On the Problem of Stability for Near to Integrable Hamiltonian Systems

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Abstract. Some recent applications and extensions of Nekhoroshev's theory on exponential stability are presented. Applications to physical systems concern on the one hand realistic evaluations of the regions where exponential stability is effective, and, on the other hand, the relaxation time for resonant states in large, possibly infinite systems. Extensions of the theory concern the phenomenon of superexponential stability of orbits in the neigbourhood of invariant KAM tori.

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1. Overview

According to Poincaré ([26], tome I, chapt. I, § 13) the general problem of dynamics is the investigation of a canonical system of differential equations with Hamiltonian

$$H(p,q,\varepsilon) = h(p) + \varepsilon f(p,q,\varepsilon) \ , \tag{1}$$

where $(p,q) \in \mathcal{G} \times \mathbf{T}^n$ are action–angle variables, $\mathcal{G} \subset \mathbf{R}^n$ is open, ε is a small parameter and n is the number of degrees of freedom. The functions h and f are assumed to be analytic in all arguments; in particular the perturbation $f(p,q,\varepsilon)$ can be expanded in power series of ε in a neighbourhood of $\varepsilon = 0$. Many physical systems may be described by a Hamiltonian of the form above; the most celebrated one is the planetary system with its natural and (which is of interest now) artificial bodies.

My aim here is to illustrate some results concerning the stability of such systems. The word "stability" is used here in a wide sense, which includes a considerable weakening of the traditional concept investigated, e.g., by Lyapounov. I will pay particular attention to quantities that remain almost constant for a time that increases faster than any inverse power of ε as $\varepsilon \to 0$. Following Littlewood, I will refer to stability estimates of this kind as *exponential stability*.

It is well known that for $\varepsilon = 0$ the unperturbed system $h(p)$ is trivially integrable, since the orbits lie on invariant tori parameterized by the actions p, and the flow is typically quasiperiodic with frequencies $\omega(p) = \frac{\partial h}{\partial p}$. It has been proven by Poincaré that for $\varepsilon \neq 0$ the system is generically non–integrable (see [26], chapt. V). This is due to the existence of resonances among the frequencies, i.e., relations of the form $\langle k, \omega(p)\rangle = 0$ with $0 \neq k \in \mathbf{Z}^n$.

It was only after the year 1954 that a significant advance of our knowledge was made with the celebrated theorem of Kolmogorov[18], Arnold[1] and Moser[23]. They proved the existence of a set of invariant tori of large relative measure, thus assuring stability in probabilistic sense. Almost at the same time, Moser[22] and Littlewood[19][20] introduced the methods leading to exponential stability. Several years later a general formulation was given by Nekhoroshev, who proved that the action variables p remain almost invariant for a time that increases exponentially with the inverse of the perturbation ε; more precisely, one has

$$\bigl|p(t) - p(0)\bigr| < B\varepsilon^b \quad \text{for} \quad |t| < T_* \exp\bigl((\varepsilon_*/\varepsilon)^a\bigr) \tag{2}$$

for some constants B, T_*, ε_*, $a \leq 1$ and $b < 1$ (see [24], [25], [3], [4], [21], [14]).

My purpose here is to report on some progress made during the last decade. I will address in particular the following points: (a) the actual relevance of exponential stability for physical systems; (b) the extension of the concept of exponential stability to systems with a very large number of degrees of freedom, and possibly to infinite systems; (c) some relations between KAM and Nekhoroshev's theory, and in particular a stronger stability result that I will call *superexponential stability.*

Both KAM theorem and Nekhoroshev's theorem apply provided the size ε of the perturbation is smaller than a critical value, ε_* say. On the other hand, the problem of finding realistic estimates for the critical value ε_* is generally a very hard one: the analytical estimates available are useless for a practical application to a physical model, and only in a few, very particular models realistic results have been obtained. One such case concerns the stability of the Lagrangian point L_4 of the restricted problem of three bodies in the Sun–Jupiter case. I discuss in sect. 2 how realistic estimates may be obtained by complementing the analytical scheme with explicit calculation of perturbation series.

For systems with a large number of degrees of freedom one is confronted with the problem that all estimates seem to indicate that Nekhoroshev's theorem looses significance for $n \to \infty$ because the constants T_*, ε_* and a tend to zero. As a typical example let us consider a system of identical diatomic molecules moving on a segment and interacting via a short range analytic potential; this may be considered as a one–dimensional model of a gas, the main simplification being that the rotational degrees of freedom of the molecules are not taken into account. The model admits a natural splitting into two subsystems, i.e., the translational motions and the internal vibrations of the molecules, with a coupling due to collisions. According to the equipartition principle, every degree of freedom would get the same average energy. However, it was already suggested by Boltzmann that this should be true only if one considers time averages over a sufficiently long time (relaxation time). Boltzmann's suggestion was that such a time would increase with the frequency of the internal vibrations, becoming of the order of days or centuries (see [9]); a few years later Jeans suggested that the relaxation time could increase exponentially with the frequency, possibly becoming of the order of billions of years (see [17]). I discuss in sect. 3 how far the suggestion of Boltzmann and Jeans may be dynamically justified if one relinquishes the request that all actions be constant, and pays attention only to the transfer of energy between the two subsystems. For a discussion of the relevance of the exponential stability in statistical mechanics see [7] and [10] and the references therein.

Finally, it is interesting from the theoretical viewpoint to investigate the behaviour of the orbits in the neighbourhood of an invariant KAM torus. I discuss this point in sect. 4 by illustrating how KAM theorem may be obtained by using Nekhoroshev's theorem as a basic iteration step. As a straightforward consequence one gets the result that in most of the phase space the orbits are stable for a time that is much longer than the exponential time predicted by Nekhoroshev. Indeed, the exponential time in (2) is replaced by $\exp\bigl(\exp(1/\varrho)\bigr)$, where ϱ is the distance from an invariant KAM torus. This is what I call *superexponential stability*.

2. The triangular Lagrangian equilibria

It is known that in a neighbourhood of an elliptic equilibrium the Hamiltonian may be given the form

$$H(x,y) = \frac{1}{2}\sum_{l=1}^{n} \omega_l \left(x_l^2 + y_l^2\right) + \sum_{s>2} H_s(x,y) \ , \tag{3}$$

where $\omega \in \mathbf{R}^n$ is the vector of the harmonic frequencies and H_s is a homogeneous polynomial of degree s in the canonical variables $(x,y) \in \mathbf{R}^{2n}$. The stability of the equilibrium $x = y = 0$ for the system (3) is a trivial matter if all frequencies ω have the same sign, e.g., they are all positive. For, in this case the classical Lyapounov's theory applies since the Hamiltonian has a minimum at the origin. This simple argument does not apply if the frequencies do not vanish but have different signs.

The stability over long times has been investigated by Birkhoff using the method of normal form going back to Poincaré (see [26], tome II, chapt. IX, § 125). Assuming that there are no resonance relations among the frequencies ω, via a near the identity canonical transformation $(x,y) \to (x',y')$ the Hamiltonian is given the normal form up to a finite order $r > 2$

$$H^{(r)}(x',y') = \frac{1}{2}\sum_{l=1}^{n} \omega_l p'_l + Z^{(r)}(p') + \mathcal{R}^{(r)}(x',y') \ , \tag{4}$$

where $p'_l = ({x'}_l^2 + {y'}_l^2)/2$ are the new actions, $Z^{(r)}$ is at least quadratic in p' and the unnormalized remainder $\mathcal{R}^{(r)}$ is a power series starting with terms of degree $r+1$ in x', y'. If we forget the remainder then the system is integrable and the motion is quasiperiodic on invariant tori, since $Z^{(r)}$ depends only on the new actions. Birkhoff's remark was that the normalized Hamiltonian $H^{(r)}$ is convergent in a neighbourhood of the origin, e.g., in some polydisk of radius ϱ (that may depend on r) and center at the origin, i.e.,

$$\Delta_\varrho = \left\{ (x,y) \in \mathbf{R}^{2n} \ : \ \sqrt{x_j^2 + y_j^2} < \varrho \right\} \ . \tag{5}$$

Hence, the size of the remainder may be estimated by $C_r \varrho^{r+1}$, with some constant C_r that Birkhoff did not try to evaluate. He concluded that the dynamics given by the integrable part of the Hamiltonian is a good approximation of the true dynamics up to a time of order $O(\varrho^{-r})$; on this remark he based his theory of *complete stability* (see [8], chapt. IV, § 2 and § 4).

It was pointed out by Poincaré that the series produced by perturbation expansions have an asymptotic character (see [26], tome II, chapt. VIII). Now this fact lies at the basis of the exponential stability. Indeed the constant C_r is expected to grow at least as $O(r!)$, so that the size of the remainder is $O\bigl(r!\varrho^{r+1}\bigr)$. Having fixed ϱ (i.e., the domain of the initial data) one chooses $r \sim 1/\varrho$, and by a straightforward use of Stirling's formula one gets $|\mathcal{R}| = O\bigl(\exp(-1/\varrho)\bigr)$. By working out the analytical estimates one gets for the unperturbed actions $p_l = (x_l^2 + y_l^2)/2$ the following bound (see [13] or [12]):

THEOREM: *Let the frequencies ω satisfy the diophantine condition*

$$\bigl|\langle k, \omega \rangle\bigr| \geq \gamma |k|^{-\tau} \quad \text{for } 0 \neq k \in \mathbf{Z}^n \ . \tag{6}$$

Then there exists a ϱ_* such that for all orbits satisfying $\bigl(x(0), y(0)\bigr) \in \Delta_\varrho$ one has

$$\bigl|p(t) - p(0)\bigr| = O(\varrho^3) \quad \text{for } |t| < T = O\bigl(\exp(1/\varrho^{1/(\tau+1)})\bigr) \ .$$

For a practical application the problem is that the estimated value of ϱ_* may be ridiculously small. A better evaluation may be obtained by explicitly calculating all series involved in the normalization process up to some (not too low) order. This just requires some elementary algebra on computer.

The Hamiltonian is expanded in power series as in (3) up to some order r, and then is given a normal form at the same order. The explicit transformation of coordinates and the new action variables p' as functions of the old coordinates can be constructed, too. Moreover, in a polydisk Δ_ϱ we may evaluate the quantity

$$D(\varrho, r) = \sup_{(x',y') \in \Delta_\varrho} \bigl|\dot{p}'\bigr| = \sup_{(x',y') \in \Delta_\varrho} \bigl|\bigl\{p', \mathcal{R}^{(r)}\bigr\}\bigr| \ ;$$

to this end, the expression of the lowest order term of the remainder $\mathcal{R}^{(r)}$ may be used. Having fixed a polydisk Δ_{ϱ_0} containing the initial data we conclude that the orbit can not escape from a polydisk Δ_ϱ, with an arbitrary $\varrho > \varrho_0$, for $|t| < \tau(\varrho_0, \varrho, r)$, where

$$\tau(\varrho_0, \varrho, r) = \frac{\varrho^2 - \varrho_0^2}{2D(\varrho, r)} \ . \tag{7}$$

This produces an estimate depending on the arbitrary quantities ϱ and r. Let ϱ_0 and r be fixed; then, in view of $D(\varrho, r) \sim C_r \varrho^{r+1}$, the function $\tau(\varrho_0, \varrho, r)$, considered as function of ϱ only, has a maximum for some value ϱ_r. This looks quite odd, because one would expect τ to be an increasing function of ϱ. However, recall that (7) is just an estimate; looking for the maximum means only that we are trying to do the best use of our poor estimate. Let us now keep ϱ_0 constant, and calculate $\tau(\varrho_0, \varrho_r, r)$ for increasing values of $r = 1, 2, \ldots$, with ϱ_r as above. Since C_r is expected to grow quite fast with r we expect to find a maximum of $\tau(\varrho_0, \varrho_r, r)$ for some optimal value $r_{\rm opt}$. Thus, we are authorized to conclude that *for every ϱ_0 we can explicitly evaluate the positive constants $\varrho(\varrho_0) = \varrho_{r_{\rm opt}}$ and $T(\varrho_0) = \tau(\varrho_0, \varrho(\varrho_0), r_{\rm opt})$ such that an orbit with initial point in the polydisk Δ_{ϱ_0} will not escape from Δ_ϱ for $|t| < T(\varrho_0)$.*

In order to show that the method above may be effective let me consider the triangular Lagrangian point L_4 of the restricted problem of three bodies, with particular reference to the Sun–Jupiter case. In the planar case the frequencies are $\omega_1 \sim 0.99676$ and $\omega_2 \sim -0.80464 \times 10^{-1}$; hence the standard Lyapounov theory does not apply.

The procedure above has been worked out by expanding all functions in power series up to order 35. One may look in particular for a value of ϱ_0 such that $T(\varrho_0)$ is the estimated age of the universe. The result is that ϱ_0 is roughly 0.127 times the distance L_4–Jupiter; this is certainly a realistic result. A comparison with the known Trojan asteroids shows that four of them are inside the region which assures stability for the age of the universe (see [16] for a complete report).

3. On the conjecture of Boltzmann and Jeans

Let us consider a canonical system with analytic Hamiltonian

$$H(p, x, \pi, \xi) = \hat{h}(p, x) + h_\omega(\pi, \xi) + f(p, x, \pi, \xi)\ , \tag{8}$$

where

$$h_\omega(\pi, \xi) = \frac{1}{2} \sum_{l=1}^{\nu} \left(\pi_l^2 + \omega_l^2 \xi_l^2\right)\ , \quad (\pi, \xi) \in \mathbf{R}^{2\nu}$$

is the Hamiltonian of a system of harmonic oscillators, $\hat{h}(p, x)$ is the Hamiltonian of a generic n–dimensional system, and $f(p, x, \pi, \xi)$ a coupling term which is assumed to be of order ξ, and so to vanish for $\xi = 0$.

This model was suggested by the numerical study of the system of diatomic molecules mentioned in sect. 1 (see [5] and [6]). In that case $\hat{h}(p, x)$ represents the translational degrees of freedom, and $h_\omega(\pi, \xi)$ describes the internal vibrations of the molecules. Since the molecules are identical, all frequencies coincide.

The identification of a perturbation parameter in the system (8) goes as follows. Write $\omega = \lambda\Omega$ with large λ and Ω of the same order of the inverse of a typical time scale of the constrained system (for example the characteristic time for the collision of two molecules, which is non zero if the interaction potential is regular); then transform the variables according to $\pi = \pi' \sqrt{\lambda\Omega}$ and $\xi = \xi' / \sqrt{\lambda\Omega}$, and assume the total energy of the subsystem h_ω to be finite, so that the variables (π',ξ') turn out to be confined in a disk of size $1/\sqrt{\lambda}$. Then the Hamiltonian may be given the form, omitting primes,

$$H(p,x,\pi,\xi,\lambda) = \hat{h}(p,x) + \lambda h_\Omega(\pi,\xi) + \frac{1}{\lambda} f_\lambda(p,x,\pi,\xi)$$
$$h_\Omega(\pi,\xi) = \frac{1}{2}\sum_{l=1}^{\nu} \Omega_l \left(\pi_l^2 + \xi_l^2\right)$$

(here, a straightforward computation would give $\lambda^{-1/2}$ in front of f, but f itself turns out to be of order $\lambda^{-1/2}$, since it vanishes for $\xi = 0$). Here too the main technical tool is the reduction of the Hamiltonian to a normal form. Precisely, via a near to identity canonical transformation $(p,x,\pi,\xi) \to (p',x',\pi',\xi')$ the Hamiltonian is given the form

$$H'(p',x',\pi',\xi',\lambda) = \lambda h_\Omega(\pi',\xi') + \hat{h}(p',x') + Z(p',x',\pi',\xi',\lambda) + \mathcal{R}(p',x',\pi',\xi',\lambda)\ ,$$

where Z is in normal form in the sense that $\{h_\Omega, Z\} = 0$. Thus h_Ω is an approximate first integral. The normalization process is performed until the remainder is exponentially small in the parameter $1/\lambda$. This requires an optimal choice of the number of normalization steps, as in the case of the elliptic equilibrium.

Theorem: *Assume that all frequencies ω are equal. Then there are positive constants T_* and λ_* such that for every $\lambda > \lambda_*$ one has*

$$\tag{9} \begin{aligned} \bigl|h_\Omega(\pi,\xi) - h_\Omega(\pi',\xi')\bigr| &= O(\lambda^{-2})\ ;\\ \bigl|h_\Omega(t) - h_\Omega(0)\bigr| &= O(\lambda^{-1}) \quad \text{for } |t| < T_* \exp\left(\frac{\lambda}{\lambda_*}\right)\ . \end{aligned}$$

The remarkable fact is that the exponent a that appears in the general form (2) of the exponential estimate is 1, no matter of the number n of degrees of freedom. This removes the worst dependence on n, and is in complete agreement with the numerical calculations in [5].

In the case of the diatomic gas there is still a dependence on n in the constants T_* and λ_*, which turn out to be $O(1/n^2)$ (the number of two–body interaction terms in the perturbation). Such a dependence could hardly be removed on a purely dynamical basis, because the possibility that all molecules collide together at all times may not be excluded. This is clearly unrealistic. A complete proof of the conjecture of Boltzmann and Jeans could perhaps be obtained by complementing the dynamical theory with statistical considerations.

The result above has been extended to further situations, including the case of infinite systems. As an example, consider a modification of the celebrated nonlinear chain of Fermi, Pasta and Ulam[11] in which the equal masses are replaced by alternating heavy and light masses. It is known that the spectrum splits into two well separated branches, called the acoustical and the optical one. Moreover the optical frequencies are very close to each other. The whole system may thus be considered as composed of two separate subsystems, and the subsystem h_ω of the optical frequencies may still be considered as a system of oscillators with the same frequency: the small difference can consistently be considered as part of the perturbation. In this case it has been proven that the exponential estimate applies also to the case of an infinite chain, provided the *total* energy is sufficiently small (see [2]). Strictly speaking, this is not enough for the application to the problem of equipartition of energy in statistical mechanics, since in that case one is interested in initial data with fixed *specific* energy. However, the discrepancy is still due to the fact that we are working on a purely dynamical basis. For, the possibility that the whole energy of the optical subsystem remains concentrated on a single oscillator for a long time is not excluded. Here too one should include statistical considerations.

4. Superexponential stability

Let us go back to considering the Hamiltonian (1). I will need to consider the action variables in a domain $\mathcal{G}_\varrho = \bigcup_{p\in\mathcal{G}} B_\varrho(p)$, where ϱ is a positive parameter, $\mathcal{G} \subset \mathbf{R}^n$ is open, and $B_\varrho(p)$ denotes the open ball of radius ϱ and center p. The phase space is $\mathcal{D} = \mathcal{G}_\varrho \times \mathbf{T}^n$.

If the unperturbed Hamiltonian $h(p)$ is non degenerate, then the construction of the normal form for the Hamiltonian can not be performed globally on the action domain $\mathcal{G}_\varrho$. For, the small denominators $\langle k, \omega(p)\rangle$ (with $k \in \mathbf{Z}^n$ and $\omega(p) = \frac{\partial h}{\partial p}$) may generically vanish in a set of points that is dense in $\mathcal{G}_\varrho$. This fact lies at the basis of Poincaré's proof of nonexistence of uniform first integrals (see [26], chapt. V).

The way out of this problem is based on: (a) a Fourier cutoff of the perturbation, i.e., only a finite number of Fourier modes is considered during the process of normalizing the Hamiltonian, and (b) the construction of the normal form in local nonresonance domains where the small denominators are far enough from zero. The burden of constructing the nonresonance domains is taken by the so called *geometric part* of the proof of Nekhoroshev's theorem: basically, the original domain $\mathcal{G}_\varrho$ is covered by subdomains corresponding to *known* resonances of different multiplicity $0, 1, \ldots, n$, where multiplicity zero corresponds to the region free from resonances. The domains so constructed are open because only a *finite* number of resonances is taken into account; this is a consequence of the Fourier cutoff. The normal form is *local* to each domain, and depends on the resonances that appear on it. Nekhoroshev's theorem on exponential stability follows by proving that every orbit is confined inside a local nonresonance domain for an exponentially long time.

The result that I'm going to illustrate is based on iteration of Nekhoroshev's theorem. Let me first state the result. Let φ^t be the canonical flow generated by the Hamiltonian (1). A n–dimensional torus $\mathcal{T}$ will be said to be (η, T)*–stable* in case one has $\mathrm{dist}(\varphi^t P, \mathcal{T}) < \eta$ for all $|t| < T$ and for every $P \in \mathcal{T}$. The formal statement is the following

THEOREM: *Consider the Hamiltonian (1), and assume that the unperturbed Hamiltonian $h(p)$ is convex. Then there exists $\varepsilon^* > 0$ such that for all $\varepsilon < \varepsilon^*$ the following statement holds true: there is a sequence $\{\mathcal{D}^{(r)}\}_{r\geq 0}$ of subsets of $\mathcal{D}$, with $\mathcal{D}^{(0)} = \mathcal{D}$, and two sequences $\{\varepsilon_r\}_{r\geq 0}$ and $\{\varrho_r\}_{r\geq 1}$ of positive numbers satisfying*

$$\varepsilon_0 = \varepsilon\ , \quad \varepsilon_r = O\bigl(\exp(-1/\varepsilon_{r-1})\bigr)\ ,$$
$$\varrho_0 = \varrho\ , \quad \varrho_r = O(\varepsilon_{r-1}^{1/4})\ ,$$

such that for every $r \geq 0$ one has:
(i) $\mathcal{D}^{(r+1)} \subset \mathcal{D}^{(r)}$ *;*
(ii) $\mathcal{D}^{(r)}$ *is a set of n–dimensional tori diffeomorphic to* $\mathcal{G}^{(r)}_{\varrho_r} \times \mathbf{T}^n$*;*
(iii) $\mathrm{Vol}(\mathcal{D}^{(r+1)}) > (1 - O(\varepsilon_r^a))\,\mathrm{Vol}(\mathcal{D}^{(r)})$ *for some positive $a < 1$;*
(iv) $\mathcal{D}^{(\infty)} = \bigcap_r \mathcal{D}^{(r)}$ *is a set of invariant tori for the flow φ^t, and moreover one has* $\mathrm{Vol}(\mathcal{D}^{(\infty)}) > (1 - O(\varepsilon_0^a))\,\mathrm{Vol}(\mathcal{D}^{(0)})$ *;*
(v) for every $p^{(r)} \in \mathcal{G}^{(r)}$ the torus $p^{(r)} \times \mathbf{T}^n \subset \mathcal{D}^{(r)}$ is $(\varrho_{r+1}, 1/\varepsilon_{r+1})$–stable;
(vi) for every point $p^{(r)} \in \mathcal{G}^{(r)}$ there exists an invariant torus $\mathcal{T} \subset B_{\varrho_r}(p^{(r)}) \times \mathbf{T}^n$.

Let me illustrate the main points of the proof (for a complete proof see [15]). A careful reading of the geometric part of Nekhoroshev's theorem allows one to extract the following information: there exists a subset $\mathcal{D}^{(1)}$ of phase space characterized by absence of resonances of order smaller than $O(1/\varepsilon)$; such a domain is the union of open balls of positive radius ϱ_1, and its complement has measure $O(\varepsilon^{1/4})$. Moreover, in this subset one may introduce new action–angle variables, (p', q') say, which give the Hamiltonian the original form (1), but with a perturbation of size $\varepsilon_1 = O\bigl(\exp(-1/\varepsilon)\bigr)$.

Nekhoroshev's theorem can be applied again to the new Hamiltonian in the open domain $\mathcal{D}^{(1)}$, thus allowing one to construct a second nonresonant domain $\mathcal{D}^{(2)}$ characterized by absence of resonances of order smaller than $O(1/\varepsilon_1) = O\bigl(\exp(-1/\varepsilon)\bigr)$. Such a procedure can be iterated infinitely many times, and this gives the sequence $\mathcal{D}^{(r)}$ of subdomains of phase space, the existence of which is stated in the theorem. Nekhoroshev's stability estimates hold in every such domain, with stability times exponentially increasing at every step.

The sequence $\mathcal{D}^{(r)}$ of domains converges to a set $\mathcal{D}^{(\infty)}$ of invariant tori. This part of the proof is just an adaptation of Arnold's proof of KAM theorem and the set of invariant tori so obtained is similar to Arnold's one.

Let me finally explain how superexponential stability arises. Properties (v) and (vi) imply that every $(\varrho^{r+1}, 1/\varepsilon_{r+1})$–stable torus is ϱ_r–close to an invariant torus. In view of the form of the sequences ϱ_r and ε_r given in the statement of our theorem one has

$$\varepsilon_{r+1} = O(1/\exp(1/\varepsilon_r)) = O(1/\exp(\exp(1/\varepsilon_{r-1}))) = O(1/\exp(\exp(1/\varrho_r)))\ .$$

In view of this remark we may say that *in the neighbourhood of an invariant torus the natural perturbation parameter is the distance ϱ from the torus, and the diffusion speed is bounded by a superexponential of the inverse of the distance from an invariant torus.*

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