Blow-up Phenomena for Critical Nonlinear Schrödinger and Zakharov Equations

Frank Merle

Abstract. In this paper, we review qualitative properties of solutions of critical nonlinear Schrödinger and Zakharov equations which develop a singularity in finite time.

1991 Mathematics Subject Classification: 35Bxx, 35Qxx
Keywords and Phrases: Blow-up, Critical, Schrodinger

I. *The Problem*

We are interested in the formation of singularities in time, in Hamiltonian systems of infinite dimension, and with infinite speed of propagation. A prototype is the nonlinear Schrödinger equation

$$\left\{\begin{array}{rcl} iu_t & = & -\Delta u - |u|^{p-1}u, \\ u(0) & = & u_0, \end{array}\right. \tag{1}$$

for $(x,t) \in R^N \times [0,T)$ and u=0 at infinity. This equation appears in various situations in physics (plasma physics, nonlinear optics,...see [20] for example). Because of its importance in physics, we are interested in the case where $p-1 = 4/N$ and $N=2$. We will consider

$$iu_t = -\Delta u - |u|^{\frac{4}{N}}u. \tag{2}$$

Equation (1) has Galilean, scaling, and translation invariances. In the case $p = \frac{4}{N}+1$, the nonlinear equation has the same structure as the linear equation: it has one more invariance (the conformal invariance): if $u(t)$ is a solution of (2) then $v(t) = \frac{1}{t^{N/2}} e^{+i\frac{|x|^2}{4t}} \overline{u}\left(\frac{1}{t}, \frac{x}{t}\right)$ is also a solution of (2). Thus, there are three invariants of the motion in this case: the mass $|u|_{L^2}$, the energy $E(u) = \frac{1}{2}\int_{R^N} |\nabla u|^2 dx - \frac{1}{\frac{4}{N}+2}\int_{R^N} |u|^{\frac{4}{N}+2} dx$, and the energy of v, $E(v)$.

A more refined physical model is also considered: the Zakharov equation (nonlinear Schrödinger equation coupled with the wave equation). Because of the coupling, all invariances disappear. The system is

$$\left\{\begin{array}{rcl} iu_t & = & -\Delta u + nu, \\ n_t & = & -\nabla \cdot v, \\ \frac{1}{c_0^2} v_t & = & -\nabla(n + |u|^2), \end{array}\right. \tag{3}$$

where $(x,t) \in R^2 \times [0,T)$.

We note formally that if $c_0 = +\infty$, system (3) reduces to equation (2) in dimension two. There are two invariants: $|u|_{L^2}$, and $H(u,n,v) = \int |\nabla u|^2 dx + \int n|u|^2 dx + \frac{1}{2}\int n^2 dx + \frac{1}{2{c_0}^2}\int |v|^2 dx$.

The first natural question concerns the local wellposedness of the equations in time. The natural spaces for this equation are spaces where the conserved quantities are defined. For the Schrödinger equation, H^1 local wellposedness has been proved in [10], [11], [14]. The use of Strichartz estimates (of space-time nature, where the role of space and time are similar) leads to the result in L^2 in [8] (L^2 is optimal in some sense, see [2]). This space will play a crucial role in the analysis below. See [4],[5] in the periodic case.

For the system (3), the coupling between the two equations creates several difficulties. In energy space, that is $(u,n,n_t) \in H_1 = H^1 \times L^2 \times L^2$, the local wellposedness was proved in [3],[9]. The problem to be solved in the analogue of L^2 for the Schrödinger equation is still open (an intermediate space was found in [9]).

The problem we are interested in concerns the description of solutions of equations (2),(3) which develop a singularity in finite time (or blow up in finite time). That is, solutions such that in the time dynamics, the nonlinear terms play an important role. This question is important from the physical point of view. Indeed, equations (2) or (3) appear as simplifications of more complex models. In particular, one hopes that the simplification is relevant for regular solutions, and that close to the singularity, the neglected terms will play a role . Blow up in finite time means that the regular regime where the approximation is carried out is unstable in time, and close to singularity, a transitory regime appears. From the description of this transitory regime, one can hope to find the new dynamics relevant from the physical point of view. In particular, a crucial question, after the *existence* of singularity in finite time, is to describe *how* this singulary forms.

For equation (2), there are two elementary results about existence of blow-up solutions.

On one hand, in 1972 Zakharov derived in [33] (see also [13],[28]) a Pohozaev type identity for the nonlinear Schrödinger equation: let $u_0 \in \Sigma$ where $\Sigma = H^1 \cap \{xu_0 \in L^2\}$; then for all t, $u(t) \in \Sigma$ and

$$\frac{d^2}{dt^2}\int |x|^2|u|^2 dx = 16E(u_0). \tag{4}$$

It follows that if $E(u_0) < 0$ then $u(t)$ blows up in finite time. Note that the power appearing in (2) is the smallest power such that blow-up occurs in H^1.

On the other hand, the elliptic theory established in the 80's ([1],[31],[17], [30]) yields the existence of one explicit solution of (2), periodic in time and of the form $P(t,x) = e^{it}Q(x)$, where Q is the unique positive solution (up to translation) of the equation

$$u = \Delta u + |u|^{\frac{4}{N}} u, \tag{5}$$

whose L^2 norm is characterized by the Gagliardo-Niremberg inequality

$$\forall v \in H^1, \frac{1}{\frac{4}{N}+2}\int |v|^{\frac{4}{N}+2}dx \le \frac{1}{2}\int |\nabla v|^2 dx \left(\frac{\int |v|^2 dx}{\int Q^2 dx}\right)^{\frac{2}{N}}. \tag{6}$$

From the conformal invariance, we have that

$$S(t,x) = \frac{1}{t^{N/2}} e^{-\frac{i}{t}+i\frac{|x|^2}{4t}} Q\left(\frac{x}{t}\right) \tag{7}$$

is a blow-up solution of equation (2). This is in some sense the only explicit blow-up solution for the critical Schrödinger equation.

Until the 90's, no rigorous results on blow-up were known for the Zakharov equation.

II. *Results for Nonlinear Critical Schrödinger Equations.*

II.1 *Characterization of the minimal blow-up solution.*

The first task is to define a notion of smallness such that u_0 small implies no blow-up. In the case $u_0 \in H^1$, energy conservation and (6) yield that if $|u_0|_{L^2} < |Q|_{L^2}$, the solution is globally defined. Moreover, we note that the blow-up solution $S(t)$ is such that $|S(t)|_{L^2} = |Q|_{L^2}$. The natural question is to characterize all minimal blow-up solutions in L^2 of equation (2).

a) *The result.*

We have the following theorem

Theorem 1 ([25]),([26])

Let $u_0 \in H^1$. Assume that $|u(t)|_{L^2} = |Q|_{L^2}$ and that $u(t)$ blows up in finite time. Then, up to invariance of equation (2),

$$u(t) = S(t). \tag{8}$$

That is, there are $x_0 \in R^N$, $x_1 \in R^N$,$T \in R$, $\theta \in R$, and $\omega \in R^+$ such that

$$u(t) = e^{i(-\frac{\omega^2}{t-T}+\frac{|x-x_0|^2}{4(t-T)})}\left(\frac{\omega}{t-T}\right)^{\frac{N}{2}} Q\left(\frac{(x-x_0)\omega}{t-T} - x_1\right). \tag{9}$$

Let us give some idea of the proof. Various arguments in the proof will apply in other contexts, giving qualitative information about blow-up solutions. Consider a blow-up solution of minimal mass $u(t)$, and denote by T its blow-up time.

- *Localization results on the singularity.* Using rough variational estimates, we show that there exist $\tilde\rho$, $\tilde\theta$, $\tilde x$ such that as t $\to$ T, $u(t) \sim e^{i\tilde\theta}\tilde\rho^{\frac{N}{2}}Q((x-\tilde x)\tilde\rho)$ in H^1. Then from refined geometrical estimates around Q, there are $\rho(t)$, $\theta(t)$, $x(t)$ such that $u(t) - e^{i\theta(t)}\rho(t)^{\frac{N}{2}}Q((x-x(t))\rho(t))$ is bounded in H^1. In particular, $|u(t,x+x(t))|^2 \rightharpoonup |Q|^2_{L^2}\delta_{x=0}$ as t $\to$ T. In the radial case, a different approach can be used to show that for all radial blow-up solutions the behavior outside the origin is mild.

- *Local virial identities.* Using time variation of $\int \psi(x)|u(t,x)|^2 dx$, where ψ is a localized function, we then show that $u(t)$ and u_0 decay at infinity. That is,

$u(t) \in \Sigma$ and $|x||u(t,x)|$ can be controlled in L^2 at infinity, uniformly in time. Moreover, it is shown that the singularity point $x(t)$ has a limit as t→ T (for example the origin).

- *Conclusion using the minimality condition.* Let us consider the polynomial in time of degree two $p(t) = \int |x|^2 |u(t,x)|^2 dx$. From the previous steps, $p(T) = 0$. Using the minimality condition, we show that $p'(T) = 0$. By explicit calculation, we check that the energy of a transformation of the initial data u_0 is zero with an L^2 norm equal to $|Q|_{L^2}$, which is the variational characterization of Q up to invariance of the elliptic equation. This concludes the proof.

b) *Application to asymptotic behavior for globally defined solutions* [26].

The conformal invariance and the nonblow-up result of Theorem 1 yield a decay result in time for solutions defined for all time. Indeed, the nonlinear term can be seen as a perturbation localized in time for the linear Cauchy problem, for initial data such that $u_0 \in \Sigma$ and $|u_0|_{L^2} \leq |Q|_{L^2}$, except for the two solutions $P(t)$ and $S(t)$ (and the ones related via the invariances). More precisely, as t $\to +\infty$, the nonlinear solution behaves as a solution of the linear Schrödinger equation (scattering theory can be carried out: $u(t,x) \sim U(t)u_\infty$ as $t \to +\infty$, where U(t) is the free semigroup).

Note that the set of initial data such that this behavior occurs is open, which implies the following: for all u_0 different from $P(t)$ and $S(t)$ such that $|u_0|_{L^2} = |Q|_{L^2}$, there is a ball in L^2 such that if the initial data is inside the ball, the solution does not blow up. It is optimal since the virial identity yields that for all $\epsilon > 0$, if $u_0 = (1+\epsilon)S(-1)$ or $(1+\epsilon)P(-1)$ then the solution blows up in finite time.

II.2 *Qualitative properties of blow-up solutions*

a) *Concentration results in L^2.*

In this subsection, we show that the blow-up phenomena may be observed in L^2 and do not depend on the space where the Cauchy theory is applied. Let us assume first that $u_0 \in H^1$, then from [22], [12], we have

- *Concentration in L^2*: there are $x(t)$ and $\rho(t) \to +\infty$ such that

$$\liminf |u(t)|_{L^2(|x-x(t)|\leq \rho(t)^{-1})} \geq |Q|_{L^2}. \tag{10}$$

- *asymptotic compactness in L^2*: for any sequence $t_n \to T$ there is a subsequence t_n and an $H \in H^1$ with $|H|_{L^2} \geq |Q|_{L^2}$ such that in $H^1 - weak$

$$\rho_n^{\frac{N}{2}} u(t_n, (x - x_n)\rho_n) \rightharpoonup H. \tag{11}$$

We do not know if H or $|H|_{L^2}$ depends on the sequence (except for some partial results in the radial case).

Let us now assume that $u_0 \in L^2$, and $N = 1, 2$; energy arguments no longer apply in this case. Nevertheless, in [6], [27], refinement of Strichartz' Inequality (implying that the Cauchy problem can be solved in $X \supset L^2$), harmonic analysis techniques, and the use of the conformal invariance allow us to obtain concentration in L^2 and asymptotic compactness properties in L^2 up to the invariance of the equation. That is, there is an $\alpha_0 > 0$ such that for a subsequence t_n and an

$H \in L^2$ with $|H|_{L^2} \geq \alpha_0$, there are parameters a_n, b_n, x_n, ρ_n, where $\rho_n \to +\infty$, such that in $L^2 - weak$

$$e^{ia_n x + ib_n |x|^2} \rho_n^{\frac{N}{2}} u(t_n, (x - x_n)\rho_n) \rightharpoonup H. \tag{12}$$

Note, from the invariance of the equation, the solution with initial data $e^{iax+ib|x|^2} c^{\frac{N}{2}} H((x-d)c)$ can be written in terms of the solution with initial data H .

It is an open problem to prove $\alpha_0 = |Q|_{L^2}$.

b) *Construction of blow-up solutions from* $S(t)$.

Here we describe constructions of solutions which behave like $S(t)$ at the blow-up point. Another problem will be to construct if possible other types of blow-up solutions (with for example a different blow-up rate, see [18],[19]). Let $x_1, \ldots, x_n$ be given points of R^N. In [21], a blow-up solution is constructed such that the blow-up set is exactly the points $x_1, \ldots, x_n$ and as t→T, $u(t) \sim \Sigma \omega_i^{\frac{N}{2}} S(t, (x - x_i)\omega_i)$ in L^2, where the ω_i are sufficiently large.

In the case $N = 2$, for u^* very regular such that $\partial^\alpha u^*(0) = 0$ for $|\alpha| \leq \alpha_0$, in [7] the existence of a solution u(t) is proved such that $u(t) \sim S(t,x) + u^*(x)$ in L^2 at the blow-up. An open problem is to reduce α_0 to 1 or 2.

c) *Giving a sense to the equation after blow-up.* [26]

We are interested in giving a sense to the equation after the blow-up time. We consider the case of a minimal blow-up solution, that is, after renormalization $u(t) = S(t,x)$ for $t < 0$.

Let $\epsilon > 0$, and set

$$u_\epsilon(t,x) = (1-\epsilon)S(-1,x) + O(\epsilon^2) \text{ in } \Sigma.$$

We have that $|u_\epsilon|_{L^2} < |Q|_{L^2}$, thus $u_\epsilon(t)$ is defined for all time. The question is what happens in the limit as $\epsilon \to 0$ after the blow-up time (for $t > 0$). Using the characterization of the minimal blow-up solution and a family of auxiliary variational problems in Σ, we have the following result:

Theorem 2 ([26]) *There is a* $\theta(\epsilon) \in R$ *continuous in* ϵ *such that*

$$\begin{array}{rcll} u_\epsilon(t) & \to & S(t) \text{ in } H^1 & \text{for} \quad t<0 \\ |u_\epsilon(0)|^2 & \rightharpoonup & |Q|^2_{L^2}\delta_{x=0} & \\ e^{-i\theta(\epsilon)}u_\epsilon(t) & \to & S(t) \text{ in } H^1 & \text{for} \quad t>0. \end{array}$$

We then prove that as $\epsilon \to 0$, the omega-limit set of $e^{i\theta(\epsilon)}$ is S^1. From this result, the omega limit set of u_ϵ is $\{u_\theta \mid \theta \in S^1\}$, where

$$u_\theta(t) = S(t) \text{ for } t<0 \text{ and } u_\theta(t) = e^{i\theta}S(t) \text{ for } t>0.$$

In particular, we first show that the singularity is unstable in time.

In addition, from the blow-up, the physical phenomenon loses its deterministic character (but just up to one parameter in S^1). In addition, this result seems in some sense independent of the approximation. Therefore, the physics (which is not understood close to the singularity) has in some sense no influence on the behavior of the solution after the blow-up, at least in the case of the minimal blow-up solution.

III. *Results for the Zakharov System.*

III.1 *No blow-up under smallness conditions*
As in the case of the critical Schrödinger equation, for initial data (u_0, n_0, v_0) in H_1, if $|u_0|_{L^2} < |Q|_{L^2}$, then there is no blow-up. Moerover for any blow-up solution, as t goes to the blow-up time, $u(t)$ concentrates in L^2 to a magnitude of at least $|Q|_{L^2}$ (see (11)).

At the critical mass level, $|u_0|_{L^2} = |Q|_{L^2}$, there is still a periodic solution (and the family it generates) given by

$$\tilde{P}(t) = (u(t), n(t), v(t)) = (P(t), -Q^2, 0). \tag{13}$$

Using the coupling between the equations, one can prove ([12]) that there are no blow-up solutions of (3) such that

$$|u_0|_{L^2} = |Q|_{L^2}. \tag{14}$$

III.2 *Existence of a family of explicit blow-up solutions* ([12]).
In fact the family of blow-up solutions of type S, for $\omega > 0$

$$S_\omega(t, x) = \omega^{\frac{N}{2}} S(t\omega^2, x\omega) \tag{15}$$

does not disappear. From bifurcation type arguments at $\omega = +\infty$ and index theory, we construct an explicit family of blow-up solutions of equation (3) of structure similar to that of $S(t)$ (where n and $|u|^2$ are of the same order): for all $\omega > 0$,

$$(u_\omega(t, x), n_\omega(t, x)) = \left((\frac{\omega}{t}) e^{-\frac{i\omega^2}{t} + i\frac{|x|^2}{4t}} P_\omega(\frac{x\omega}{t}), \frac{\omega^2}{t^2} N_\omega(\frac{x\omega}{t}) \right), \tag{16}$$

where (P_ω, N_ω) are radial solutions of the following equation, where $r = |x|$

$$\begin{cases} P + NP = \Delta P, \\ \frac{1}{{c_0}^2\omega^2}(r^2 \frac{\partial^2 N}{\partial r^2} + 6r\frac{\partial N}{\partial r} + 6N) - \Delta N = \Delta P^2. \end{cases} \tag{17}$$

Note that when $\omega = +\infty$ (17) reduces to (5). It is then proved that $\{|P_\omega|_{L^2}\} = (|Q|_{L^2}, +\infty)$, which has several consequences:

- *There are no minimal blow-up solutions in L^2 for the Zakharov equation.* Indeed, for all $\epsilon > 0$, there is a blow-up solution such that $|u_0|_{L^2} = |Q|_{L^2} + \epsilon$ and there are no blow-up solutions such that $|u_0|_{L^2} \leq |Q|_{L^2}$. The situation is different from the Schrödinger equation.

- *Any $c > |Q|_{L^2}$ can be a concentration mass*: there is a blow-up solution such that at the blow-up, $|u(t, x)|^2 \rightharpoonup c\delta_{x=0}$.

- Using these explicit solutions as ω becomes large, we are able to prove that the periodic solution $\tilde{P}(t)$ *is unstable* in the following sense: in all neighborhoods of it, there is a blow-up solution.

III.3 *Existence of a large class of blow-up solutions* [24].

As for the critical Schrödinger equation, a natural question to ask is, For Hamiltonians $H_0 < 0$, does the solution blow-up? (which will produce a large class of blow-up solutions). No Pohozaev identity was known until recently. In [24], the following identity was derived

$$\frac{d^2}{dt^2}M(t) = 2H(u_0, n_0, v_0) + \frac{1}{{c_0}^2}\int |v|^2 dx, \tag{18}$$

where

$$M(t) = \frac{1}{4}\int |x|^2|u|^2 dx + \frac{1}{{c_0}^2}\int_{R^2\times[0,t]} n(x.v)dxdt. \tag{19}$$

Note that if $c_0 = +\infty$ then relation (19) reduces to (4). The nature of the obstruction to global existence is slightly different from that in equation (2). Indeed, in [23], it is shown that for any blow-up solution, $M(t) \to -\infty$ as t goes to the blow-up time. Nevertheless, by localization techniques, it is proved in the radial case that if $H_0 < 0$, then the solution blows up in finite time or infinite time (and is concentrated in L^2 at the blow-up).

As a corollary, all periodic solutions of type $(u, n) = (e^{it}W(x), -W^2)$ where W is a solution of (5) are unstable since $H(e^{it}W(x), -W^2, 0) = 0$.

III.4 *Toward the structural stability of* S [23].

Let us measure the blow-up rate by the H^1 norm $|\nabla u(t)|_{L^2}$. An important problem is to understand the type of rates at the blow-up time and their stability. For equation (2), the blow-up of S (that is of the minimal blow-up solution) is $\frac{1}{|t|}$. We expect that minimality is related to stability. It seems not to be the case; in [18], [19] a blow-up rate of the type $\frac{|Log|log|t|||^{\frac{1}{2}}}{|t|^{\frac{1}{2}}}$ is observed numerically.

Nevertheless, we show the following result for the Zakharov equation (relating minimality to structural stability). Consider any blow-up solution of (3) (with any finite c_0), then

$$|\nabla u(t)|_{L^2} \geq \frac{c}{|t|}. \tag{20}$$

We note that this lower bound is optimal since the solution (u_ω, n_ω) blows up with this rate. Therefore, if we consider the refined equation from the physical point of view, the solution with blow-up rate $\frac{|Log|log|t|||^{\frac{1}{2}}}{|t|^{\frac{1}{2}}}$ disappears (even if c_0 is very large).

In [29], the same blow-up rate that was observed for S is seen, and seems numerically stable. It is an open problem to prove that all blow-up solutions of the Zakharov equation blow up with the same rate as S (the upper bound remains to be proved).

References

[1] Berestycki, H.; Lions, P.-L. Nonlinear scalar field equations.I, II. Arch. Rational Mech. Anal. 82 (1983), no. 4, 313–345 and 347–375.

[2] Birnir, B.; Kenig, C.; Ponce, G.; Svanstedt, N.; Vega, L.; On the ill-posedness of the IVP for the generalized Korteweg-de Vries and nonlinearSchrödinger equations. J. London Math. Soc. (2) 53 (1996), no. 3, 551–559.

[3] Bourgain, J.; Colliander, J. On wellposedness of the Zakharov system. Internat. Math. Res. Notices 1996, no. 11, 515–546.

[4] Bourgain, J.; Harmonic analysis and nonlinear partial differential equations. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zurich, 1994), 31–44, Birkhäuser, Basel, 1995.

[5] Bourgain, J.; Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I.Schrodinger equations. Geom. Funct. Anal. 3 (1993), no. 2, 107–156.

[6] Bourgain, J.; Refinement of Strichartz' Inequality and applications to 2D-NLS with critical Nonlinearity, I.R.R.N., n.5, (1998) 253-283.

[7] Bourgain, J.; Wang, Preprint.

[8] Cazenave, T.; Weissler, F.; Some remarks on the nonlinear Schrödinger equation in the critical case. Nonlinear semigroups, partial differential equations and attractors (Washington, DC, 1987), 18–29, Lecture Notes in Math., Springer, Berlin-New York, 1989.

[9] Ginibre, J.; Tsutsumi, Y.; Velo, G.; On the Cauchy problem for the Zakharov system. J. Funct. Anal. 151 (1997), no. 2, 384–436.

[10] Ginibre, J.; Velo, G.; The global Cauchy problem for the nonlinear Schrödinger equation revisited. Ann. Inst. H. Poincare Anal. Non Lineaire 2 (1985), no. 4, 309–327.

[11] Ginibre, J.; Velo, G.; On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case. J. Funct. Anal. 32 (1979), no. 1, 1–32.

[12] Glangetas, L.; Merle, F. Existence of self-similar blow-up solutions for Zakharov equation in dimension two. I., Concentration properties of blow-up solutions and instability results for Zakharov equation in dimension two. II. Comm. Math. Phys. 160 (1994), no.1 and 2, 173–215 and 349–389.

[13] Glassey, R. T. On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations. J. Math. Phys. 18 (1977), no. 9, 1794–1797.

[14] Kato, T.;On nonlinear Schrödinger equations. Ann. Inst. H. Poincare Phys. Theor. 46 (1987), no. 1, 113–129.

[15] Kenig, C.; Ponce, G.; Vega, L.; On the Zakharov and Zakharov-Schulman systems. J. Funct. Anal. 127 (1995), no. 1, 204–234.

[16] Kenig, C.; Ponce, G.; Vega, L.; Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle. Comm. Pure Appl. Math. 46 (1993), no. 4, 527–620.

[17] Kwong, M. K.; Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbf{R}^n$. Arch. Rational Mech. Anal. 105 (1989), no. 3, 243–266.

[18] Landman, M. J.; Papanicolaou, G. C.; Sulem, C.; Sulem, P.-L.; Rate of blowup for solutions of the nonlinear Schrödinger equation at critical dimension. Phys. Rev. A (3) 38 (1988), no. 8, 3837–3843.

[19] LeMesurier, B. J.; Papanicolaou, G. C.; Sulem, C.; Sulem, P.-L.; Local structure of the self-focusing singularity of the nonlinear Schrödinger equation. Phys. D 32 (1988), no. 2, 210–226.

[20] Mac laughlin D., Lecture in Park-city, I.A.S. summer school 1995.

[21] Merle, F.; Construction of solutions with exactly k blow-up points for the Schrödinger equation with critical nonlinearity. Comm. Math. Phys. 129 (1990), no. 2, 223–240.

[22] Merle, F.; Tsutsumi, Yoshio L^2 concentration of blow-up solutions for the nonlinear Schrödinger equation with critical power nonlinearity. J. Differential Equations 84 (1990), no. 2, 205–214.

[23] Merle, F.; Lower bounds for the blowup rate of solutions of the Zakharov equation in dimension two. Comm. Pure Appl. Math. 49 (1996), no. 8, 765–794.

[24] Merle, F.; Blow-up results of virial type for Zakharov equations. Comm. Math. Phys. 175 (1996), no. 2, 433–455.

[25] Merle, F. Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power. Duke Math. J. 69 (1993), no. 2, 427–454.

[26] Merle, F.; On uniqueness and continuation properties after blow-up time of self-similar solutions of nonlinear Schrödinger equation with critical exponent and critical mass. Comm. Pure Appl. Math. 45 (1992), no. 2, 203–254.

[27] Merle, F.; Vega,L.; Compactness at blow-up time for L^2 solutions of the critical Nonlinear Schrödinger equation in 2D, I.M.R.N. (1998), no. 8, 399–425.

[28] Ogawa, T.; Tsutsumi, Y.; Blow-up of H^1 solution for the nonlinear Schrödinger equation. J. Differential Equations 92 (1991), no. 2, 317–330.

[29] Papanicolaou, G. C.; Sulem, C.; Sulem, P.-L.; Wang, X. P.; Singular solutions of the Zakharov equations for Langmuir turbulence. Phys. Fluids B 3 (1991), no. 4, 969–980.

[30] Strauss, W.; Existence of solitary waves in higher dimensions. Comm. Math. Phys. 55 (1977), no. 2, 149–162.

[31] Weinstein, M. I.; Nonlinear Schrödinger equations and sharp interpolation estimates. Comm. Math. Phys. 87 (1982/83), no. 4, 567–576.

[32] Weinstein, M. I.; On the structure and formation of singularities in solutions to nonlinear dispersive evolution equations. Comm. Partial Differential Equations 11 (1986), no. 5, 545–565.

[33] Zakharov, V.E.; Collapse of langmuir waves, J.E.T.P 35 (1972), 908-914.

Frank Merle
Mathematiques
Université de Cergy-Pontoise
2, Avenue A. Chauvin, Pontoise
95302 Cergy-Pontoise
France