Scaling Algebras in Local Relativistic Quantum Physics

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Abstract. The novel method of scaling algebras allows one to compute and classify the short distance (scaling) limit of any local relativistic C^*–dynamical system and to determine its symmetry structure. The approach is based on an adaptation of ideas from renormalization group theory to the C^*–algebraic setting.

Local relativistic quantum physics [1] in a pseudo-Riemannian spacetime manifold $(\mathcal{M}, g)$ can conveniently be described by C^*–dynamical systems $(\mathcal{A}, \alpha)$, where $\mathcal{A}$ is a C^*–algebra, describing the physical observables in $\mathcal{M}$, and α a representation of the isometry group of $(\mathcal{M}, g)$ by automorphisms of $\mathcal{A}$. The principle of Einstein causality is implemented in this setting by specifying a net (pre-cosheaf) of subalgebras of $\mathcal{A}$ which are labelled by the open, relatively compact regions $\mathcal{O} \subset \mathcal{M}$,

$$\mathcal{O} \longmapsto \mathcal{A}(\mathcal{O}),$$

such that algebras corresponding to causally disjoint regions commute with each other. A theory is fixed by specifying a dynamical system which is subject to these constraints.

In high energy physics the structure of the observables in very small spacetime regions $\mathcal{O}$ (at "small spacetime scales") is of great interest. It can be explored with the help of scaling algebras which have been introduced in [2] by adopting ideas from the theory of the renormalization group. We outline this method for the case where $(\mathcal{M}, g)$ is d–dimensional Minkowski space $\mathbb{R}^{\mathrm{d}}$, equipped with its standard Lorentzian metric. The corresponding isometry group is the Poincaré group $\mathcal{P}_+^\uparrow$ whose elements (Λ, x) are composed of Lorentz transformations and spacetime translations.

For the analysis of the short distance properties of a theory one first proceeds from the given net and automorphisms $(\mathcal{A}, \alpha)$ at spacetime scale $\lambda = 1$ (in appropriate units) to the corresponding nets $(\mathcal{A}^{(\lambda)}, \alpha^{(\lambda)})$ describing the theory at arbitrary scale $\lambda \in \mathbb{R}_+$. This is accomplished by setting for given λ

$$\mathcal{O} \longmapsto \mathcal{A}^{(\lambda)}(\mathcal{O}) \doteq \mathcal{A}(\lambda\mathcal{O}), \quad \alpha^{(\lambda)}_{\Lambda,x} \doteq \alpha_{\Lambda,\lambda x}.$$

The latter nets are in general not isomorphic to each other for different values of λ, so they are to be regarded as different theories.

In addition one needs a way of comparing observables at different scales. To this end one considers certain specific functions $\underline{A}$ of the scaling parameter whose values $\underline{A}_\lambda$ are, for given λ, regarded as observables in the theory $(\mathcal{A}^{(\lambda)}, \alpha^{(\lambda)})$. With this idea in mind one is led to the following construction.

Consider the C^*–algebra $\mathrm{L}^\infty(\mathcal{A})$ of functions $\underline{A} : \mathbb{R}_+ \longmapsto \mathcal{A}$ for which the algebraic operations are pointwise defined and which have finite norm $||\underline{A}|| = \sup_\lambda ||\underline{A}_\lambda||$. The Poincaré group $\mathcal{P}_+^\uparrow$ acts on $\mathrm{L}^\infty(\mathcal{A})$ by automorphisms $\underline{\alpha}_{\Lambda,x}$ which are given by

$$\big(\underline{\alpha}_{\Lambda,x}(\underline{A})\big)_\lambda \doteq \alpha^{(\lambda)}_{\Lambda,x}(\underline{A}_\lambda).$$

We restrict attention to the subalgebra of $\mathrm{L}^\infty(\mathcal{A})$ on which these automorphisms act strongly continuously. Moreover, we introduce a local net structure on this subalgebra by setting

$$\mathcal{O} \longmapsto \underline{\mathcal{A}}(\mathcal{O}) \doteq \{\underline{A} : \underline{A}_\lambda \in \mathcal{A}^{(\lambda)}(\mathcal{O}),\ \lambda \in \mathbb{R}_+\}.$$

The *scaling algebra* $\underline{\mathcal{A}}$ is then defined as the inductive limit of the local algebras $\underline{\mathcal{A}}(\mathcal{O})$. It is easily checked that $(\underline{\mathcal{A}}, \underline{\alpha})$ is again a local C^*–dynamical system which is completely fixed by the given net.

The physical states in the underlying theory are described by a folium of positive linear and nomalized functionals $\omega \in \mathcal{A}^*$ which are locally normal with respect to each other [1]. Their structure at small spacetime scales can be analyzed with the help of the scaling algebra as follows. Given ω, one defines its lift to the scaling algebra at scale $\lambda \in \mathbb{R}_+$ by setting

$$\underline{\omega}_\lambda(\underline{A}) \doteq \omega(\underline{A}_\lambda), \quad \underline{A} \in \underline{\mathcal{A}}.$$

If $\underline{\pi}_\lambda$ denotes the GNS–representation of $\underline{\mathcal{A}}$ induced by $\underline{\omega}_\lambda$ one considers the net

$$\mathcal{O} \longmapsto \underline{\mathcal{A}}(\mathcal{O})/\ker \underline{\pi}_\lambda, \quad \underline{\alpha}^{(\lambda)},$$

where ker means "kernel" and $\underline{\alpha}^{(\lambda)}$ is the induced action of the Poincaré transformations $\underline{\alpha}$ on this quotient. It is important to notice that this net is isomorphic to the underlying theory $(\mathcal{A}^{(\lambda)}, \alpha^{(\lambda)})$ at scale λ. This insight leads to the following canonical definition of the scaling limit of the theory: One first considers the limit(s) of the net of states $\{\underline{\omega}_\lambda\}_{\lambda \searrow 0}$. By standard compactness arguments, this net has always a non–empty set $\{\underline{\omega}_0\}$ of limit points. The following facts about these limit states have been established in [2]:

Proposition 1. The set $\{\underline{\omega}_0\}$ does not depend on the chosen physical state ω.

Proposition 2. Each $\underline{\omega}_0$ is a vacuum state on $(\underline{\mathcal{A}}, \underline{\alpha})$, i.e. a ground state with respect to the time evolution which is invariant under Poincaré transformations. Moreover, in d > 2 dimensional Minkowski space theories these vacua are pure states.

Denoting the GNS–representation corresponding to given $\underline{\omega}_0$ by $(\underline{\pi}_0, \underline{\mathcal{H}}_0)$ one then defines in complete analogy to the case $\lambda > 0$ the net

$$\mathcal{O} \longmapsto \mathcal{A}^{(0)} \doteq \underline{\mathcal{A}}(\mathcal{O})/\ker \underline{\pi}_0, \quad \alpha^{(0)} \doteq \underline{\alpha}^{(0)}$$

which is to be interpreted as *scaling limit of the underlying theory.* The various steps in this construction can be summarized in the diagram

$$(\mathcal{A}, \alpha) \longrightarrow (\mathcal{A}^{(\lambda)}, \alpha^{(\lambda)}) \longrightarrow (\underline{\mathcal{A}}, \underline{\alpha}) \longrightarrow \{(\mathcal{A}^{(0)}, \alpha^{(0)})\}.$$

It is now possible to classify the scaling limits as follows [2].

CLASSIFICATION: Let $(\mathcal{A}, \alpha)$ be a net with properties specified above. There are the following mutually exclusive possibilities for the structure of the scaling limit theory induced by the corresponding scaling limit states $\{\underline{\omega}_0\}$.

1. The nets $(\mathcal{A}^{(0)}, \alpha^{(0)})$ are all isomorphic to the trivial net $(\mathbb{C}\cdot 1, \mathrm{id})$ (classical scaling limit)
2. The nets $(\mathcal{A}^{(0)}, \alpha^{(0)})$ are all isomorphic and the algebras $\mathcal{A}^{(0)}$ are non–abelian (quantum scaling limit)
3. Not all of the nets $(\mathcal{A}^{(0)}, \alpha^{(0)})$ are isomorphic (degenerate scaling limit)

Theories with a quantum scaling limit are of primary physical interest. For this class there holds the following statement on the enhancement of symmetries at small scales [2].

PROPOSITION 3. The scaling limit nets $(\mathcal{A}^{(0)}, \alpha^{(0)})$ of theories with a quantum scaling limit admit an automorphic action of the scaling transformations $\mathbb{R}_+$.

Simple examples in this class are nets generated by non–interacting quantum fields in d = 3 and 4 dimensions [3]. For a discussion of the other cases see [4].

The fact that the scaling limit theories $(\mathcal{A}^{(0)}, \alpha^{(0)})$ exhibit all features of local nets of observable algebras allows one to apply standard methods for their analysis and physical interpretation. For the determination of the symmetries appearing in the scaling limit one can rely in the case of d > 2 dimensional Minkowski space on the Doplicher–Roberts reconstruction theorem [5]. The necessary prerequisite for its application is the following result established in [2].

PROPOSITION 4. If a local net complies with the special condition of duality (modular covariance) of Bisognano and Wichmann, the same holds true for its scaling limit.

By the results of Doplicher and Roberts [5] one can then recover from the outer local endomorphisms of the scaling limit net $(\mathcal{A}^{(0)}, \alpha^{(0)})$ a compact group $G^{(0)}$ whose irreducible representations are in one–to–one correspondence to the set of physical states which appear in the scaling limit and carry a localizable charge. Moreover, there exists an extension of the scaling limit net to a field net $(\mathcal{F}^{(0)}, \alpha^{(0)})$ on which $G^{(0)}$ acts by automorphisms and which implements the action of the local endomorphisms. The vacuum representation of this field net describes the charged physical states and can be used to analyse their detailed properties. In particular one can determine from it the particle content of the scaling limit theory, which corresponds to the set of non–trivial irreducible representations of the Poincaré group $\mathcal{P}_+^\uparrow$ appearing in the vacuum sector of $(\mathcal{F}^{(0)}, \alpha^{(0)})$.

Of special interest is the comparison of the particle and symmetry content of the underlying theory and of its scaling limit [6]. Depending on the theory, there may be particles at finite scales which disappear in the scaling limit, particles which survive in this limit and particles which only come into existence at very small scales. These possibilities correspond exactly to the features of the various particle like structures which are observed in high energy collision experiments. Intuitive physical notions, such as quark, gluon, colour symmetry and confinement thus acquire an unambiguous mathematical meaning in the present setting [6].

The extension of the short distance analysis to local nets on spacetimes $(\mathcal{M}, g)$ with a large isometry group, such as de Sitter space, is straightforward. For theories on spacetimes with small isometry groups it is however less clear how to define corresponding scaling algebras which consist of sufficiently regular elements. An interesting proposal to solve this problem has been made by Verch [7]. In this approach the resulting scaling limits turn out to be local C^*-dynamical systems in the (Minkowskian) fibers of the tangent bundle of $(\mathcal{M}, g)$. For a further classification of these theories it would be of interest to analyze the transport between the corresponding dynamical systems in the various fibers which is induced by the underlying dynamics. This "quantum connection" should also contain relevant information on the presence of local gauge symmetries in the underlying theory.

References

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