Minimal Regularity Solutions of Nonlinear Wave Equations

Hans Lindblad

Abstract.

Inspired by the need to understand the complex systems of non-linear wave equations which arise in physics, there has recently been much interest in proving existence and uniqueness for solutions of nonlinear wave equations with low regularity initial data.

We give counterexamples to local existence with low regularity data for the typical nonlinear wave equations. In the semi-linear case these are sharp, in the sense that with slightly more regularity one can prove local existence.

We also present join work with Georgiev and Sogge proving global existence for a certain class of semi-linear wave equation. This result was a conjecture of Strauss following an initial result of Fritz John. We develop weighted Strichartz estimates whose proof uses techniques from harmonic analysis taking into account the symmetries of the wave equation.

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Introduction.

Recently there has been much interest in proving existence and uniqueness of solutions of nonlinear wave equations with low regularity initial data. One reason is that many equations from physics can be written as a system of nonlinear wave equations with a conserved energy norm. If one can prove local existence and uniqueness assuming only that the energy norm of initial data is bounded then global existence and uniqueness follow. Therefore it is interesting to find the minimal amount of regularity of the initial data needed to ensure local existence for the typical nonlinear wave equations.

We give counterexamples to local existence with low regularity data for the typical nonlinear wave equations. In the semi-linear case the counterexamples are sharp, in the sense that with slightly more regularity one can prove local existence. It is natural to look for existence in Sobolev spaces, since the Sobolev norms are more or less the only norms that are preserved for a linear wave equation. The counterexamples involve constructing a solution that develops a singularity along a characteristic for all positive times. In the quasi-linear case it also involves controlling the geometry of the characteristic set. The norm is initially bounded but becomes infinite for all positive times, contradicting the existence of a solution in the Sobolev space. The counterexamples are half a derivative more regular than what is predicted by a scaling argument. The scaling argument use the fact that the equations are invariant under a scaling to obtain a sequence of solutions for which initial data is bounded in an appropriate Sobolev norm. The counterexamples were not widely expected since for several nonlinear wave equations one does obtain local existence down to the regularity predicted by scaling.

On the other hand, the classical local existence theorems for nonlinear wave equations are not sharp in the semi-linear case. These results were proved using just the energy inequality and Sobolev's embedding theorem. Recently they were improved using space-time estimates for Fourier integral operators known as Strichartz' estimates, and generalizations of these. There are many recent results in this field, for example work by Klainerman-Machedon[13-15], Lindblad-Sogge[24], Grillakis[6] Ponce-Sideris[26] and Tataru. In particular, Klainerman-Machedon proved that for equations satisfying the 'null condition', one can go down to the regularity predicted by the scaling argument mentioned above. In joint work with Sogge[24] we prove local existence with minimal regularity for a simple class of model semi-linear wave equations. There are related results for KdV and nonlinear Schrödinger equations, for example in work by Bourgain and Kenig-Ponce-Vega.

Whereas the techniques of harmonic analysis were essential in improving the local existence results, the Strichartz estimates are not the best possible global estimates since they do not catch the right decay as time tends to infinity if the initial data has compact support. The classical method introduced by Klainerman [11,12] to prove global existence for small initial data is to use the energy method with the vector fields coming from the invariances of the equation. However, this method requires much regularity of initial data and also the energy method alone does not give optimal estimates for the solution since it is an estimate for derivatives. We will present joint work with Georgiev and Sogge giving better global estimates using techniques from harmonic analysis taking into account the invariances or symmetries of the wave equation. We obtain estimates with mixed norms in the angular and spherical variables, with Sogge[24], and weighted Strichartz' estimates with Georgiev and Sogge[4]. Using these new estimates we prove that a certain class of semi-linear wave equations have global existence in all space dimensions. This was a conjecture by Strauss, following an initial result by John.

1. Counterexamples to local existence.

We study quasi-linear wave equations and ask how regular the initial data must be to ensure that a local solution exists. We present counterexamples to local existence for typical model equations. Greater detail of the construction can be found in Lindblad [20-23]. In the semi-linear case the counter examples are sharp in the sense that for initial data with slightly more regularity a local solution exists. This was shown recently in Klainerman-Machedon [13-15], Ponce-Sideris[26] and Lindblad-Sogge[24] using space time estimates know as Strichartz' estimates and refinements of these. However for quasi-linear equations it is still unknown what the optimal result is; there is a gap between the counterexamples and a recent improvement on the existence result by Tataru[42] and Bahouri-Chemin[1].

Consider the Cauchy problem for a quasi-linear wave equation:

$$\begin{aligned} \Box u &= G(u, u', u''), \quad (t,x) \in S_T = [0,T) \times \mathbb{R}^n, \\ u(0,x) &= f(x), \quad u_t(0,x) = g(x), \end{aligned} \tag{1.1}$$

where G is a smooth function which vanishes to second order at the origin and is linear in the third variable u''. (Here $\Box = \partial_t^2 - \sum_{i=1}^n \partial_{x_i}^2$.) Let $\dot{H}^\gamma$ denote the homogeneous Sobolev space with norm $\|f\|_{\dot{H}^\gamma} = \| \, |D_x|^\gamma f\|_{L^2}$ where $|D_x| = \sqrt{-\Delta_x}$ and set

$$||u(t,\cdot)||_\gamma^2 = \int \Big(\big| \, |D_x|^{\gamma-1} u_t(t,x)\big|^2 + \big| \, |D_x|^\gamma u(t,x)\big|^2 \Big) \, dx. \tag{1.2}$$

We want to find the smallest possible γ such that

$$(f,g) \in \dot{H}^\gamma(\mathbb{R}^n) \times \dot{H}^{\gamma-1}(\mathbb{R}^n), \tag{1.3}$$

$$\operatorname{supp} f \cup \operatorname{supp} g \subset \{x; |x| \le 2\} \tag{1.4}$$

implies that we have a local distributional solution of (1.1) for some $T > 0$, satisfying

$$(u, \partial_t u) \in C_b\big([0,T]; \dot{H}^\gamma(\mathbb{R}^n) \times \dot{H}^{\gamma-1}(\mathbb{R}^n) \big). \tag{1.5}$$

To avoid certain peculiarities concerning non-uniqueness we also require that u is a *proper solution*:

Definition 1.1. We say that u is a *proper solution* of (1.1) if it is a distributional solution and if in addition u is the weak limit of a sequence of smooth solutions u_ε to (1.1) with data $(\phi_\varepsilon * f, \phi_\varepsilon * g)$, where $\phi_\varepsilon(x) = \phi(x/\varepsilon)\varepsilon^{-n}$ for some function ϕ satisfying $\phi \in C_0^\infty$, $\int \phi \, dx = 1$.

Even if one has smooth data and hence a smooth solution there might still be another distributional solution which satisfies initial data in the space given by the norm (1.2). In fact, $u(t,x) = 2H(t-|x|)/t$ satisfies $\Box u = u^3$ in the sense of distribution theory. If $\gamma < 1/2$ then $||u(t,\cdot)||_\gamma \to 0$ when $t \to 0$ by homogeneity. Since $u(t,x) = 0$ is another solution with the same data it follows that we have non-uniqueness in the class (1.5) if $\gamma < 1/2$. Definition 1.1 picks out the smooth solution if there is one.

Our main theorem is the following:

Theorem 1.2. *Consider the problem in 3 space dimensions, $n = 3$, with*

$$\begin{aligned} \Box u &= \big(D^l u \big) D^{k-l} u, \quad D = (\partial_{x_1} - \partial_t), \\ u(0,x) &= f(x), \quad u_t(0,x) = g(x), \end{aligned} \tag{1.6}$$

where $0 \le l \le k-l \le 2$, $l = 0,1$. Let $\gamma = k$. Then there are data (f,g) satisfying (1.3)-(1.4), with $\|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}}$ arbitrarily small, such that (1.6) does not have any proper solution satisfying (1.5) in $S_T = [0,T) \times \mathbb{R}^3$ for any $T > 0$.

Remark 1.3. It follows from the proof of the theorem above that the problem is ill-posed if $\gamma = k$. In fact there exists a sequence of data $f_\varepsilon, g_\varepsilon \in C_0^\infty(\{x; |x| \le 1\})$

with $\|f_\varepsilon\|_{\dot H^\gamma}+\|g_\varepsilon\|_{\dot H^{\gamma-1}}\to 0$ such that if T_ε is the largest number such that (1.6) has a solution $u_\varepsilon\in C^\infty([0,T_\varepsilon)\times\mathbb{R}^3)$, we have that either $T_\varepsilon\to 0$ or else there are numbers $t_\varepsilon\to 0$ with $0<t_\varepsilon<T_\varepsilon$ such that $\|u_\varepsilon(t_\varepsilon,\cdot)\|_\gamma\to\infty$. It also follows from the proof of the Theorem that either there is no distributional solution satisfying (1.5) with $\gamma=k$ or else we have non-uniqueness of solutions in (1.5).

Remark 1.4. By a simple scaling argument one gets a counterexample to well-posedness, but it has lower regularity than our counterexamples:

$$\gamma<k+\frac{n-4}{2}. \tag{1.7}$$

Indeed, if u is a solution of (1.6) which blows up when $t=T$ then $u_\varepsilon(t,x)=\varepsilon^{k-2}u(t/\varepsilon,x/\varepsilon)$ is a solution of the same equation with lifespan $T_\varepsilon=\varepsilon T$ and $\|u_\varepsilon(0,\cdot)\|_\gamma=\varepsilon^{k-2+n/2-\gamma}\|u(0,\cdot)\|_\gamma\to 0$ if γ satisfies (1.7). By contrast, our counterexamples are designed to concentrate in one direction, close to a characteristic. It appears that our construction has a natural generalization to any number of space dimensions n, with the initial data lying in $\dot H^\gamma$,

$$\gamma<k+\frac{n-3}{4}. \tag{1.8}$$

Remark 1.5. In Klainerman-Machedon[13,15] it was proved that for semi-linear wave equations satisfying the "null condition" one can in fact get local existence for data having the regularity (1.7) predicted by the scaling argument.

Now, there is a unique way to write (1.6) in the form

$$\sum_{j,k=0}^{3} g^{jk}(u)\partial_{x_j}\partial_{x_k}u=F(u,Du) \tag{1.9}$$

where $x_0=t$ and $g^{jk}(u)$ are symmetric. In the semi-linear case $g^{jk}=m^{jk}$, where m^{jk} is given by (1.10). We now define the notion of a *domain of dependence.*

Definition 1.6. Assume that $\Omega\subset\mathbb{R}_+\times\mathbb{R}^3$ is an open set equipped with a Lorentzian metric $g_{jk}\in C(\Omega)$ such that inverse g^{jk} satisfies

$$\sum_{j,k=0}^{3}|g^{jk}-m^{jk}|\le 1/2,\quad\text{where}\quad \begin{cases} m^{00}=1,\ \ m^{jj}=-1,\ j>0\\ m^{jk}=0,\quad\text{if}\quad j\neq k\end{cases}. \tag{1.10}$$

Then Ω is said to be a *domain of dependence for the metric g_{ij}* if for every compact subset $K\subset\Omega$ there exists a smooth function $\phi(x)$ such that the open set $\mathcal{H}=\{(t,x);\, t<\phi(x)\}$ satisfies

$$\overline{\mathcal{H}}\subset\Omega,\qquad K\subset\mathcal{H} \tag{1.11}$$

and $\partial\mathcal{H}$ is space-like, i.e.

$$\sum_{j,k=0}^{3} g^{jk}(t,x)N_j(x)N_k(x)>0,\quad\text{if}\quad t=\phi(x),\quad N(x)=\big(1,-\nabla_x\phi(x)\big). \tag{1.12}$$

Since a solution u to (1.6) gives rise to a unique metric g_{jk} we say that Ω is a *domain of dependence for the solution u* if it is a domain of dependence for g_{jk}.

LEMMA 1.7. *There is an open set* $\Omega \subset \mathbb{R}_+ \times \mathbb{R}^3$ *and a solution* $u \in C^\infty(\Omega)$ *of (1.6) such that* Ω *is a domain of dependence and writing*

$$\Omega_t = \{x; (t,x) \in \Omega\}, \tag{1.13}$$

we have that $\partial\Omega_0$ *is smooth,*

$$\int_{\Omega_t} \big((\partial_{x_1} - \partial_t)^k u(t,x)\big)^2\, dx = \infty, \quad t > 0, \qquad \textit{and} \tag{1.14}$$

$$\sum_{|\beta|\le k} \int_{\Omega_t} \big(\partial^\beta u(t,x)\big)^2\, dx < \infty, \quad \textit{when} \quad t = 0, \quad \textit{where} \tag{1.15}$$

$\beta = (\beta_0, ..., \beta_3)$ *and* $\partial^\beta = \partial^{\beta_0}/\partial x_0^{\beta_0} \cdots \partial^{\beta_3}/\partial x_3^{\beta_3}$. *Furthermore in the quasi-linear case,* $k - l = 2$, *the norms* $\|D^l u\|_{L^\infty(\Omega)}$ *can be chosen to be arbitrarily small.*

Proof of Theorem 1.2. By Lemmas 1.7 we get a solution $\overline{u}$ in a domain of dependence Ω with initial data $\overline{u}(0,x) \in H^k(\Omega_0)$ and $\overline{u}_t(0,x) \in H^{k-1}(\Omega_0)$. We can extend these to $f \in H^k(\mathbb{R}^3)$ and $g \in H^{k-1}(\mathbb{R}^3)$, see Stein[36]. If there exist a proper solution u of (1.6) in $S_T = [0,T] \times \mathbb{R}^3$ with these data, it follows from Definition 1.1 and Lemma 1.8 that u is equal to $\overline{u}$ in $S_T \cap \Omega$, contradicting (1.5).

LEMMA 1.8. *Suppose* $u \in C^\infty(\Omega)$ *is a solution to (1.6) where* Ω *is a domain of dependence. In the quasi-linear case,* $k - l = 2$, *assume also that* $\|D^l u\|_{L^\infty(\Omega)} \le \delta$. *Suppose also that* $u_\varepsilon \in C^\infty(S_T)$, *where* $S_T = [0,T) \times \mathbb{R}^3$, *and* u_ϵ *are solutions of (1.6) with data* $(f_\varepsilon, g_\varepsilon)$ *where* $f_\varepsilon \to f$ *and* $g_\varepsilon \to g$ *in* $C^\infty(K_0)$ *for all compact subsets of* K_0 *of* $\Omega_0 = \{x; (0,x) \in \Omega\}$. *Then* $u_\varepsilon \to u$ *in* $\Omega \cap S_T$.

It is essential that Ω is a domain of dependence for Lemma 1.8 to be true; one needs exactly the condition (1.12) in order to be able to use the energy method.

Let us now briefly describe how to construct the solution u and the domain of dependence Ω in Lemma 1.7. First we find a solution $u_1(t, x_1)$ for the corresponding equation in one space dimension, (1.16), which develops a certain singularity along a non time like curve $x_1 = \mu(t)$, with $\mu(0) = 0$. The initial data (1.17)-(1.18) has a singularity when $x_1 = 0$ and because of blow-up for the nonlinear equations, the singularity that develops for $t > 0$ is stronger than the singularity of data. Then $u(t,x) = u_1(t,x_1)$ is a solution of (1.6) in the set $\{(t,x); x_1 > \mu(t)\}$. The singularity of data is however too strong for the integral in (1.15) over this set to be finite when $t = 0$. Therefore we will construct a smaller domain of dependence, Ω, satisfying (1.20), such that the curve $x_1 = \mu(t)$, $x_2 = x_3 = 0$, still lies on $\partial\Omega$.

One can find rather explicit solution formulas for the one dimensional equations;

$$(\partial_{x_1} + \partial_t)(\partial_{x_1} - \partial_t)u_1(t,x_1) + (\partial_{x_1} - \partial_t)^l u_1(t,x_1)(\partial_{x_1} - \partial_t)^{k-l} u_1(t,x_1) = 0. \tag{1.16}$$

By choosing particular initial data

$$\begin{aligned} &u_1(0,x_1) = \chi''(x_1), && \partial_t u_1(0,x_1) = 0, && \text{if } k = 0, l = 0,\\ &u_1(0,x_1) = -\chi'(x), && \partial_t u_1(0,x_1) = \chi''(x_1) + \chi'(x_1)^2, && \text{if } k = 1, l = 0,\\ &u_1(0,x_1) = 0, && \partial_t u_1(0,x_1) = -\chi^{(3-k)}(x_1), && \text{if } k \ge 2, \end{aligned} \tag{1.17}$$

$$\text{where} \quad \chi(x_1) = \int_0^{x_1} -\varepsilon|\log|s/4||^\alpha\, ds, \quad 0 < \alpha < 1/2, \varepsilon > 0 \tag{1.18}$$

we get a solution

$$(1.19)\qquad u_1 \in C^\infty(\Omega^1), \quad \text{where} \quad \Omega^1 = \{(t,x_1); \mu(t) < x_1 < 2-t\} \subset \mathbb{R}_+ \times \mathbb{R}^1$$

for some function $\mu(t)$ with $\mu(0) = 0$, such that Ω^1 is a domain of dependence and such that $u_1(t,x_1)$ has a singularity along $x_1 = \mu(t)$. One sees this from the solution formulas which can be found in Lindblad[22,23]. Essentially what is happening is that the initial data (1.17)-(1.18) has a singularity when $x_1 = 0$. For the linear equation, $u_{tt} - u_{x_1x_1} = 0$, the singularity would just have propagated along a characteristic, however the nonlinearity causes the solution to increase and this strengthens the singularity for $t > 0$. (This is the same phenomena that causes blow-up for smooth initial data.)

Define $\Omega \subset \mathbb{R}_+ \times \mathbb{R}^3$ to be the largest domain of dependence for the metric obtained from the solution $u(t,x) = u_1(t,x_1)$ (see (1.9)), such that

$$(1.20)\qquad \Omega \subset \Omega^1 \times \mathbb{R}^2, \quad \Omega_0 = \{x; (0,x) \in \Omega\} = B_0 = \{x; |x - (1,0,0)| < 1\}.$$

(It follows from Definition 1.6 that the union and intersection of a finite number of domains of dependence is a domain of dependence so indeed a maximal domain exists.) It follows that $u(t,x) = u_1(t,x_1)$ is a solution of (1.6) in Ω satisfying (1.17) in Ω_0. The initial data (1.17)-(1.18) was chosen so that (1.15) just is finite if $t = 0$

Let Ω_t be as in (1.13) and

$$(1.21)\qquad S_t(x_1) = \{(x_2,x_3) \in \mathbb{R}^2;\, (x_1,x_2,x_3) \in \Omega_t\}, \quad a_t(x_1) = \int_{S_t(x_1)} dx_2\, dx_3.$$

With this notation the integral in (1.14) becomes

$$(1.22)\qquad \int_{\mu(t)}^{2-t} a_t(x_1)\big((\partial_{x_1} - \partial_t)^k u_1(t,x_1)\big)^2\, dx_1.$$

The proof that this integral is infinite consists of estimating the two factors in the integrand from below, close to $x_1 = \mu(t)$.

In the semi-linear case the metric g^{jk} is just m^{jk} so Ω^1 is a domain of dependence if and only if $\mu'(t) \geq 1$ and it follows that $\Omega = \Omega^1 \times \mathbb{R}^2 \cap \Lambda$, where $\Lambda = \{(t,x); |x - (1,0,0)| + t < 1\}$. Hence for $x_1 > \mu(t)$; $S_t(x_1) = \{(x_2,x_3);\, (x_1-1)^2 + x_2^2 + x_3^2 < (1-t)^2\}$ so then $a_t(x_1) = \pi(2-t-x_1)(x_1-t)$. Also, the specific solution formulas are relatively simple. In particular if $k - l = l = 1$ then its easy to verify that

$$(1.23)\qquad (\partial_{x_1} - \partial_t)u_1(t,x_1) = \frac{\chi'(x_1-t)}{1 + t\chi'(x_1-t)}, \quad u_1(0,x) = 0$$

satisfies (1.16)-(1.17) when $1 + t\chi'(x_1 - t) > 0$. Since $\chi'(0+) = -\infty$ and $\chi'' > 0$ it follows that there is a function $\mu(t)$, with $\mu'(t) > 1$ and $\mu(0) = 0$, such that $1 + t\chi'(x_1 - t) = 0$, when $x_1 = \mu(t)$. Hence $1 + t\chi'(x_1 - t) \leq C(t)(x_1 - \mu(t))$ so

$$(1.24)\qquad \int_{\mu(t)}^{1/2} a_t(x_1)\big((\partial_{x_1} - \partial_t)u_1(t,x_1)\big)^2\, dx_1 \geq \int_{\mu(t)}^{1/2} \frac{(x_1-t)\, dx_1}{C(t)^2(x_1-\mu(t))^2} = \infty.$$

However, in the quasi-linear case, estimating $a_t(x_1)$ from below requires a detailed analysis of the characteristic set $\partial\Omega$ for the operator (1.25), see Lindblad[23].

$$(1.25)\qquad \partial_t^2 - \sum_{i=1}^{3} \partial_{x_i}^2 - V(\partial_{x_1} - \partial_t)^2, \text{ where } V = (\partial_{x_1} - \partial_t)^l u_1.$$

2. Global Existence

We will present sharp global existence theorems in all dimensions for small-amplitude wave equations with power-type nonlinearities. For a given "power" $p > 1$, we shall consider nonlinear terms F_p satisfying

$$\left| (\partial/\partial u)^j F_p(u) \right| \leq C_j |u|^{p-j}, \; j = 0, 1. \tag{2.1}$$

The model case, of course, is $F_p(u) = |u|^p$. If $\mathbb{R}^{1+n}_+ = \mathbb{R}_+ \times \mathbb{R}^n$, and if $f, g \in C_0^\infty(\mathbb{R}^n)$ are fixed, we shall consider Cauchy problems of the form

$$\begin{cases} \Box u = F_p(u), \;\; (t,x) \in \mathbb{R}^{1+n}_+ \\ u(0,x) = \varepsilon f(x), \;\; \partial_t u(0,x) = \varepsilon g(x), \end{cases} \tag{2.2}$$

where $\Box = \partial^2/\partial t^2 - \Delta_x$. Our goal is to find, for a given n, the range of powers for which one always has a global weak solution of (2.2) if $\varepsilon > 0$ is small enough.

In 1979, John [9] showed that for $n = 3$, (2.2) has global solutions if $p > 1 + \sqrt{2}$ and $\varepsilon > 0$ is small. He also showed that when $p < 1 + \sqrt{2}$ and $F_p(u) = |u|^p$ there is blow-up for most small initial data, see also [17]. It was shown later by Schaeffer [28] that there is blowup also for $p = 1 + \sqrt{2}$. After Johns work, Strauss made the conjecture in [38] that when $n \geq 2$, global solutions of (2.2) should always exist if ε is small and p is greater than a critical power p_c that satisfy

$$(n-1)p_c^2 - (n+1)p_c - 2 = 0, \;\; p_c > 1. \tag{2.3}$$

This conjecture was shortly verified when $n = 2$ by Glassey [5]. John's blowup results were then extended by Sideris [30], showing that for all n there can be blowup for arbitrarily small data if $p < p_c$. In the other direction, Zhou [43] showed that when $n = 4$, in which case $p_c = 2$, there is always global existence for small data if $p > p_c$. This result was extended to dimensions $n \leq 8$ in Lindblad and Sogge [25]. Here it was also shown that, under the assumption of spherical symmetry, for arbitrary $n \geq 3$ global solutions of (2.2) exist if $p > p_c$ and ε is small enough. For odd spatial dimensions, the last result was obtained independently by Kubo [16]. The conjecture was finally proved in all dimensions by Georgiev-Lindblad-Sogge[4]. Here we will present that argument.

We shall prove Strauss conjecture using certain "weighted Strichartz estimates" for the solution of the linear inhomogeneous wave equation

$$\begin{cases} \Box w(t,x) = F(t,x), \quad (t,x) \in \mathbb{R}^{1+n}_+ \\ w(0, \cdot\,) = \partial_t w(0, \cdot\,) = 0. \end{cases} \tag{2.6}$$

This idea was initiated by Georgiev [3]. We remark that we only have to consider powers smaller than the conformal power $p_{\text{conf}} = (n+3)/(n-1)$ since it was already known that there is global existence for larger powers. See, e.g., [24].

Let us, however, first recall the inequality for (2.6), that John [9] used;

$$\|t(t-|x|)^{p-2} w\|_{L^\infty(\mathbb{R}^{1+3}_+)} \leq C_p \|t^p (t-|x|)^{p(p-2)} F\|_{L^\infty(\mathbb{R}^{1+3}_+)},$$
$$\text{if } \; F(t,x) = 0, \; t - |x| \leq 1, \; \text{and } \; 1 + \sqrt{2} < p \leq 3.$$

Unfortunately, no such pointwise estimate can hold in higher dimensions due to the fact that fundamental solutions for $\Box$ are no longer measures when $n \geq 4$. Despite this, it turns out that certain estimates involving simpler weights which are invariant under Lorentz rotations (when $R = 0$) hold;

THEOREM 2.1. *Suppose that $n \geq 2$ and that w solves the linear inhomogeneous wave equation* (2.6) *where $F(t,x) = 0$ if $|x| \geq t + R - 1$, $R \geq 0$. Then*

$$\|((t+R)^2 - |x|^2)^{\gamma_1} w\|_{L^q(\mathbb{R}^{1+n}_+)} \leq C_{q,\gamma} \|((t+R)^2 - |x|^2)^{\gamma_2} F\|_{L^{q/(q-1)}(\mathbb{R}^{1+n}_+)}, \tag{2.7}$$

provided that $2 \leq q \leq 2(n+1)/(n-1)$ and

$$\gamma_1 < n(1/2 - 1/q) - 1/2, \text{ and } \gamma_2 > 1/q. \tag{2.8}$$

One should see (2.7) as a weighted version of Strichartz [39,40] estimate;

$$\|w\|_{L^{2(n+1)/(n-1)}(\mathbb{R}^{1+n}_+)} \leq C \|F\|_{L^{2(n+1)/(n+3)}(\mathbb{R}^{1+n}_+)}. \tag{2.9}$$

If one interpolates between this inequality and (2.7), one finds that the latter holds for a larger range of weights (see also our remarks for the radial case below). However, for the sake of simplicity, we have only stated the ones that we will use.

Let us now give the simple argument showing how our inequalities imply the proof of Strauss conjecture. Let $u_{-1} \equiv 0$, and for $m = 0, 1, 2, 3, \ldots$ let u_m be defined recursively by requiring

$$\begin{cases} \Box u_m = F_p(u_{m-1}) \\ u_m(0,x) = \varepsilon f(x), \ \partial_t u_m(0,x) = \varepsilon g(x), \end{cases}$$

where $f, g \in C_0^\infty(\mathbb{R}^n)$ vanishing outside the ball of radius $R - 1$ centered at the origin are fixed. Then if $p_c < p \leq (n+3)/(n-1)$, we can find γ satisfying

$$1/p(p+1) < \gamma < ((n-1)p - (n+1))/2(p+1). \tag{2.10}$$

Set

$$A_m = \|((t+R)^2 - |x|^2)^{\gamma} u_m\|_{L^{p+1}(\mathbb{R}^{1+n}_+)}. \tag{2.11}$$

Because of the support assumptions on the data, domain of dependence considerations imply that u_m, and hence $F_p(u_m)$, must vanish if $|x| > t + R - 1$. It is also standard that the solution u_0 of the free wave equation $\Box u_0 = 0$ with the above data satisfies $u_0 = O(\varepsilon(1+t)^{-(n-1)/2}(1 + |t - |x||)^{-(n-1)/2})$. Using this one finds that $A_0 = C_0\varepsilon < \infty$. It follows from (2.10) that

$$\gamma < n(1/2 - 1/q) - 1/2, \text{ and } p\gamma > 1/q, \quad \text{if} \quad q = p+1, \tag{2.12}$$

so if we apply (2.7) to the equation $\Box(u_m - u_0) = F_p(u_{m-1})$ we therefore obtain

$$\begin{aligned} &\|((t+R)^2 - |x|^2)^{\gamma} u_m\|_{L^{p+1}} \\ &\quad \leq \|((t+R)^2 - |x|^2)^{\gamma} u_0\|_{L^{p+1}} + C_1 \|((t+R)^2 - |x|^2)^{p\gamma} |u_{m-1}|^p\|_{L^{(p+1)/p}} \\ &\qquad = \|((t+R)^2 - |x|^2)^{\gamma} u_0\|_{L^{p+1}} + C_1 \|((t+R)^2 - |x|^2)^{\gamma} u_{m-1}\|^p_{L^{p+1}}, \end{aligned}$$

i.e. $A_m \le A_0 + C_1 A_{m-1}^p$. From this we can inductively deduce that $A_m \le 2A_0$, for all m, if $A_0 = C_0\varepsilon$ is so small that $C_1(2A_0)^p \le A_0$. Similarly, we can get bounds for differences showing that $\{u_m\}$ is a Cauchy sequence in the space associated with the norm (2.11), so the limit exists and satisfies (2.2).

The proof of Theorem 2.1 uses a decomposition into regions, where the weights $(t^2 - |x|^2)$ are essentially constant, together with the invariance of the norms and the equation under Lorentz transformations. In each case we get the desired estimate by using analytic interpolation, Stein[35], between an $L^1 \to L^\infty$ and an $L^2 \to L^2$ estimate with weights, for the Fourier integral operators associated with the wave equation. See [4] for the complete proof and further references. In [4] we also prove a stronger scale invariant weighted Strichartz estimate under the assumption of radial symmetry. This assumption was later removed by Tataru[41]

Theorem 2.2. *Let n be odd and assume that F is spherically symmetric and supported in the forward light cone $\{(t,x) \in \mathbb{R}^{1+n} : |x| \le t\}$. Then if w solves* (2.6) *and if* $2 < q \le 2(n+1)/(n-1)$

$$\|(t^2 - |x|^2)^{-\alpha} w\|_{L^q(\mathbb{R}^{1+n}_+)} \le C_\gamma \|(t^2 - |x|^2)^{\beta} F\|_{L^{q/(q-1)}(\mathbb{R}^{1+n}_+)}, \quad \text{if } \beta < 1/q, \quad \alpha + \beta + \gamma = 2/q, \quad \text{where} \quad \gamma = (n-1)(1/2 - 1/q). \tag{2.13}$$

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Hans Lindblad
UCSD Department of Mathematics
9500 Gilman Drive
La Jolla, CA 92093-0112
USA
lindblad@math.ucsd.edu