Viscosity Solutions of Elliptic Partial Differential Equations

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Abstract. In my talk and its associated paper I shall discuss some recent results connected with the uniqueness of viscosity solutions of nonlinear elliptic and parabolic partial differential equations. By now, most researchers in partial differential equations are familiar with the definition of viscosity solution, introduced by M. G. Crandall and P. L. Lions in their seminal paper, "Condition d'unicité pour les solutions generalisées des équations de Hamilton-Jacobi du premier order," *C. R. Acad. Sci. Paris* 292 (1981), 183–186. Initially, the application of this definition was restricted to nonlinear first order partial differential equations—i.e., Hamilton-Jacobi-Bellman equations—and it was shown that viscosity solutions satisfy a maximum principle, implying uniqueness. In 1988 an extended definition of viscosity solution was applied to second order partial differential equations, establishing a maximum principle for these solutions and a corresponding uniqueness result. In the following years numerous researchers obtained maximum principles for viscosity solutions under weaker and weaker hypotheses. However, in all of these papers it was necessary to assume some minimal modulus of spatial continuity in the nonlinear operator, depending on the regularity of the solution, and to assume either uniform ellipticity or strong monotonicity in the case of elliptic operators. The results I shall discuss are related to attempts to weaken these assumptions on the partial differential operators—e.g., operators with only measurable spatial regularity, and operators with degenerate ellipticity.

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1 Viscosity solutions: a brief history

Although the history of viscosity solutions begins in 1981/83, depending on your individual bias, an important precursor is found in the work of S. N. Kruzkov. In fact, it's noted in [12] that, "analogies with S. N. Krukov's theory of scalar conservation laws ([29]) provided guidance for the notion [of viscosity solutions] and its presentation." In this context one should also mention L. C. Evans [16], which developed techniques that serendipitously anticipated the introduction of viscosity solutions.

M. G. Crandall and P. L. Lions announced the discovery of viscosity solutions in 1981 ([10]). Complete proofs and details were presented shortly after this in their landmark paper [11]. However, the definition of viscosity solution used in this paper bears little resemblance to any of those we now employ. It is in M. G. Crandall, L. C. Evans and P. L. Lions [9] where we first see a systematic use of one of the now familiar definitions of viscosity solutions. P. L. Lions was quick to grasp the potential in extending the notion of viscosity solutions to more general PDEs—[10] and [11] only deal with first order Hamilton-Jacobi-Bellman equations. His papers, [30] and [31], are the first attempts to extend the first order results of [11] to second order equations. Using stochastic control theory, he was able to prove a maximum principle for viscosity solutions of convex (or concave) nonlinear second order Hamilton-Jacobi equations.

It was five years later that methods were developed which extended the theory of viscosity solutions to fully nonlinear second order elliptic PDEs. In the first of these papers R. Jensen [24] proved a maximum principle for Lipschitz viscosity solutions to the fully nonlinear second order elliptic PDE on a bounded domain $\Omega \subset \mathbf{R}^n$

$$F(u, Du, D^2u) = 0 \quad \text{in} \quad \Omega \tag{1}$$

Next, in a short note R. Jensen, P. .L. Lions, and P. E. Souganidis [28] removed the hypothesis of Lipschitz continuity from the viscosity solution. At about the same time, using the ideas in [24], N. Trudinger proved $C^{1,\alpha}$ regularity for viscosity solutions of uniformly elliptic problems ([35]), and a maximum principle for such solutions ([36]). Then H. Ishii [20] made an important contribution by removing the assumption of spatial independence in the PDE. I.e., the maximum principle could now be applied to viscosity solutions of

$$F(x, u, Du, D^2u) = 0 \quad \text{in} \quad \Omega \tag{2}$$

Finally, in concurrently developed papers H. Ishii and P. L. Lions [22], and R. Jensen [25] significantly extended [20] giving very general (and in [25], a rather complicated technical) conditions under which a maximum principle holds for viscosity solutions of (2). In particular, suppose the functions $F(x,t,p,M)$ appearing in (2) is given by the formula

$$F(x,t,p,M) = \min_{\beta\in\mathcal{B}} \left\{ \max_{\gamma\in\mathcal{C}} \left\{ a_{il}^{\beta\gamma}(x) a_{jl}^{\beta\gamma}(x) m_{ij} + b_i^{\beta\gamma}(x) p_i - c^{\beta\gamma}(x) t - h^{\beta\gamma}(x) \right\} \right\} \tag{3}$$

where $M = (m_{ij}),\ p = (p_1, \ldots, p_n)$ and summation is implicit over the indices $i, j,$ and l. Then we have from [25]

COROLLARY 5.11. *Let F be the function defined by* (3) *and assume $\left\{\left(a_{rs}^{\beta\gamma}(x)\right)\right\}$ are uniformly Lipschitz continuous in $\overline{\Omega}$, $\left\{\left(b_i^{\beta\gamma}(x)\right)\right\}$ are uniformly Lipschitz continuous in $\overline{\Omega}$, and $\left\{\left(c^{\beta\gamma}(x)\right)\right\}$ and $\left\{\left(h^{\beta\gamma}(x)\right)\right\}$ are equicontinuous in $\overline{\Omega}$. If u is a* *viscosity subsolution of* (2)*, v is a viscosity supersolution of* (2)*, and*

$$F(x,t,p,M)-F(x,s,q,N)\leq \max\left\{K_1\text{trace}(M-N),K_2(s-t)\right\}+K_3|p-q| \quad (4)$$

then

$$\sup_{\Omega}(u-v)^+\leq \sup_{\partial\Omega}(u-v)^+ \quad (5)$$

We also have two other corollaries from [25] which demonstrate the link between the spatial dependence of F and the regularity of the viscosity solution.

Corollary 5.14. *Let F be the function defined by* (3) *and assume $\left\{\left(a_{rs}^{\beta\gamma}(x)\right)\right\}$ are uniformly Hölder continuous with exponent $\gamma(>1/2)$ in $\overline{\Omega}$, $\left\{\left(b_i^{\beta\gamma}(x)\right)\right\}$ are uniformly Hölder continuous with exponent $2\gamma-1$ in $\overline{\Omega}$, and $\left\{\left(c^{\beta\gamma}(x)\right)\right\}$ and $\left\{\left(h^{\beta\gamma}(x)\right)\right\}$ are equicontinuous in $\overline{\Omega}$. If u is a viscosity subsolution of* (2)*, v is a viscosity supersolution of* (2)*, either u or v is Hölder continuous with exponent $\alpha>2-2\gamma$, and* (4) *holds, then*

$$\sup_{\Omega}(u-v)^+\leq \sup_{\partial\Omega}(u-v)^+ \quad (6)$$

Corollary 5.16. *Let F be the function defined by* (3) *and assume $\left\{\left(a_{rs}^{\beta\gamma}(x)\right)\right\}$ are uniformly Hölder continuous with exponent $\gamma(\leq 1/2)$ in $\overline{\Omega}$, $\left\{\left(b_i^{\beta\gamma}(x)\right)\right\}$ are equicontinuous in $\overline{\Omega}$, and $\left\{\left(c^{\beta\gamma}(x)\right)\right\}$ and $\left\{\left(h^{\beta\gamma}(x)\right)\right\}$ are also equicontinuous in $\overline{\Omega}$. If u is a viscosity subsolution of* (2)*, v is a viscosity supersolution of* (2)*, either u or v is in $C^{1,\alpha}(\Omega)$ for some $\alpha\geq\frac{1-2\gamma}{1-\gamma}$, and* (4) *holds, then*

$$\sup_{\Omega}(u-v)^+\leq \sup_{\partial\Omega}(u-v)^+ \quad (7)$$

While the preceding results are not sharp, they do indicate how the assumption of greater regularity of the viscosity solution allows us to reduce the regularity in the spatial dependence of F necessary to prove a maximum principle. Specifically, in conjunction with regularity results about the gradient (e.g., [35]), one obtains a fairly general maximum principle (compare [36]).

It was also during this period that L. Caffarelli's famous paper [3] on interior *a priori* estimates for viscosity solutions appeared. It was in this paper that Caffarelli extended the classical $W^{2,p}$, $C^{1,\alpha}$, and$C^{2,\alpha}$ interior estimates, using the Aleksandov-Bakelman-Pucci maximum principle, the Calderon-Zygmund decomposition lemma, and an extremely clever application of the Krylov-Safonov Harnack inequality. By eschewing the traditional approach used for linear PDEs—singular integral operator theory—he obtains results which are powerful enough to apply to fully nonlinear uniformly elliptic operators.

2 Viscosity solutions: recent results

The two most exciting (or depressing, depending on your point of view) recent results are a pair of counterexamples due to N. Nadirashvili. The first ([32]), is an example of nonuniqueness for linear uniformly elliptic PDEs with bounded, measurable coefficients. I.e., consider the equation

$$\left.\begin{array}{l}\displaystyle\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} = f(x) \quad \text{in} \quad \Omega \subset \mathbf{R}^n \\ u|_{\partial\Omega}(x) = g(x)\end{array}\right\} \tag{8}$$

If $(a_{ij}(x))$ are bounded, measurable and uniformly elliptic, f is bounded and measurable, and g is bounded and continuous we may define a solution of (8) as a limit of solutions of

$$\left.\begin{array}{l}\displaystyle\sum_{i,j=1}^{n} a^k_{ij}(x)\frac{\partial^2 u^k}{\partial x_i \partial x_j} = f(x) \quad \text{in} \quad \Omega \subset \mathbf{R}^n \\ u^k|_{\partial\Omega}(x) = g(x)\end{array}\right\} \tag{9}$$

where $\left\{\left(a^k_{ij}(x)\right)\right\}$ are smooth and converge almost everywhere to $(a_{ij}(x))$. The sequence $\left\{u^k\right\}$ is equicontinuous due to Krylov's Hölder continuity estimates. Hence, the sequence has accumulation points. We may view these accumulation points as "good" solutions of (8). If there is only one accumulation point no matter what approximating sequence we use, then (in some sense) the "good" solution of (8) is unique.

Under certain conditions it is possible to prove that "good" solutions of (8) are unique. For example, M. C. Cerutti, L. Escauriaza, and E. B. Fabes [6] prove this if the set of discontinuities of $(a_{ij}(x))$ is countable with at most one accumulation point. M. Safonov [34] proves uniqueness if the set of discontinuities of $(a_{ij}(x))$ has sufficiently small Hausdorff dimension. In this connection R. Jensen [27] defines a measure theoretic notion of viscosity solution and proves that viscosity solutions and "good" solutions are equivalent. A continuous function $u \in C(\overline{\Omega})$ is a viscosity subsolution of (8) if for any $\phi \in C^2(\Omega)$ such that $(u-\phi)(x) \geq (u-\phi)(y)$ for all $y \in \Omega$ and for all $\eta > 0$

$$\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon^n} \int_{\mathrm{B}(x,\varepsilon)} \left[\sum_{i,j=1}^{n} a_{ij}(y)\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) + \eta\delta_{ij}\right) - f(y)\right]^+ dy > 0 \tag{10}$$

it's a viscosity supersolution if for any $\phi \in C^2(\Omega)$ such that $(u-\phi)(x) \leq (u-\phi)(y)$ for all $y \in \Omega$ and for all $\eta > 0$

$$\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon^n} \int_{\mathrm{B}(x,\varepsilon)} \left[\sum_{i,j=1}^{n} a_{ij}(y)\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) - \eta\delta_{ij}\right) - f(y)\right]^- dy > 0 \tag{11}$$

and it's a viscosity solution if it's both a subsolution and a supersolution. It's relatively easy to see that a "good" solution is always a viscosity solution. Amazingly, it's also possible to show that if u is a viscosity solution of (8), then there is a sequence of coefficients $\{(a_{ij}^k(x))\}$ converging to $(a_{ij}(x))$ such that the solutions $\{u^k\}$ of (9) converge to u.

It follows that viscosity solutions is the "right" or "natural" space to work in when studying solutions of (8). The counterexample of [32] shows that multiple viscosity solutions of (8) *do* can exist. I.e., viscosity solutions of (8) are *not* unique. Still, [27] has some interesting consequences. For example, suppose $(a_{ij}(x))$ are continuous. Then we know from the general theory of linear PDEs that there is a solution $w \in W^{2,p}(\Omega) \cap C(\overline{\Omega})$ for any $p > n$. Such solutions are unique and stable. It now follows from [27] that if u is a viscosity solution of (8), then $u = w$. Thus, if $(a_{ij}(x))$ are continuous, then viscosity solutions of (8) are in $W^{2,p}(\Omega)$. In a pair of papers related to [27], [5] and [8], L. Caffarelli, , M. G. Crandall, M. Kocan, P. Soravia, and A. Święch examine the notion of a L^p-viscosity solutions. In the context of (8) a function $u \in W^{2,p}(\Omega)$ for $p > n/2$ is a L^p-viscosity subsolution of (8) if for any $\phi \in W^{2,q}_{loc}(\Omega)$ such that $q > p$ and $(u-\phi)(y)$ has a local max at $y = x$ then

$$\operatorname{ess} \limsup_{y \to x} \left\{ \sum_{i,j=1}^n a_{ij}(y) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(y) - f(y) \right\} \geq 0 \tag{12}$$

it's a L^p-viscosity supersolution if for any $\phi \in W^{2,q}_{loc}(\Omega)$ such that $q > p$ and $(u - \phi)(y)$ has a local min at $y = x$ then

$$\operatorname{ess} \liminf_{y \to x} \left\{ \sum_{i,j=1}^n a_{ij}(y) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(y) - f(y) \right\} \leq 0 \tag{13}$$

and it's a L^p-viscosity solution if it's both a subsolution and a supersolution. The authors prove a variety of interesting results concerning such solutions. In particular they they show that that such solutions are twice differentiable almost everywhere, they examine the relationship between various definitions of viscosity solutions (in the measurable context), and they extend and generalize the results in [27]. One of the tools in their analysis is the interesting paper of L. Escauriaza ([15]), which extends the classical Aleksandrov-Bakelman-Pucci maximum principle.

Nadirashvili's second counterexample, [33], shows that there is a smooth function F such that the solution of (2) is *not* C^2. This is important because this result shows that the $C^{2,\alpha}$ regularity theory—the Schauder estimates—of linear PDEs doesn't hold for fully nonlinear PDEs, underscoring the importance of the theory of viscosity solutions to elliptic PDEs. Applications of viscosity solutions to degenerate elliptic and parabolice PDEs also underscore their importance. One of the more widely known applications has been to the problem of motion by mean curvature. The idea of embedding the hypersurface as a level set of some initial value and evolving the initial data by the appropriate degenerate parabolic PDE goes back to L. C. Evans and J. Spruck [19], and Y. G. Chen, Y. Giga, and S. Goto [7]. Showing that the level set's evolution was independent of the particular initial data used, they were able to prove existence and uniqueness results for the motion by mean curvature problem. These results have been expanded on and generalized in L. C. Evans [18], and H. Ishii, and P. E. Souganidis [23].

In a different vein R. Jensen [26] studied a highly nonlinear degenerate elliptic PDE in the context of L^∞ minimization and the limit of the p-Laplacian as p goes to infinity. Recently this operator has also been connected to the Monge-Kantorovich problem of optimal transport, and (I have been told) to image processing. The problem studied in [26] is to find the "best" Lipschitz extension into Ω of the boundary data $g(x)$. This is reduced to the problem of existence and uniqueness of the nonlinear PDE

$$\left.\begin{aligned} &\sum_{i,j=1} \frac{\frac{\partial u}{\partial x_i}}{|Du|}(x)\frac{\frac{\partial u}{\partial x_j}}{|Du|}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} = 0 \quad \text{in} \quad \Omega \subset \mathbf{R}^n \\ &u|_{\partial\Omega}(x) = g(x) \end{aligned}\right\} \tag{14}$$

It is easy to see that (14) is both degenerate elliptic and singular at $Du(x) = 0$. Never the less, it was shown that viscosity solutions of (14) exist and also satisfy a maximum principle. Hence, they are unique. Furthermore, for this problem there are also counterexamples to the existence of classical solutions. In fact, the best regularity for this problem appears to be $C^{1,\alpha}$, but a proof of this remains open.

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