if there exists a periodic interval J around the critical point of period $p \geq 2$, i.e., $f^p(J) \subset J$ and the interior of the intervals $J, f(J), \ldots, f^{p-1}(J)$ are pairwise disjoint. Hence, the restriction of f^p to J is again a fold map. To a renormalizable map f we can associate the set of positive integers $\mathcal{P}_f = \{2 \leq q_1 < q_2, \ldots\}$ of periods of renormalization. We say that f is infinitely renormalizable with bounded combinatorial type if the cardinality of $\mathcal{P}_f$ is infinite and the quotient of any two consecutive elements of $\mathcal{P}_f$ is bounded by some integer N. For ∞-renormalizable C^2 maps without periodic attractors, the critical orbit has an even stronger role since its closure is the global attractor: the

ω-limit set of Lebesgue almost all points in the dynamical interval is the closure of the critical orbit.

By combining the results of Sullivan [Su], McMullen, [Mc], [Mcb] and Lyubich [Ly], we prove in [dMP] the following rigidity result:

Theorem 2.1 (Rigidity in phase space). *If f and g are C^2 ∞-renormalizable fold maps with the same bounded combinatorial type, then there exists a $C^{1+\alpha}$ diffeomorphism $h\colon \mathbf{R} \to \mathbf{R}$ such that $h(f^i(c_f)) = g^i(c_g)$ for all $i \in \mathbf{N}$, where the Hölder exponent $\alpha > 0$ depends only on the bound on the combinatorial type.*

Remark 1.. An example in [FM1] can be adapted to show that the above result is false if the combinatorial type is not bounded.

Remark 2.. Even if the maps are very smooth we cannot expect the mapping h to be much smoother. This is in contrast with the case of circle diffeomorphisms treated by Herman, where, if the combinatorics is correct, the conjugacy is C^∞ if the diffeomorphisms are C^∞.

Let us give a geometric interpretation of the above theorem. Consider the complement of the closure of the critical orbit in the complex plane, $S_f = \mathbf{C} \setminus \text{Closure}\,(\{f^n(c_f), n \geq 0\})$, endowed with the hyperbolic metric (complete Riemannian metric of constant curvature -1). If f has bounded combinatorics, then there exists a family of closed geodesics that partition S_f in a countable number of pairs of pants and the lengths of the geodesics are uniformly bounded from above and from below (Corollary 3.1 of section 3). S_f is a tree of pairs of pants connected by the closed geodesics in the boundary, so each pair of pants has a "height" in this tree. If two map f and g have the same combinatorial type, then the partition of S_f and S_g into pairs of pants are isormorphic: there exists a homeomorphisms between the two surfaces respecting the partition and the height. Now, Theorem 2.1 implies that the differences between the lengths of the corresponding geodesics converge to zero exponentially fast so that the corresponding pair of pants becomes closer and closer to being isometric.

Let us formulate a rigidity conjecture in the parameter space.

Conjecture (Rigidity in the parameter space). *Let q_a , $1 \leq a \leq 2$ be the quadratic family $q_a(x) = -ax^2 + 1$. Given $N \geq 2$ and a typical family f_t of C^r fold maps, there exists a $C^{1+\epsilon}$ map $k_N\colon \mathbf{R} \to \mathbf{R}$ such that*

a) f_t is ∞-renormalizable with combinatorial type bounded by N if and only if $q_{k_N(t)}$ has the same combinatorial type as f_t;

b) k_N is piecewise monotone with a finite number of turning points corresponding to maps that are hyperbolic (the critical point of the mapping belongs to the basin of attraction of a periodic point).

Another way to formulate this conjecture is to say that the space of maps of combinatorial type bounded by N is laminated by smooth codimension-one submanifolds consisting of maps with the same combinatorial type and the holonomy of this lamination is $C^{1+\epsilon}$. The quadratic family intersects transversally each leaf of the lamination in a unique point and intersects the lamination in a Cantor set of Hausdorff dimension bigger than 0 and smaller than 1. A typical family is also transversal to the leaves and intersects each leaf in at most a bounded number of points. This conjecture was verified for analytic families of quadratic like maps in [Ly].

We are still far from proving this conjecture in the setting of smooth maps. In this direction, by combining the results of A. M. Davie [D] and of Lyubich [Ly], we get as a consequence of [FMP]:

Theorem 2.2. *If r is big enough then, in the space of C^r fold maps, the set of ∞-renormalizable fold maps with the same combinatorial type bounded by N is a C^1 codimension-one Banach submanifold.*

The combinatorial behavior of a critical circle mapping without periodic points is characterized by a unique real number since any such a map f is combinatorially equivalent to a rigid rotation $R_\alpha\colon x \mapsto x+\alpha \mod 1$, where the irrational number α is called the rotation number of f. Even more related to the dynamics is the set of positive integers that give the continued fraction decomposition of $\alpha = [a_0, a_1, \dots, a_n, \dots]$,

$$[a_0, a_1, \dots, a_n, \dots] = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cfrac{\cdots}{a_n + \cfrac{1}{\cdots}}}}},$$

We say that the combinatorial type of f is bounded by N if $a_i \le N$ for all i. The main result of [FM1] and [FM2] is

Theorem 2.3. *If f and g are real analytic critical circle mappings with the same bounded combinatorial type, then there exists a $C^{1+\alpha}$ conjugacy between f and g where $\alpha > 0$ depends only on the bound on the combinatorial type.*

As in the case of fold maps we cannot expect to have a much better regularity of the conjugacy. We expect the result to hold also for smooth critical circle maps, except when the combinatorics is unbounded in which case we have a counter example in [FM1]. However, Yoccoz proved in [Yob] that two critical circle mappings with the same irrational rotation number are topologically conjugate and in fact, as he proved later in an unpublished paper (see [FM1]), the conjugacy is always quasi-symmetric. This is again in contrast to the situation of circle diffeomorphisms where the conjugacy is not in general quasi-symmetric if the rotation number is Liouville.

The same type of rigidity in parameter space is expected for critical circle maps: for typical one parameter families, the rotation number is a piecewise monotone function of the parameter and the correspondence between parameters corresponding to maps having bounded combinatorial type should be $C^{1+\epsilon}$.

In [Mcc], McMullen proved a result similar to Theorem 2.3 in the context of Siegel discs of quadratic-like maps, see also his article in this volume.

3. The renormalization operator

Any C^r fold map is smoothly conjugate to a C^r fold map $f: [-1,1] \to [-1,1]$ of the form $f = \phi \circ q$ where $q(x) = x^2$ and $\phi: [0,1] \to [-1,1]$ is a C^r embedding with $\phi(0) = 1$. Hence we can restrict our attention to the space $\mathcal{F}^r$ of C^r fold maps so normalized. Let $\mathcal{D}$ be the set of maps in $\mathcal{F}$ that are renormalizable. If $f \in \mathcal{D}$ and its minimum period of renormalization is p we set

$$\mathcal{R}(f)(x) = \frac{1}{\lambda} f^p(\lambda x) \quad \text{where} \quad \lambda = f^p(0) \tag{3}$$

The mapping $\mathcal{R}: \mathcal{D} \to \mathcal{F}$ is the *renormalization operator*. To each $f \in \mathcal{D}$ with minimum renormalization period p we can associate a permutation γ of $\{1, 2, \ldots, p\}$ as follows. Let $I_1, \ldots, I_p$ be a labeling of the intervals J, with end-points $f^p(0), -f^p(0)$, $f(J), \ldots, f^{p-1}(J)$ compatible with their ordering in the real line. Then γ is defined by $f(I_j) \subset I_{\gamma(j)}$. We call γ a unimodal permutation because it has the following properties: i) if we plot the graph of γ and connect the consecutive points by a line segment we get a unimodal map; ii) the iterates of any point by γ is the whole set; iii) there is no partition of the domain in disjoint subsets that are permuted by γ. Conversely, if γ is a unimodal permutation there exists a function in the quadratic family that is renormalizable and has γ as the associated permutation. Therefore, we can write the domain $\mathcal{D}$ of the renormalization operator as a disjoint union of $\mathcal{D}_{\gamma_i}$, where $\{\gamma_i, \quad i = 1, 2 \ldots\}$ is the set of all unimodal permutations. If $r \leq \infty$, then $\mathcal{D}_{\gamma_i}$ are clearly the connected components of $\mathcal{D}$. The intersection of each $\mathcal{D}_{\gamma_i}$ with the quadratic family is an interval (see [MS], pp. 194). From a profound theorem of Yoccoz, [Hu], it follows also that the intersection of $\mathcal{D}$ with the quadratic family is dense in the interval $(0, 1]$.

Two $\infty-$renormalizable fold maps f and g have the same combinatorial type if and only if $\mathcal{R}^n(f)$ and $\mathcal{R}^n(g)$ belong to the same $\mathcal{D}_{\gamma(i)}$ for every $n \geq 0$. Hence, each combinatorial type is given by a sequence $\gamma(i), \quad i \geq 0$, of unimodal permutations and vice-versa.

The non-linearity of a C^2 interval map ϕ at a point $x \in$ is defined as

$$N\phi(x) = \frac{D^2 f(x)}{Df(x)}.$$

Theorem 3.1 (Sullivan [Su]). *There exists a universal constant $d > 0$ such that if $f \in \mathcal{F}^r, \quad r \geq 2$, is infinitely renormalizable with bounded combinatorial type then $\mathcal{R}^n(f) = \phi_n \circ q$, where the non-linearity of ϕ_n is bounded by d for all $n \geq n_0 = n_0(f)$.*

Notice that this is a compactness kind of result since, by Ascoli's theorem, it implies the existence of a subset $\mathcal{K} \subset \mathcal{F}$, which is compact in the C^k topology for $k < r$, such that the iterates of any map by the renormalization operator eventually belong to $\mathcal{K}$. If f is at least C^3, we can also prove theorem 3.1 for maps with unbounded combinatorics.

There is an important consequence on the geometry of the critical orbit. We say that a Cantor set in the complex plane has bounded geometry if there exists a family of closed geodesics of the hyperbolic metric of the complement of the Cantor set that are uniformly bounded from above and from below and that separates the space into pair of pants.

Corollary 3.1. *If f is infinitely renormalizable of bounded combinatorial type, then the closure of the critical orbit is a Cantor set of bounded geometry. In particular, its Hausdorff dimension is bigger than 0 and smaller than one.*

Another important consequence of Theorem 3.1 is that the limit set of the renormalization operator restricted to the subset $\mathcal{D}^{(N)} \subset \mathcal{D}$ of fold maps with renormalizable minimum period $\leq N$, is a compact set in a much stronger topology. To state this result we need the basic definition below.

Definition 3.1. A fold map $f = \phi \circ q$ belongs to the Epstein class $\mathcal{E}$, if ϕ has a holomorphic extension Φ to a topological disc such that: 1) Φ is one-to-one; 2) the image of ϕ is equal to the topological disc $\mathbf{C}(L) = (\mathbf{C} \setminus \mathbf{R}) \cup L$, where L is an interval containing the image of ϕ.

For holomorphic maps defined on topological discs we may consider the Carathéodory topology [Mc]. This is a very strong topology. Indeed, if a holomorphic extension F_n of a fold map f_n converges to a holomorphic extension F of f, then there exist a neighborhood U of the dynamical interval which is contained in the domains of all the maps and such that the restriction of the sequence to U converges uniformly to the restriction of f. In particular f_n converges to f in the C^r topology for any r, namely $|f_n - f|_r \to 0$, where $|f_n - f|_r = \sup\{|f_n(x) - f(x)|, \ldots, |D^r f_n(x) - D^r f(x)|\}$.

Corollary 3.2. *There exists a compact subset $\mathcal{C}_N \subset \mathcal{E}$ in the Carathéodory topology such that if f is an ∞-renormalizable C^2 fold map of combinatorial type bounded by N , there are constants $C > 0$ and $0 < \lambda < 1$ and a sequence f_n having a holomorphic extension in $\mathcal{C}_N$ such that $|\mathcal{R}^n(f) - f_n|_0 \leq C\lambda^n$. In particular all the limit set of iterates of the renormalization operator at maps of combinatorial type bounded by N is contained in $\mathcal{C}_N$.*

The proof of Theorem 3.1 and its corollaries involve only real analysis and the main ingredient is the Koebe's distortion theorem and the control of the distortion of the cross ratio under iteration [MS].

The maps of the Epstein class have the important property that each branch of its inverse contracts the hyperbolic map of the upper half space by the Schwarz Lemma. This is the basic tool in the proof of the next theorem which is a bridge between real and complex dynamics.

Definition 3.2. A quadratic-like map is a holomorphic map $F\colon U \to V$ between topological discs U and V such that F is proper, two-to-one, and the open disc V contains the closure of U. The modulus of F is defined as the conformal modulus of the annulus $V \setminus \text{Closure}(U)$.

Theorem 3.2 (Sullivan [Su]). *For each N, there exists a subset $\mathcal{S}$ of quadratic-like maps which is compact in the Carathéodory topology and $0 < \lambda < 1$ such that if f is an infinitely renormalizable C^2 fold map of combinatorial type bounded by N, then there exists a sequence of fold maps f_n having holomorphic extensions in $\mathcal{S}$ and a positive constant $C > 0$ so that $|\mathcal{R}^n(f) - f_n|_0 \leq C\lambda^n$. In particular all maps in the limit set of the renormalization operator restricted to the set of maps of combinatorial type bounded by N have holomorphic extensions in $\mathcal{S}$.*

If $F: U \to V$ is a quadratic-like map then the set $K_F = \{z \in U; F^n(z) \in U \ \forall n \geq 0\}$ is called the filled-in Julia set of F and its boundary is the Julia set of F. If F is the extension of a fold map, then K_F contains the dynamical interval $[-1, 1]$ and all its pre-images. If f is infinitely renormalizable the filled-in Julia set has empty interior and is equal to the Julia set. Theorem 3.1 combined with Sullivan's pull-back argument gives the following:

Corollary 3.3. *If F and G are quadratic-like holomorphic extensions of ∞-renormalizable fold maps of the same bounded combinatorial type then there exists a quasi-conformal conjugacy between F and G in a neighborhood of the Julia set.*

Using Corollary 3.3 and the measurable Riemann mapping theorem, one can perform quasi-conformal deformations of germs of quadratic-like maps that are extensions of fold maps. Using this and some extensions of the Teichmüler theory to Riemann surfaces laminations connected to such germs, Sullivan arrived at the theorem below that describes completely the dynamics of the renormalization operator in the space of maps of bounded combinatorial type. Let P_N be the finite set of unimodal permutations of length at most N. Let Σ_N be the set of biinfinite sequences $\theta: \mathbf{Z} \to P_N$ endowed with the product topology and let $\sigma: \Sigma_N \to \Sigma_N$ be the shift homeomorphism $\sigma(\theta)(i) = \theta(i+1)$.

Theorem 3.3 (Sullivan [Su]). *There exists a one to one continuous mapping $\theta \in \Sigma_N \mapsto f_\theta \in \mathcal{F}$ with the following properties:*

1 f_θ is ∞-renormalizable of combinatorial type $(\theta(0), \theta(1), \dots)$ and has a quadratic like extension that belongs to the compact set $\mathcal{S}$.

2 $\mathcal{R}(f_\theta) = f_{\sigma(\theta)}$

3 If f is a C^2 infinitely renormalizable map of the same combinatorial as f_θ then $|\mathcal{R}^n f - \mathcal{R}^n f_\theta|_0$ converges to zero as $n \to \infty$.

The relevance of the convergence of the renormalization operator to the rigidity problem in phase space is given by the following result, which is proved using again Theorem 3.1 (see [MS], pp. 546):

Theorem 3.4. *Let f and g be two C^2, infinitely renormalizable, fold maps with the same bounded combinatorial type. If $|\mathcal{R}^n(f) - \mathcal{R}^n(g)|_0$ converges to zero exponentially fast, then there exists a $C^{1+\alpha}$ diffeomorphism of the real line that conjugates the two maps along the critical orbits.*

The compact set $\Lambda_N = \{f_\theta; \theta \in \Sigma_N\}$ is called the renormalization limit set since, by Theorem 3.3, the union of the basin of attraction of the functions in Λ_N is equal to the set of all infinitely renormalizable maps of combinatorial type bounded by N. However Theorem 3.3 does not give yet a rate of convergence. The first main step in getting the exponential convergence needed in Theorem 3.4 comes from the work of McMullen. Using Sullivan's compactness theorem and a rigidity result a la Mostow in the geometric limit of renormalization, he was able to prove that the quasi-conformal conjugacy in Corollary 3.3 is in fact $C^{1+\alpha}$ at the critical point (see [Mc], [Mcb] and his article in this volume) proving the following:

Theorem 3.5 (McMullen). *Let f and g be ∞-renormalizable fold maps with the same bounded combinatorial type. If f and g have quadratic-like extensions then $|\mathcal{R}^n(f) - \mathcal{R}^n(g)|_0$ converges to zero exponentially fast.*

The final basic step is Lyubich's hyperbolicity of the renormalization limit set Λ_N in the space of germs of quadratic like maps [Ly]. His result implies that the iterates of the renormalization operator expand exponentially the distance between two maps with different combinatorics. Even the precise formulation of this statement is subtle because there is no natural domain of definition for the holomorphic extensions of the maps in the Cantor set Λ_N and, therefore, we do not have a Banach space of maps where the operator acts smoothly. One of the major achievements is Lyubych's paper is to provide a natural complex analytic structure to the set of germs of quadratic-like maps (modeled in a direct set of Banach spaces) and a complex holomorphic operator with respect to this structure that restricts to the renormalization operator. It is with respect to this structure that he formulates and proves the hyperbolicity of the Λ_N.

In [dMP] we use the expanding direction of Λ_N, in the space of germs of quadratic-like maps, to improve Theorem 3.2: the iterates of the renormalization operator at an infinitely renormalizable map f of bounded combinatorial type can be exponentially approximated by a map f_n in $\mathcal{S}$ with the same combinatorics as $\mathcal{R}^n(f)$. Combining this with McMullen's exponential contraction we arrive at Theorem 2.1.

In [FMP], we translate Lyubich's hyperbolicity statement in terms of an operator acting on an open set of a finite union of Banach spaces of holomorphic functions containing the renormalization limit set Λ_N as a hyperbolic set in the usual sense and such that the new operator restricts to an iterate of the renormalization operator. Extending to this setting the analytic estimates of [D], we show that the hyperbolicity feature persists in the space of C^r fold maps for r big enough with C^1 local stable manifolds (and real analytic local unstable manifolds given by Lyubich).

In a remarkable paper [Lyb], Lyubich proved the hyperbolicity of the renormalization operator in the space of all renormalizable maps including those of unbounded type. In particular he was able to extend Theorem 3.5 to maps with any combinatorial type. However, this is not sufficient to establish a rigidity result which, as we pointed out before, is false at least for smooth mappings.

4. Renormalization of critical circle mappings

Let us consider a critical circle mapping f without periodic points whose rotation number has the following continued fraction expansion: $\rho(f) = [a_0, \dots a_n, \dots]$. When the partial quotients a_n are bounded, we say that $\rho(f)$ is a number of *bounded type*

The denominators of the convergents of $\rho(f)$, defined recursively by $q_0 = 1$, $q_1 = a_0$ and $q_{n+1} = a_n q_n + q_{n-1}$ for all $n \geq 1$, are the *closest return times* of the orbit of any point to itself. We denote by Δ_n the closed interval containing c whose endpoints are $f^{q_n}(c)$ and $f^{q_{n+1}}(c)$. We also let $I_n \subseteq \Delta_n$ be the closed interval whose endpoints are c and $f^{q_n}(c)$. Observe that $\Delta_n = I_n \cup I_{n+1}$. The most important combinatorial fact in the study of the geometry of a circle map is that for each n the collection of intervals

$$\mathcal{P}_n = \Big\{ I_n,\, f(I_n), \ldots ,\, f^{q_{n+1}-1}(I_n) \Big\} \cup \Big\{ I_{n+1},\, f(I_{n+1}), \ldots ,\, f^{q_n-1}(I_{n+1}) \Big\}$$

constitutes a partition of the circle (modulo endpoints), called *dynamical partition of level* n of the map f. Note that, for all n, $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$.

Of course, these definitions make sense for an arbitrary homeomorphism of the circle. For a rigid rotation, we have $|I_n| = a_{n+1}|I_{n+1}| + |I_{n+2}|$. Therefore, if a_{n+1} is very large then I_n is much longer than I_{n+1}. It is a remarkable fact, first proved by Świątek and Herman, that this cannot happen for a critical circle map! Indeed, the dynamical partitions $\mathcal{P}_n$ have *bounded geometry*, in the sense that adjacent atoms have comparable lengths.

Commuting pairs and renormalization. Let f be a critical circle map as before, and let $n \geq 1$. The first return map $f_n : \Delta_n \to \Delta_n$ to $\Delta_n = I_n \cup I_{n+1}$, called the *$n$-th renormalization of f without rescaling*, is determined by a pair of maps, namely $\xi = f^{q_n} : I_{n+1} \to \Delta_n$ and $\eta = f^{q_{n+1}} : I_n \to \Delta_n$. This pair (ξ, η) is what we call a *critical commuting pair*. Each f_{n+1} is by definition the *renormalization without rescaling* of f_n. Conjugating f_n by the affine map that takes the critical point c to 0 and I_n to $[0,1]$ we obtain $\mathcal{R}^n(f)$, the n-th renormalization of f.

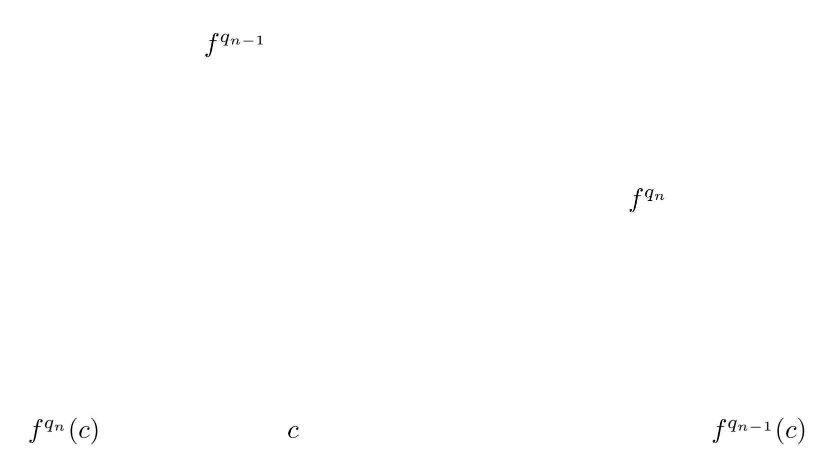

Figure 1. Two consecutive renormalizations of f.

Holomorphic pairs. The concept of *holomorphic commuting pair* was introduced in E. de Faria's thesis,[dF], and plays a crucial role in the proof of theorem 2.2. We recall the definition and some of the relevant properties of these objects, henceforth called simply *holomorphic pair*. Assume we are given a configuration of four simply-connected domains $\mathcal{O}_\xi, \mathcal{O}_\eta, \mathcal{O}_\nu, \mathcal{V}$ in the complex plane, called a *bowtie*, such that

(a) Each $\mathcal{O}_\gamma$ is a Jordan domain whose closure is contained in $\mathcal{V}$;

(b) We have $\overline{\mathcal{O}}_\xi \cap \overline{\mathcal{O}}_\eta = \{0\} \subseteq \mathcal{O}_\nu$;

(c) The sets $\mathcal{O}_\xi \setminus \mathcal{O}_\nu$, $\mathcal{O}_\eta \setminus \mathcal{O}_\nu$, $\mathcal{O}_\nu \setminus \mathcal{O}_\xi$ and $\mathcal{O}_\nu \setminus \mathcal{O}_\eta$ are non-empty and connected.

A holomorphic pair with domain $\mathcal{U} = \mathcal{O}_\xi \cup \mathcal{O}_\eta \cup \mathcal{O}_\nu$ is the dynamical system generated by three holomorphic maps $\xi : \mathcal{O}_\xi \to \mathbb{C}$, $\eta : \mathcal{O}_\eta \to \mathbb{C}$ and $\nu : \mathcal{O}_\nu \to \mathbb{C}$ satisfying the following conditions.

[H_1] Both ξ and η are univalent onto $\mathcal{V} \cap \mathbb{C}(\xi(J_\xi))$ and $\mathcal{V} \cap \mathbb{C}(\eta(J_\eta))$ respectively, where $J_\xi = \mathcal{O}_\xi \cap \mathbb{R}$ and $J_\eta = \mathcal{O}_\eta \cap \mathbb{R}$. (Notation: $\mathbb{C}(I) = (\mathbb{C} \setminus \mathbb{R}) \cup I$.)

[H_2] The map ν is a 3-fold branched cover onto $\mathcal{V} \cap \mathbb{C}(\nu(J_\nu))$, where $J_\nu = \mathcal{O}_\nu \cap \mathbb{R}$, with a unique critical point at 0.

[H_3] We have $\mathcal{O}_\xi \ni \eta(0) < 0 < \xi(0) \in \mathcal{O}_\eta$, and the restrictions $\xi|[\eta(0), 0]$ and $\eta|[0, \xi(0)]$ constitute a critical commuting pair.

[H_4] Both ξ and η extend holomorphically to a neighborhood of zero, and we have $\xi \circ \eta(z) = \eta \circ \xi(z) = \nu(z)$ for all z in that neighborhood.

[H_5] There exists an integer $m \geq 1$, called the *height* of Γ, such that $\xi^m(a) = \eta(0)$, where a is the left endpoint of J_ξ; moreover, $\eta(b) = \xi(0)$, where b is the right endpoint of J_η. The interval $J = [a, b]$ is called the *long dynamical interval* of Γ, whereas $\Delta = [\eta(0), \xi(0)]$ is the *short dynamical interval* of Γ. They are both forward invariant under the dynamics. The *rotation number* of Γ is by definition the rotation number of the critical commuting pair of Γ (condition H_3).

In the figure below the solid lines are mapped into the real axis and the heavier solid lines are mapped into the interval J. Notice that ν is not a polynomial-like map of degree three because the interval to the left of $\eta(0)$ in the domain of ν is in the image of the boundary of $\mathcal{O}_\nu$.

Figure 2. Holomorphic commuting pair.

A fundamental step in the proof of Theorem 2.3 is the statement that a sufficiently high renormalization of a real analytic critical circle map has holomorphic extension belonging to a compact set of holomorphic pairs. From that we prove that the limit set of this pair is "chaotic" and that the critical point is a deep point in the limit set in the sense of [Mcb]. From this point on we use McMullen's machinery, developed in chapter 9 of [Mcb], to prove Theorem 2.3.

REFERENCES

[CT] P. Coullet and C. Tresser, *Itération d'endomorphismes et groupe de renormalization.*, C. R. Acad. Sci. Paris **287A** (1978), 577–580.

[D] A.M. Davie, *Periodic doubling for $C^{2+\epsilon}$ mappings*, Commun. Math. Phys. **176** (1996), 262–272.

[DH] A. Douady and J.H. Hubbard, *On the dynamics of polynomial-like maps*, Ann.Sc.Éc.Norm.Sup. **18** (1985), 287–346.

[dF] E. de Faria, *Asymptotic rigidity of scaling ratios for critical circle mappings*, IMS Stony Brook preprint 96/13, to appear in Erg.Th.Dynam.Sys..

[FM1] E. de Faria and W. de Melo, *Rigidity of critical circle mappings I*, IMS Stony Brook preprint 97/16.

[FM2] E. de Faria and W. de Melo, *Rigidity of critical circle mappings II* (1997), IMS Stony Brook preprint 97/17.

[FMP] E. de Faria W. de Melo and A. A. Pinto, *Global hyperbolicity of renormalization for C^r unimodal mappings*, In preparation..

[F] M. J. Feigenbaum, *Quantitative universality for a class of non-linear transformations*, J. Stat. Phys. **19** (1978), 25–52.

[FKS] M. J. Feigenbaum L. P. Kadanoff and S. J. Shenker, *Quasiperiodicity in dissipative systems: a renormalization group analysis.*, Phys. D **5** (1982), 370–386.

[H] M. Herman, *Sur la conjugaison differentiable des difféomorphismes du cercle a des rotations*, Publ. Math. IHES **49** (1979), 5–234.

[Hu] J. H. Hubbard, *Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz*, Topological Methods in Modern Mathematics, A Symposium in Honor of John Milnor, Publish or Perish, Inc, 1993, pp. 467–511.

[La1] O. Lanford III, *A computer assisted proof of the Feigenbaum conjectures.*, Bull. Amer. Math. Soc. **6** (1982), 427–434.

[La2] O. Lanford III, *Renormalization group methods for critical circle mappings with general rotation number*, VIIIth International Congress on Mathematical Physics (Marseille, 1986), World Sci. Publishing, Singapore, 1987, pp. 532–536.

[Ly] M. Lyubich, *Feigenbaum-Coullet-Tresser Universality and Milnor's Hairiness Conjecture*, Annals of Mathematics, to appear (1998).

[Lyb] M. Lyubich, *Almost every real quadratic map is either regular or stochastic*, IMS Stony Brook preprint 97/8 (1998).

[M] M.. Martens, *The periodic points of renormalization.*, IMS Stony Brook preprint 96/3, to appear in Annals of Math. (1996).

[Mc] C. McMullen, *Complex dynamics and renormalization*, Annals of Math Studies **135** (1994), Princeton University Press, Princeton.

[Mcb] ——, *Renormalization and 3-manifolds which fiber over the circle*, Annals of Math Studies **142** (1996), Princeton University Press, Princeton.

[Mcc] ——, *Self-similarity of Siegel disks and the Hausdorff dimension of Julia sets.*, Acta Mathematica, to appear.

[dMP] W. de Melo and A. A. Pinto, *Rigidity of C^2 infinitely renormalizable quadratic maps*, In preparation..

[MS] W. de Melo and S. van Strien, *One dimensional dynamics*, Springer-Verlag, Berlin and New York, 1993.

[Ra] D. Rand, *Global phase-space universality, smooth conjugacies and renormalization: the $C^{1+\alpha}$ case*, Nonlinearity **1** (1988), 181–202.

[SK] Ya. G. Sinai, K.M Khanin, *Smoothness of conjugacies of diffeomorphisms of the circle with rotations.*, Russian Math. Surveys **45** (1989), 69–99.

[Su] D. Sullivan, *Bounds, quadratic differentials and renormalization conjectures*, Mathematics into the Twenty-First Century, Amer. Math. Soc. Centennial Publication, vol.2, Amer. Math. Soc., Providence, RI, 1991.

[W] K. G. Wilson, *The renormalization group and critical phenomena*, Reviews of Modern Physics **55** (1983), 583–600.

[Yoa] J. C. Yoccoz, *Conjugaison differentiable des diffeomorphismes du cercle dont le nombre de rotation verifie une condition Diophantienne*, Ann. Sci. de l'Ec. Norm. Sup. **17** (1984), 333–361.

[Yob] J. C. Yoccoz, *Il n'y a pas de contre-example de Denjoy analytique,*, C. R. Acad. Sci. Paris **298** (1984), 141–144.

W. de Melo
IMPA,
Estrada Dona Castorina 110,
Jardim Botânico, CEP22460-320
Rio de Janeiro RJ - Brasil
email: demelo@impa.br