Lattice Point Problems and the Central Limit Theorem in Euclidean Spaces

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Abstract. A number of problems in probability and statistics lead to questions about the actual error in the asymptotic approximation of nonlinear functions of the observations. Recently new methods have emerged which provide optimal bounds for statistics of quadratic type. These tools are adaptions of methods which provide sharp bounds in some high dimensional lattice point remainder problems and solve some problems concerning the distribution of values of quadratic forms.

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1. Introduction.

Let $X_1, \ldots, X_n$ denote independent and identically distributed random vectors in R^d, $d \geq 1$.

Example 1.1 Assume that X_1 takes values in the finite set $\{-1,1\}^d \subset \mathbb{R}^d$ with equal probability 2^{-d}. Write

$$S_n = n^{-1/2}(X_1 + \ldots + X_n).$$

By the Central Limit Theorem (CLT) the sequence of random vectors S_n converges in distribution to a multivariate Gaussian distribution with mean zero and identity covariance matrix. Let $|m|^2 = \langle m, m\rangle$ denote the d-dimensional Euclidean norm and scalar-product. A number of statistical problems require to determine asymptotic approximations for the distribution of test statistics of type

$$T_n = |S_n|^2.$$

It is well known that the distribution function (d.f.) $\mathbf{P}\{T_n \leq v\}$ converges to the χ^2–distribution function with d degrees of freedom, say $\chi(v)$, for all $v \in \mathbb{R}$. In order to measure the error of this approximation we shall use the Kolmogorov distance and would like to determine the optimal exponents $\alpha > 0$ such that for a constant $c > 0$ independent of n

$$\delta_n = \sup_{v \geq 0} |\mathbf{P}\{T_n \leq v\} - \chi(v)| \leq c n^{-\alpha}. \tag{1.1}$$

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Here $T_n \le v$ means that the sum $\sqrt{n}\, S_n$ is contained in a ball $B_{vn} = \{|x| \le \sqrt{v\, n}\}$.

General estimates in the multivariate CLT (Sazonov [Sa], Bhattacharya and Rao [BR]) established the rate $\alpha = 1/2$ uniformly in the class of convex sets. Hence for balls and ellipsoids the achievable rate α should be at least $1/2$.

In Example 1.1 the sum S_n takes values in a lattice. By the local limit theorem its discrete density may be approximated by a Gaussian density such that

$$\mathbf{P}\big\{S_n = \frac{m}{\sqrt{n}}\big\} = \varphi_n(m)\big(1 + \mathcal{O}(n^{-1})\big), \quad \varphi_n(m) := \frac{1}{(2\pi n)^{d/2}} \exp\big\{ - \frac{|m|^2}{2n} \big\}.$$

Hence bounds in (1.1) can be derived from estimates of

$$\sup_v \Big| \sum_{m \in B_{vn} \cap \mathbb{Z}^d} \varphi_n(m) - \chi(v) \Big|.$$

Since the weights $\varphi_n(m)$ are 'smoothly' depending on m, the problem might be further reduced to the case of constant weights, which leads to a problem about counting the lattice points in B_{vn}. In this way Esseen [E] and Yarnold [Y] have proved

Theorem 1.2.

$$\mathbf{P}\{T_n \le v\} - \chi(v) = \exp\{-v/2\}\, \Delta(B_{vn}) + \mathcal{O}(n^{-1}). \tag{1.2}$$

Here $\Delta(A)$ denotes the relative lattice point remainder given by

$$\Delta(A) := \frac{\mathrm{vol}_{\mathbb{Z}} A - \mathrm{vol}\, A}{\mathrm{vol}\, A}, \tag{1.3}$$

with $\mathrm{vol}_{\mathbb{Z}} A$ and $\mathrm{vol}\, A$ denoting the number of points of the standard lattice $\mathbb{Z}^d$ in A and the volume of A respectively.

The relation (1.2) obviously establishes for Example 1.1 an equivalence between bounds in the lattice point remainder problem for ellipsoids and bounds of type (1.1) in the multivariate CLT. Indeed, Landau [L1] and Esseen [E] proved

$$\Delta(B_s) = \mathcal{O}\big(s^{-d/(d+1)}\big) \quad \text{resp.} \quad \delta_n = \mathcal{O}\big(n^{-d/(d+1)}\big).$$

Note though that Esseen's bound holds for balls and *arbitrary* i.i.d. random vectors X_j with finite fourth moment and identity covariance operator, where an equivalence of type (1.2) is not known.

Example 1.1 provides as well lower bounds for the error. Notice that nT_n assumes integer values in the interval $[-dn, dn]$. Distributing probability 1 among these values there exists an integer j such that

$$\mathbf{P}\{T_n = j\, n^{-1}\} \ge c\, n^{-1}, \quad c = 1/(2d+1).$$

Comparing the piecewise constant function $v \mapsto \mathbf{P}\{T_n \le v\}$ with the smooth limit $v \mapsto \chi(v)$, we find the lower bound $\delta_n \ge cn^{-1}$. Hence the rates α in (1.1) are restricted to $1/2 \le \alpha \le 1$.

This lecture is organized as follows. Section 2 contains results in the CLT for quadratic statistics in Euclidean spaces. Corresponding results in lattice point problems are described in section 3. Section 4 contains applications to distributions of values of positive definite and indefinite forms. Finally, in Section 5 we describe inequalities for trigonometric sums which are essential for these results.

A major part of the results in this lecture represents joint work with V. Bentkus.

2. Approximations in the CLT for Quadratic Statistics.

The CLT in Euclidean Spaces. Let $X, X_1, X_2, \dots$ be a sequence of i.i.d. random vectors taking values in the d-dimensional Euclidean space $\mathbb{R}^d$ including the case $d = \infty$ of infinite dimensional real Hilbert spaces. We assume that X has mean zero and $|X|$ has a finite second moment. Then the sums S_n converge weakly to a mean zero Gaussian random vector, say $G,$ with covariance equal to the covariance of X. Assume that G is not concentrated on a proper subspace of $\mathbb{R}^d$. Let Q denote a bounded linear operator on $\mathbb{R}^d$. Consider the quadratic form $\mathbb{Q}[x] = \langle Qx, x\rangle$ and assume that Q is non-degenerated, that is $\ker Q = \{0\}$.

The distribution of the quadratic form $\mathbb{Q}[G]$ is determined by its distribution function, say $\chi(v)$, and may be represented up to a shift as the distribution of a finite (resp. infinite) weighted sum of squares of i.i.d. standard Gaussian variables.

Rates of approximation in (1.1) in the CLT for $T_n = \mathbb{Q}[S_n]$ have been intensively studied especially in the infinite dimensional case in view of applications to non parametric goodness-of-fit statistics based on empirical distributions. Unfortunately the techniques of multivariate Fourier inversion of earlier results like that of Esseen [E] cannot be applied here. Several approaches have been developed for this problem.

A probabilistic approach is based on the Skorohod embedding resp. the KMT–method and provided bounds of order $\alpha = 1/4$, Kiefer [Ki], resp. $\mathcal{O}(n^{-1/2}\log n)$, Csörgö [Cs]. An analytic approach is based on a Weyl type inequality for characteristic functions, see (5.4). Using this technique, rates $\alpha = 1 - \varepsilon$ for any $\varepsilon > 0$ have been proved in (1.1), see [G1] and for refinements Bentkus and Zalesskii [BZ] and Nagaev and Chebotarev [NC]. Moreover, using methods like (5.4) the approximation $\chi(v)$ may be refined by asymptotic expansions in (1.1) up to an error of order $\mathcal{O}(n^{-k/2+\varepsilon})$ for *polynomials* of S_n of degree $k \geq 2$, see[G3].

Results providing *optimal* bounds of order $\alpha = 1$ are based on techniques used in related bounds for the corresponding lattice point problems. For *diagonal* quadratic forms and vectors X with *independent* coordinates the rate $\alpha = 1$ was proved for $d \geq 5$ in [BG1]. Here the additive structure of $\mathbb{Q}[x]$ allows to apply a discretization of type (5.5) and a version of the Hardy-Littlewood method of analytic number theory.

New tools described in (5.5)–(5.6) lead to the following result.

THEOREM 2.1. [BG2]. *Let $\mathbf{E}\, X = 0$ and $\beta_4 = \mathbf{E}\,|X|^4 < \infty$. Assume that $d \geq 9$ or $d = \infty$. Then*

$$\sup_v \Big| \mathbf{P}\big\{ \mathbb{Q}[S_n] \leq v \big\} - \mathbf{P}\big\{ \mathbb{Q}[G] \leq v \big\} \Big| = \mathcal{O}\big(n^{-1}\big). \tag{2.4}$$

The constant in this bound depends on β_4, the eigenvalues of Q and the covariance operator of G only.

REMARK 2.2.

1) For $d = 8$ the bound $\mathcal{O}(n^{-1}\ln^{\delta} n)$ holds with some $\delta > 0$.

2) Similar results like (2.4) hold for $\mathbb{Q}[x-a]$ involving an arbitrary center $a \in \mathbb{R}^d$. Here the approximation by the limit d.f. $\mathbf{P}\big\{\mathbb{Q}[G-a] \leq v\big\}$ needs to be improved by a further expansion term, say $n^{-1/2}\chi_1(v; a)$, which vanishes for $a = 0$.

3) For dimensions $d > 9$ including the case $d = \infty$ *uniform* bounds in (2.4), for $Q = \mathrm{Id}$ say, depend on moments of X and on lower bounds for a *finite* number, say m, of the largest eigenvalues of the covariance operator of X. For such bounds the minimal number $m \le d$ of eigenvalues needed has recently been determined to be $m = 12$, see [GU].

These results can be extended as follows.

U-Statistics. Let $X, X_1, \ldots, X_n$ be i.i.d. random variables taking values in an arbitrary measurable space $(\mathcal{X}, \mathcal{B})$ and let $g : \mathcal{X} \to \mathbb{R}$, $h : \mathcal{X}^2 \to \mathbb{R}$ denote real-valued measurable functions. Assume that $h(x, y) = h(y, x)$, for all $x, y \in \mathcal{X}$ and $\mathbf{E}\, h(x, X) = 0$ for almost all $x \in \mathcal{X}$. Consider the so called degenerated U-statistic

$$T_n = \frac{1}{n} \sum\nolimits_{1 \le i < j \le n} h(X_i, X_j) + \frac{1}{\sqrt{n}} \sum\nolimits_{1 \le i \le n} g(X_i), \tag{2.5}$$

and write $\beta_s = \mathbf{E}\, \big| g(X) \big|^s$ and $\gamma_s = \mathbf{E}\, \big| h(X_1, X_2) \big|^s$. Assuming that γ_2 is positive and $\beta_2 + \gamma_2$ is finite, the U-statistic T_n converges to a weighted χ^2–type distribution, say χ. Using a further expansion term, say χ_1, the problem is to derive explicit estimates for the error

$$\delta_n = \sup_v |\mathbf{P}\{T_n \le v\} - \chi(v) - n^{-1/2} \chi_1(v)|. \tag{2.6}$$

Rates of order $\delta_n = o(n^{-1/2})$ have been proved by Korolyuk and Borovskich [KB]. Moreover, for degenerated U-statistics of *any* degree $k \ge 2$ asymptotic approximations have been established up to errors $\delta_n = \mathcal{O}(n^{-k/2+\epsilon})$ in [G2].

Using similar techniques as in Theorem 2.1 the following explicit bound with optimal rate $\alpha = 1$ holds.

Theorem 2.3. [BG4]. *Let q_j denote the eigenvalues (ordered by decreasing absolute value) of the Hilbert-Schmidt operator induced on $L^2(\mathcal{X})$ by the kernel h. Write $\gamma_{s,r} = \mathbf{E}\big(\mathbf{E}\big(\big|h(X_1, X_2)\big|^s \,\big|\, X_2\big)\big)^r$ and $\sigma^2 := \gamma_2$. If $q_{13} \ne 0$,*

$$\delta_n \le \frac{C}{n} \Big(\frac{\beta_4}{\sigma^4} + \frac{\beta_3^2}{\sigma^6} + \frac{\gamma_3}{\sigma^3} + \frac{\gamma_{2,2}}{\sigma^4} \Big), \qquad \textit{where } C \le \exp\Big\{ \frac{c\sigma}{|q_{13}|} \Big\}. \tag{2.7}$$

Remark 2.4. 1) In cases where the expansion term χ_1 vanishes the condition $q_9 \ne 0$ suffices to prove a similar bound.

2) The result can be extended to von Mises statistics, i.e. statistics including diagonal terms $h(X_j, X_j) := d(X_j)$, where $d(X)$ has mean zero. This allows to consider as well statistics like $T_n := |S_n - a|^2$.

It is likely that improvements in lattice point approximation problems (see the Conjecture in Section 3) allow to prove error bounds of order $\mathcal{O}(n^{-1})$ in Theorems 2.1 and 2.3 for dimensions $5 \le d \le 8$ as well.

3. Lattice Point Problems.

For a symmetric positive definite matrix Q consider the quadratic form $\mathbb{Q}[x] = \langle Qx, x \rangle$ on R^d and the corresponding ellipsoid

$$E_s := \big\{ x \in \mathbb{R}^d : \mathbb{Q}[x] \le s \big\}, \qquad \text{for } s \ge 0.$$

Special Ellipsoids. Using similar arguments as for δ_n in Section 1 a corresponding lower bound can be shown for the lattice point remainder $\Delta(E_s)$ (for $Q = Id$), namely

$$\Delta(E_s) = \Omega(s^{-1}), \quad d \geq 1. \tag{3.1}$$

For balls of dimensions $2 \leq d \leq 4$, the lattice point remainder $\Delta(E_s)$ admits sharper lower bounds, e.g.

$$\Omega(s^{-3/4} \log^{1/4} s),\ d = 2, \quad \Omega(s^{-1} \log^{1/2} s),\ d = 3, \quad \text{and} \quad \Omega(s^{-1} \log\log s),\ d = 4,$$

due to Hardy [Ha], Szegö [Sz] and Walfisz [W2] respectively. The upper bound

$$\Delta(E_s) = \mathcal{O}(s^{-1}), \quad d \geq 5 \tag{3.2}$$

has been shown in a number of special cases. It holds for ellipsoids which are *rational*, that is the matrix Q is a multiple of a matrix with rational coefficients. Otherwise Q is called *irrational*. This result is due to Landau [L2] and Walfisz [W1] and depends on the rational coefficients in a non uniform way. For a detailed discussion see the monograph by Walfisz [W2].

For *diagonal* forms $\mathbb{Q}[x] = \sum_{j=1}^{d} q_j x_j^2$ with arbitrary $q_j > 0$, (3.2) is due to Jarnik [J1]. Moreover, if Q is irrational, Jarnik and Walfisz [JW] have shown that the bound

$$\Delta(E_s) = o(s^{-1}), \quad d \geq 5 \tag{3.3}$$

holds and is best possible for general irrational numbers q_j.

General Ellipsoids. For this class Landau [L1] obtained $\Delta(E_s) = \mathcal{O}(s^{-1+\lambda})$ with $\lambda = 1/(d+1)$ for $d \geq 1$, using Dirichlet series methods. His result has been extended by Hlawka [Hl] to convex bodies with smooth boundary and strictly positive Gaussian curvature, and improved to $\mathcal{O}(s^{-1+\lambda})$, with some $\lambda = \lambda(d) > 0$, $\lambda < 1/(d+1)$, by Krätzel and Nowak [KN1, KN2].

Assume without loss of generality that the smallest eigenvalue of Q is 1 and denote the largest eigenvalue by q. Hence $q \geq 1$. The following results provide optimal *uniform* bounds of type (3.2) resp. (3.3) for general ellipsoids.

THEOREM 3.1. [BG3, BG5]. *There is a constant $c > 0$ depending on d only and a function $\rho(s) \in [0, 2]$, depending on Q, see (5.2), such that for all $s \geq 1$*

$$\sup_{a \in \mathbb{R}^d} \Delta(E_s + a) \leq c\, q^d\, s^{-1} \big(s^{-\lambda} + \rho(s)\big), \qquad \text{for } d \geq 9, \tag{3.5}$$

where $\lambda \overset{\text{def}}{=} \frac{1}{2} \left[\frac{d-1}{2} \right] - 1$, *and*

$$\lim_{s \to \infty} \rho(s) = 0 \qquad \text{if and only if} \qquad Q \text{ is irrational.}$$

If $d = 8$ the bound $\sup_{a \in \mathbb{R}^d} \Delta(E_s + a) \leq c\, q^8 s^{-1} \ln^2 (s+1)$ still holds.

The error for generic forms $\mathbb{Q}[x]$ should be much smaller than for rational forms, which can be seen by the following *heuristic* argument. Let $C(m)$ denote the cube of side length 1 centered at a lattice point $m \in \mathbb{Z}^d$ and let I_s denote the indicator function of E_s. Define ξ_m as function of a randomly chosen Q as $\xi_m = I_s(m) - \int_{C(m)} I_s(x)\,dx$. Then $|\xi_m| \le 1$ and we may assume that the ξ_m have mean zero. Let D_s denote the set of lattice points m such that $C(m)$ intersects ∂E_s. Note that $\xi_m = 0$ for $m \notin D_s$. Then

$$\Delta(E_s)\ \mathrm{vol}\, E_s = \sum_{m\in\mathbb{Z}^d} \xi_m = \sum_{m\in D_s} \xi_m. \tag{3.4}$$

Since E_s has diameter proportional to $r = \sqrt{s}$, the sum in (3.4) extends over $\mathcal{O}(r^{d-1})$ nonzero summands only. If the random variables ξ_m are approximately independent the CLT implies for $r \to \infty$ with probability tending to 1 that (3.4) is smaller than $r^{(d-1)/2}\log r$. Hence one would expect that $\Delta(E_s) = \mathcal{O}(s^{-(d+1)/4}\log s)$. Indeed, Jarnik [J2] proved for $d \ge 4$ an upper bound of order $\mathcal{O}(s^{-d/4+\varepsilon})$ for Lebesgue almost all diagonal forms. For generic forms Landau [L3] established $\Delta(E_s) = \Omega(s^{-(d+1)/4})$. The results described so far suggest the following hypothesis about worst and generic case errors.

Conjecture. For any $\epsilon > 0$ the relative lattice point remainder is of order

$$\begin{aligned}
\Delta(E_s + a) &= \mathcal{O}(s^{-1}), && d \ge 5, && \text{for all } Q \text{ and } a,\\
&= o(s^{-1}), && d \ge 5, && \text{for irrational } Q,\\
&= \mathcal{O}(s^{-(d+1)/4+\epsilon}), && d \ge 2, && \text{for Lebesgue almost all } Q \text{ and } a.
\end{aligned}$$

4. Distribution of Values of Quadratic Forms.

Positive Definite Forms. For fixed $\delta > 0$ consider the shells $E_{s+\delta} \setminus E_s = \{x \in \mathbb{R}^s : s \le \mathbb{Q}[x] \le s+\delta\}$. Theorem 3.1 implies

Corollary 4.1. *For $d \ge 9$ and irrational Q we have*

$$\lim_{s\to\infty} \frac{\mathrm{vol}_{\mathbb{Z}}\big(E_{s+\delta} \setminus E_s\big)}{\mathrm{vol}(E_{s+\delta} \setminus E_s)} = 1. \tag{4.1}$$

This result may be applied as well to shrinking intervals of size $\delta = \delta(s) \to 0$ as s tends to infinity. The quantity $\mathrm{vol}(E_{s+\delta} \setminus E_s)$ measures the number of values of a positive quadratic form in an interval $(s, s+\delta]$, counting these values according to their multiplicities.

Let s and $n(s)$ denote successive elements of the ordered set $\mathbb{Q}[\mathbb{Z}^d]$ of values of $\mathbb{Q}[m]$. Davenport and Lewis [DL] conjectured that the distance between successive values, that is $n(s) - s$, converges to zero as s tends to infinity for irrational quadratic forms $\mathbb{Q}[x]$ and dimensions $d \ge 5$. They proved in [DL] that there exists a dimension d_0 such that for all $d \ge d_0$ and any given $\varepsilon > 0$ and any lattice point m with sufficiently large norm $|m|$ there exist another lattice point $\overline{m} \in \mathbb{Z}^d$ such that $\big|\mathbb{Q}[m+\overline{m}] - \mathbb{Q}[\overline{m}]\big| < \varepsilon$. This does not rule out the possibility of arbitrary large gaps between possible clusters of values $\mathbb{Q}[m]$, $m \in \mathbb{Z}^d$. This result has been improved by Cook and Raghavan [CR], providing the bound $d_0 \le 995$. Corollary 4.1 now solves this problem for $d \ge 9$.

Define the maximal gap between the values $\mathbb{Q}[m-a]$, $m \in \mathbb{Z}^d$ in the interval $[\tau, \infty)$ as $d(\tau; \mathbb{Q}, a) = \sup_{s\ge\tau}\big(n(s) - s\big)$. Then (4.1) implies

COROLLARY 4.2. [BG5]. *Assume that* $d \geq 9$ *and that* $\mathbb{Q}[x]$ *is positive definite. If* Q *is irrational then* $\sup_{a \in \mathbb{R}^d} d(\tau; \mathbb{Q}, a) \to 0$*, as* $\tau \to \infty$.

Indefinite Forms and the Oppenheim conjecture. Assume that Q is irrational and *indefinite*. Consider the infimum value of $\mathbb{Q}[m]$ for nonzero lattice points $m \in \mathbb{Z}^d$

$$M(Q) = \inf \Big\{ \big| \mathbb{Q}[m] \big| : \; m \neq 0, \; m \in \mathbb{Z}^d \Big\}.$$

Oppenheim [O1] conjectured that $M(Q) = 0$, for $d \geq 5$ and irrational *indefinite* Q, and has shown that this implies that the set $\mathbb{Q}[\mathbb{Z}^d]$ is dense in $\mathbb{R}$ for $d \geq 3$, see [O2]. This conjecture has been proved, e.g. for diagonal forms and $d \geq 5$ by Davenport and Heilbronn [DH] and for general forms and $d \geq 21$ by Davenport [Da]. For a review, see Margulis [Mar2]. It has been finally established for all dimensions $d \geq 3$ by Margulis [Mar1].

Let C_s denote a d–dimensional cube of side length $\sqrt{s}$ and center 0. The results of Theorem 3.1 are a consequence of more general asymptotic expansion of $\mu_s\{\mathbb{Q}[x] \leq \beta\}$ in powers of s^{-1} for certain 'smooth' distributions μ_s on $\mathbb{Z}^d$ with support in C_s, see [BG5, Theorem 2.1]. For indefinite forms this result yields the following refinement of Oppenheim's conjecture for dimensions $d \geq 9$.

For a sufficiently small positive constant, say $c_0 = c_0(d)$, let $d(s)$ denote the maximal gap in the finite set of values $\mathbb{Q}[m]$ such that $-c_0 s \leq \mathbb{Q}[m] \leq c_0 s$ and $m \in C_{s/c_0^2} \cap \mathbb{Z}^d$. Then

THEOREM 4.3. [BG5]. *For* $d \geq 9$ *the maximal gap satisfies*

$$d(s) \ll_d q^{3d/2} \big(s^{-\lambda} + \rho(s) \big) \qquad \text{for } s \geq c_0^{-1} \, q^{3d/2},$$

with $\rho(s) \leq 2$ *defined in (5.2) and* λ *given in Theorem 3.1.*

The *quantitative* version of Oppenheim's conjecture by Dani and Margulis [DM] describes the uniformity of the distribution of the set of values $\mathbb{Q}[\mathbb{Z}^d \cap C_s]$ for star-shaped sets like the cubes C_s introduced above. For a fixed interval $[\alpha, \beta]$ let $V_{\alpha,\beta}$ denote the set of $x \in \mathbb{R}^d$ such that $\mathbb{Q}[x] \in [\alpha, \beta]$. Eskin, Margulis and Mozes proved the following result using ergodic theory for unipotent groups.

THEOREM 4.4. [EMM]. *For any irrational indefinite form* Q *of signature* (p, q) *with* $q \geq 3$,

$$\frac{\operatorname{vol}_{\mathbb{Z}}\big(V_{\alpha,\beta} \cap C_s\big)}{\operatorname{vol}\big(V_{\alpha,\beta} \cap C_s\big)} = 1 + o(1), \qquad \text{as } s \to \infty. \tag{4.3}$$

In particular (4.3) holds for all indefinite irrational forms with $d \geq 5$.

Using expansion results for arbitrary forms, the error term in this convergence result can be explicitly estimated for $d \geq 9$, see [BG5, Theorem 2.6].

5. INEQUALITIES FOR CHARACTERISTIC FUNCTIONS AND TRIGONOMETRIC SUMS.

In order to prove the results of Sections 2–4, characteristic functions of $\mathbb{Q}[S_n]$ and weighted trigonometric sums, say $f(t)$, are used. In the latter case the weights are given by a uniform distribution on the lattice points in the cube C_{2s} smoothed at the boundary of C_{2s} by convolutions with uniform distributions on some sufficiently small cubes, retaining constant weights in the center part $C_s \subset C_{2s}$. A simplified version of these weighted trigonometric sums, used in the explicit bounds of Theorems 3.1 and 4.3, is defined as follows. Let

$$\varphi_a(t;s) = \Big| \big(\mathrm{vol}_{\mathbb{Z}}\, C_s\big)^{-3} \sum_{x_j \in \mathbb{Z}^d \cap C_s} \exp\big\{ i\, t\, \mathbb{Q}[x_1 + x_2 + x_3 - a] \big\} \Big|. \tag{5.1}$$

Note that $\varphi_a(t;s)$ is normalized so that $|\varphi_a(t;s)| \le \varphi_a(0;s) = 1$. Define

$$\gamma\big(s,T\big) = \sup_a \sup_{s^{-1/2} \le t \le T} \varphi_a(t;s).$$

It can be shown that $\lim_{s\to\infty} \gamma\big(s,T\big) = 0$ iff Q is irrational. Finally, given $d \ge 9$ and ε with $0 < \varepsilon < \kappa := 1 - 8/d$, the characteristic $\rho(s)$ of Theorem 3.1 and 4.3 is given by

$$\rho(s) = \inf_{T \ge 1} \Big(T^{-1} + \gamma\big(s,T\big)^{\kappa - \varepsilon}\, T^{\varepsilon} \Big). \tag{5.2}$$

The connections between the probability resp. counting problems and $f(t)$ are made by means of Fourier inversion inequalities based on Beuerling type functions, see Prawitz [Pr], which bound δ_n resp. $|\Delta(E_s)|$ by

$$\int_{-1}^{1} |f(t) - g(t)|\, t^{-1}\, dt + \int_{-1}^{1} (|f(t)| + |g(t)|)\, dt. \tag{5.3}$$

Here $g(t)$ is the continuous approximation to $f(t)$ replacing the distribution of S_n by a Gaussian distribution resp. the counting measure by the Lebesgue measure.

In the CLT the following version of Weyl's [We] difference scheme for sums of $\mathbb{R}^d$–valued, independent random vectors, say U, V (with identical distribution) and Z, W is used. Let $\bar{X}$ denote an independent copy of X and let $\widetilde{X} = X - \bar{X}$ be its symmetrization. The inequality

$$\Big| \mathbf{E} \exp\{ i\, t\, \mathbb{Q}[U + V + Z + W] \} \Big|^2 \le \mathbf{E} \exp\{ 2\, i\, t\, \langle Q\widetilde{U}, \widetilde{Z} \rangle \}, \tag{5.4}$$

now reduces the estimation of $f(t)$ in (5.3) to bounds of order $\mathcal{O}(n^{-1+\varepsilon})$ for conditional linear forms, but in a restricted domain $|t| \le n^{-\epsilon}$ only. This leads to rates $a = 1 - \varepsilon$ in (1.1), see [G1].

In order bound the integral (5.3) by $\mathcal{O}(n^{-1})$, this Weyl step is followed by a discretization step for positive definite functions $H : \mathbb{R}^d \to \mathbb{R}$. For even $n = 2\,l$ and binomial weights $p_n(k) = \binom{n}{l-k}/2^n$, bounds like

$$\mathbf{E}\, H(S_n) \le \frac{1}{n} \sum_{j=1}^{n} \mathbf{E}\, \Big(\sum_{|k| \le l} p_n(k) H\big(k\, n^{-1/2}\, \widetilde{X}_j \big) \Big), \tag{5.5}$$

reduce the support of X to $\mathbb{Z}^d$ and replace characteristic functions of S_n by weighted trigonometric sums.

Finally, for general Q, the desired bounds for weighted trigonometric sums, say $f(t)$, of type (5.1), are based on the following 'correlation' bound

$$\big|f(t)f(t+\varepsilon)\big| \le c\, q^d \left(\left(\varepsilon\, s\right)^{-d/2} + \varepsilon^{d/2}\right) \text{ for all } \ t \in \mathbb{R} \text{ and } \ \varepsilon \ge 0. \tag{5.6}$$

For $t = 0$ we have $f(t) = 1$ and (5.6) becomes a 'double large sieve' estimate for distributions on the lattice, see e.g. Bombieri and Iwaniec [BI]. The inequality (5.6) implies for $t_0 \le t_1$ with $0 < \delta \le |f(t_0)|, |f(t_1)| \le 2\,\delta$ that either

$$|t_0 - t_1| \le \lambda_r = c_1 \delta^{-4/d} s^{-1} \qquad \text{or} \qquad |t_0 - t_1| \ge \kappa = c_2 \delta^{-4/d}.$$

Thus either the arguments t_0 and t_1, where the trigonometric sums are of the same (large) order δ, nearly coincide or their distance has to be 'large' (dependent on δ and d). Hence the set of arguments t, where $f(t)$ assumes values in an interval $[\delta, 2\delta]$ like $A_\delta = \{t \ge v \,:\, \delta \le |f(t)| \le 2\delta\}$ with $v := s^{-2/d}$, may be roughly described as a set of intervals of size at most δ_r separated by 'gaps' of size at least κ. This allows to estimate part of (5.3) approximately as

$$\int_{A_\delta} |f(t)|\, \frac{dt}{t} \ll \sum_{l=0}^{L} \delta\, \lambda_r \frac{1}{v + l\kappa} \ll s^{-1} \delta^{1-8/d} \log \frac{1}{\delta},$$

with some L such that $L\kappa \le 1$. The sum of these parts for $\delta = 2^{-l}$, $l \in \mathbb{N}$ is now of order $\mathcal{O}(s^{-1})$, provided that $d > 8$, which explains the dimensional restriction of this method.

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