Brownian Motion and Random Obstacles

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Abstract. The investigation of Brownian motion and random obstacles exhibits a rich phenomenology and displays paradigms which appear in several other areas of random media. We provide here a brief survey of some recent developments.

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0. Introduction

Much effort has been devoted to the investigation of random media over the last two decades. This field offers a broad selectio n of surprising effects and represents a mathematical challenge. The above applies in particular to the topic of Brownian motio n and random obstacles, which has given rise to new ideas, results and techniques. We shall now explain what the subject is abo ut.

A common example of random obstacles are the soft Poissonian potentials:

$$V(x,\omega) = \sum_i W(x - x_i),\ x \in \mathbb{R}^d\ , \tag{0.1}$$

where $\omega = \sum_i \delta_{x_i}$ is a typical cloud configuration for the Poisson measure $\mathbb{P}$ with constant intensity $\nu > 0$, and $W(\cdot)$ is a bounded measurable nonnegative function, compactly supported and not a.e. equal to 0. Of central i nterest is the investigation of the interaction of Brownian motion with the random obstacles. Several path measures of interest arise in this context, for instance

- Brownian motion in a Poissonian potential, described by:

$$Q_{t,\omega} = \frac{1}{S_{t,\omega}}\ \exp\Big\{ - \int_0^t V(Z_s,\omega)ds\Big\}\, P_0,\ \ \text{(quenched measure)}, \tag{0.2}$$

with ω a $\mathbb{P}$-typical cloud configuration, $Z_.$ the canonical d-dimensional Brownian motion, P_0 the Wiener measure, $S_{t,\omega}$ the normalizing constant,

$$Q_t = \frac{1}{S_t}\ \exp\Big\{ - \int_0^t V(Z_s,\omega)ds\Big\}\, P_0 \otimes \mathbb{P},\ \ \text{(annealed measure)}\ , \tag{0.3}$$

with S_t the normalizing constant,

- Brownian crossings in a Poissonian potential, described by:

$$\hat{P}^{\lambda}_{x,\omega} = \frac{1\{H(x) < \infty\}}{e_\lambda(0,x,\omega)} \, \exp\Big\{ - \int_0^{H(x)} (\lambda + V(Z_s,\omega))ds \Big\} \, P_0, \quad \text{(quenched measure)} \,, \tag{0.4}$$

with ω as in (0.2), $\lambda \geq 0$, $x \in \mathbb{R}^d$, $H(x)$ the entrance time of $Z_.$ in the unit ball around x, $e_\lambda(0,x,\omega)$ the normalizing constant,

$$\hat{P}^{\lambda}_{x} = \frac{1\{H(x) < \infty\}}{\bar{e}_\lambda(x)} \, \exp\Big\{ - \int_0^{H(x)} (\lambda + V(Z_s,\omega))ds \; g \; P_0 \otimes \mathbb{P}, \quad \text{(annealed measure)}, \tag{0.5}$$

with $\bar{e}_\lambda(x)$ the normalizing constant.

Trapping problems provide natural interpretations for these path measures. In this light, $V(x,\omega)$ can be viewed as the random rate of absorption at location x for a particle diffusing in the environment ω. Thus (0.2), (0.3) govern the so-calle d quenched and annealed behaviors of a particle conditioned to survive absorption up to a (long) time t, whereas (0.4), (0.5) govern the quenched and annealed behaviors of a particle conditioned to perform a (long) crossing without being absorbed. Ther e are other physical interpretations, and for instance (0.2) also comes as a model of "flux lines in dirty-high-temperature su perconductors", cf. Section 4.6.3 of Krug [13], or Krug-Halpin Healy [14]. In this case t represents the transvers al thickness of a material with "columnar defects", rather than time. Discrete analogues of the above path measures also aris e in the literature, see for instance Bolthausen [3], Khanin et al. [12]. It may be helpful to mention that quenched measures describe the evolution in a $\mathbb{P}$-typical environment of a particle starting at the origin, whereas for the annealed m easures the $\mathbb{P}$-integration should be viewed as the result of an ergodic average over the starting point of the particle. It i s a recurrent theme of random media that quenched and annealed behaviors can be substantially different.

I. Normalizing constants for (0.2), (0.3)

Analyzing the principal asymptotic behavior of normalizing constants is a first step in the understanding of the path measures attached to Brownian motion in a Poissonian potential.

With the help of the Feynman-Kac formula, the normalizing constants $S_{t,\omega}$ and S_t can respectively be expressed as:

$$S_{t,\omega} = u_\omega(t,0) \;\text{ and }\; S_t = \mathbb{E}[u_\omega(t,0)] \,, \tag{1.1}$$

where $u_\omega(t,x)$ is the bounded solution of

$$\begin{cases} \partial_t \, u_\omega & = \; \frac{1}{2} \, \Delta u_\omega - V u_\omega \,, \\ u_\omega(0,x) & = \; 1 \,. \end{cases} \tag{1.2}$$

Their principal asymptotic behaviors as $t \to \infty$, are governed by:

$$\text{(1.3)} \qquad \mathbb{P}\text{–a.s.,} \quad S_{t,\omega} = \exp\{-c(d,\nu)\, t(\log t)^{-2/d}(1+o(1))\}\ ,$$

$$\text{(1.4)} \qquad S_t \quad = \exp\{-\widetilde{c}(d,\nu)\, t^{\frac{d}{d+2}}(1+o(1))\}\ .$$

The constants c and $\widetilde{c}$ are "explicit", and independent of the specific choice of $W(\cdot)$ in (0.1). If $\lambda(U)$ and $|U|$ respectively denote the principal Dirichlet eigenvalue of $-\frac{1}{2}\,\Delta$ in U and the volume of U, one has:

$$\text{(1.5)} \qquad c(d,\nu) = \lambda(B(0,R_0)), \ \text{ with } \ R_0 = \Big(\frac{d}{\nu|B(0,1)|}\Big)^{1/d}, \ \text{ whereas}$$

$$\text{(1.6)} \qquad \begin{aligned} \widetilde{c}(d,\nu) = \ & \inf_{U \text{ open}} \{\nu|U| + \lambda(U)\} = \nu|B(0,\widetilde{R}_0)| + \lambda(B(0,\widetilde{R}_0)), \ \text{ with} \\ & \widetilde{R}_0 = \Big(\frac{2\lambda(B(0,1))}{d\nu\,|B(0,1)|}\Big)^{\frac{1}{d+2}}\ . \end{aligned}$$

The annealed asymptotics (1.4) goes back to Donsker-Varadhan [5], where it was obtained as an application of large deviati on theory for occupation times of Brownian motion on a torus. Both asymptotics have also been derived through the analysis of p rincipal Dirichlet eigenvalues of $-\frac{1}{2}\,\Delta + V(\cdot,\omega)$ in large boxes, and the method of enlargement of obstacl es, cf. [24], [33], [36]. Sharper versions of (1.3), (1.4) can also be found in [36].

Intuitively for the quenched asymptotics, the contribution in the Feynman-Kac formula

$$\text{(1.7)} \qquad S_{t,\omega} = E_0\Big[\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big]$$

of Brownian paths going to some obstacle-free ball of radius of order $R_0(\log t)^{1/d}$, typically occurring within distance slightly less than t from the origin, and staying there up to time t, has the principal asymptotic behavior (1.3). On the other hand for the annealed asymptotics, the contribution in the representation

$$\text{(1.8)} \qquad S_t = \mathbb{E} \otimes E_0\Big[\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big]\ ,$$

of largely deviant environments, for which an obstacle-free ball of radius of order $\widetilde{R}_0\, t^{\frac{1}{d+2}}$ contains the origin, and of Brownian trajectories, which stay in the ball up to time t, has the principal behavior (1.4). Of course, under standing whether and up to what point these heuristics truly govern the quenched and annealed path measures (0.2), (0.3) is qui te another matter. As it turns out, the loose concept of *pockets of low local principal Dirichlet eigenvalue* for $-\frac{1}{2}\,\Delta + V(\cdot,\omega)$, plays an important role in the analysis of (0.2), (0.3). The predominance of atypical "pocket s of abnormally low eigenvalues" locally describing a system is a recurrent paradigm of random media, which for instance shows up in models of intermittency, cf. Gärtner-Molchanov [8], [9], Molchanov [17], in random walks in random envi ronment, cf. [4], [19], [20], [35], or in stochastic dynamics of spin systems with random interactions, cf. [16] and references therein.

II. Pinning effect and confinement property

The large t behavior of the quenched path measure $Q_{t,\omega}$ is governed by a "competition" between the various "pockets o f low local eigenvalues", resulting in a pinning effect: the path tends to get attracted to near minima of a certain random va riational problem. The discussion of the real pinning effect would go beyond the scope of this expository article, and we restr ict here to a simplified version. We refer to [32] or [36] for the "real story". We denote by $\lambda_\omega(U)$ the principal D irichlet eigenvalue of $-\frac{1}{2}\,\Delta + V(\cdot,\omega)$ in U, and for sufficiently small $\chi > 0$, consider the random function on $\mathbb{R}^d$.

$$F_t(x,\omega) = \alpha_0(x) + t\lambda_\omega(B(x,R_t)) \,, \tag{2.1}$$

with $R_t = \exp\{(\log t)^{1-\chi}\}$, a "small scale" growing slower than any positive power of t, and $\alpha_0(\cdot)$ a certain deterministic norm, the so-called quenched 0-th Lyapunov coefficient, see Section IV below, (the role of $\alpha_0(\cdot)$ is somewhat cosmetic in the simplified pinning effect we discuss here). Minimizing $F_t(\cdot,\omega)$ induces a competitio n between distance to the origin and occurrence of pockets of low local eigenvalues. One can show, cf. [32], [36], that

$$\mathbb{P}\text{-a.s.},\ \inf F_t(\cdot,\omega) \sim c(d,\nu)\, t(\log t)^{-2/d}, \ \text{ as } \ t \to \infty \,, \tag{2.2}$$

with $c(d,\nu)$ as in (1.5). Defining a skeleton of near minima of $F_t(\cdot,\omega)$ via

$$\mathcal{L}_{t,\omega} = \Big\{x \in \frac{1}{\sqrt{d}}\,\mathbb{Z}^d,\ F_t(x,\omega) \le \inf F_t(\cdot,\omega) + t(\log t)^{-\chi-\frac{2}{d}}\Big\} \,, \tag{2.3}$$

it can be shown that this set "lies almost at distance t" from the origin. The (simplified) pinning effect asserts that

THEOREM: *For small $\chi > 0$,*

$$\mathbb{P}\text{-}a.s.,\ \lim_{t\to\infty} Q_{t,\omega}(C) = 1, \ \ \textit{where} \tag{2.4}$$

$$C = \ \{Z_{\cdot} \textit{ comes before time } t \textit{ within distance } 1 \textit{ of some } x \in \mathcal{L}_{t,\omega} \textit{ from which it then does not move further away than } R_t \textit{ up to time } t\} \,. \tag{2.5}$$

As a by-product of the proof one also has the refinement of (1.3):

$$\mathbb{P}\text{-a.s.},\ \log S_{t,\omega} + \inf F_t(\cdot,\omega) = o(t(\log t)^{-\chi-\frac{2}{d}}) \,. \tag{2.6}$$

The true pinning effect is substantially sharper but involves certain random scales which would take too long to introduce here . In particular in the one-dimensional case it can be shown that for $\epsilon > 0$, with $\mathbb{P} \times Q_{t,\omega}$-probability ten ding to 1 as $t \to \infty$, $Z_.$ gets pinned within time ϵt in scale $t(\log t)^{-3}$ within an interva l of length $2(\log t)^{2+\epsilon}$, cf. [32], [36].

Loosely speaking, in the quenched situation the particle "goes the extra mile" to find an adequate pocket of low local eigenv alue. The annealed situation is quite different and favours a "good location" for the starting point of the path which then t ends to remain "confined" there. For instance in the case of hard obstacles, i.e. for the path measure

$$Q_t = \mathbb{P} \otimes P_0[\,\cdot\,|T > t]\;, \tag{2.7}$$

with T the entrance time of $Z_.$ in the obstacle set $\bigcup_i x_i + K$, $\omega = \sum_i \delta_{x_i}$ and K a fixed nonpola r compact set, one has the confinement property:

Theorem: *For any $d \geq 1$,*

$$\lim_{t\to\infty} Q_t[\sup_{0\leq u\leq t} |Z_u| \leq 2\, t^{\frac{1}{d+2}}\, (\widetilde{R}_0 + \epsilon(t))] = 1\;, \tag{2.8}$$

with $\widetilde{R}_0$ as in (1.6), and $\epsilon(t)$ a suitable function tending to 0, when t tends to ∞.

Thus the path "typically lives in scale $t^{\frac{1}{d+2}}$ under Q_t". The result is considerably harder to prove when $d \geq 2$. The two-dimensional case goes back to [26]. The case of dimension $d \geq 3$ was proved by Povel [21], who used a recent version of the method of enlargement of obstacles (cf. next section), and certain isoperimetric controls of R.R. Hall [?], which play the role of the Bonnesen's inequality in the two-dimensional proof. In fact in the two-dimensional case, it was proved in [26] that

Theorem: *($d = 2$)*

(2.10) *There exists a measurable map $D_t(\omega)$, $B(0, t^{1/4}\,(\widetilde{R}_0 + \epsilon(t)))$-valued, such that with Q_t-probability tending to 1, as $t \to \infty$, $Z_{[0,t]}$ is included in $B(D_t, t^{1/4}(\widetilde{R}_0 + \epsilon(t)))$ and no obstacle fall in $B(D_t, t^{1/4}(\widetilde{R}_0 - \epsilon(t)))$.*

In the case of the simple random walk on $\mathbb{Z}^2$, Bolthausen proved in [3] a version of this result using a refined version of D onsker-Varadhan's large deviation principles. It is also possible to obtain further information on the "spherical clearing" w here the process lives, cf. Schmock [23], when $d = 1$, [26], when $d = 2$, and [21], when $d \geq 3$:

(2.11) As $t \to \infty$, $t^{-\frac{1}{d+2}} Z_{.\, t^{\frac{2}{d+2}}}$ converges in law under Q_t, to the mixture with weight $\psi(x)/\int \psi$ of the laws of Brownian motion starting from 0 conditioned not to exit $B(x, \widetilde{R}_0)$, with ψ the principal Dirichlet eigenfunction of $-\frac{1}{2}\,\Delta$ in $B(0, \widetilde{R}_0)$.

III. THE METHOD OF ENLARGEMENT OF OBSTACLES

As mentioned above, in many questions related to Brownian motion in a Poissonian potential, the analysis of local principal Dir ichlet eigenvalues of $-\frac{1}{2}\,\Delta + V(\cdot,\omega)$ plays in important role. Indeed these numbers control in a very qu antitative fashion the decay properties of the Dirichlet-Schrödinger semigroup. This is illustrated by the estimat e:

$$\text{(3.1)}\quad \sup_x E_x\Big[\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\},\, T_U > t\Big] \le c(1+(\lambda_\omega(U)t)^{d/2})\, e^{-\lambda_\omega(U)t}\ ,$$

with c a merely dimension dependent constant and T_U the exit time of $Z_.$ from U, cf. [36]. The method of enlargement o f obstacles in particular provides an efficient way of deriving uniform controls on the numbers $\lambda_\omega(U)$ close to 0 (i .e. the bottom of the spectrum of $-\frac{1}{2}\,\Delta + V(\cdot,\omega)$ in $\mathbb{R}^d$), as U and ω vary. The rough idea is to remodel the region $V > 0$, and construct a coarse grained picture with lower combinatorial complexity than the original clo ud configuration, which for probabilistic purpose is simpler to analyze, but still has principal eigenvalues close to the origi nal objects. This remodeling of the region $V > 0$ brings into play a trichotomy of $\mathbb{R}^d$. In a first region, true obstacles a re quickly sensed by Brownian motion, and obstacles can be "enlarged" by imposing Dirichlet condition on this set. A second r egion where obstacles are insufficiently present and where enlargement of obstacles could possibly influence eigenvalues is sho wn to have little volume and thus little effect on probabilistic estimates. The third and last region receives no point of the cloud. In a sense, this parallels the trichotomy associated to any compact set K by considering the set of regular points of K, the set of irregular points of K and the complement of K.

Specifically after scaling the problem so that ϵ represents the size of the true obstacles, 1 the size of the pocket s of interest in the scaled cloud configurations (still denoted by ω), one constructs a density set $\mathcal{D}_\epsilon(\ o)$ where obstacles are enlarged and a bad set $\mathcal{B}_\epsilon(\omega)$ where obstacles are untouched, so that:

(3.2)

i) $\mathcal{D}_\epsilon(\omega),\ \mathcal{B}_\epsilon(\omega),\ \mathbb{R}^d\backslash(\mathcal{D}_\epsilon(\omega)\cup\mathcal{B}_\epsilon(\omega))$; partition $\mathbb{R}^d$,

ii) no point of ω falls in $\mathbb{R}^d\backslash(\mathcal{D}_\epsilon(\omega)\cup\mathcal{B}_\epsilon(\omega))$,

iii) for each box C of size 1, the maps $\omega\to C\cap\mathcal{D}_\epsilon(\omega)$ and $\omega\to C\cap\mathcal{B}_\epsilon(\omega)$ have range of cardinality smaller then $2^{\epsilon^{-d\beta}}$, with $\beta\in(0,1)$ a fixed number.

Denoting by $V_\epsilon(\cdot,\omega)=\sum_i \epsilon^{-2}\,W(\frac{\cdot - x_i}{\epsilon})$ the scaled potential, the construct ion can be done so that for a suitable $\alpha\in(0,\beta)$, Brownian motion, when starting on $\overline{\mathcal{D}}_\epsilon(\omega)$, strongly feels the obstacles before moving at distance ϵ^α:

THEOREM A_0: *(pointwise absorption estimate). There exists $\rho_0 > 0$, such that*

$$\text{(3.3)}\quad \overline{\lim_{\epsilon\to 0}}\,\epsilon^{-\rho_0} \sup_{\omega,x\in\omega verline\mathcal{D}_\epsilon(\omega)} E_x\Big[\exp\Big\{-\int_0^{H_{\epsilon^\alpha}} V_\epsilon(Z_s,\omega)ds\Big\}\Big] < 1,\ \ \textit{with}$$
$$H_{\epsilon^\alpha} = \inf\{s\ge 0,\ |Z_s - Z_0|\ge\epsilon^\alpha\}\ ,$$

and on the other hand the bad set has small volume:

Theorem B: *(volume estimate)*

(3.4) $$\exists\kappa > 0, \quad \overline{\lim_{\epsilon\to 0}} \sup_{\text{riptsize } C \text{ box of size } 1, \omega} \epsilon^{-\kappa}\, |\mathcal{B}_\epsilon(\omega)\cap C| < 1\ .$$

The construction of the trichotomy (3.3) i) relies on a type of quantitative Wiener test involving a series of capacities of a skeleton of the true obstacles at scales intermediate between ϵ^β and ϵ^α. In a sense (3.4), (3.5) parallels the Wiener test characterization of regular points of a compact set and the Kellog-Evans theorem on the smallness of the set of ir regular points of a compact set. As an application of the pointwise absorption estimates (3.3) one can in particular obtain eig envalue estimates:

Theorem A: *(eigenvalue estimate)*

(3.5)
$$\exists\rho > 0,\ \forall M > 0,\ \lim_{\epsilon\to 0} \epsilon^{-\rho} \sup_{\omega, U} (\lambda^\epsilon psilon_\omega(U\backslash\overline{\mathcal{D}}_\epsilon(\omega))\wedge M - \lambda^\epsilon_\omega(U)\wedge M) = 0\ ,$$

with $\lambda^\epsilon_\omega(O) =$ *principal Dirichlet eigenvalue of* $-\frac{1}{2}\,\Delta + V_\epsilon(\cdot,\omega)$ *in* O.

In other words this shows that in the asymptotic regime, provided $\lambda^\epsilon_\omega(U)$ has value of order unit, an addition al Dirichlet condition on $\overline{\mathcal{D}}_\epsilon(\omega)$ does not essentially increase the principal eigenvalue.

The method of enlargement of obstacles has numerous applications to the quenched and annealed situation, cf. [36]. The method e asily applies to non-Poissonian obstacles (uniformity of controls in ω is very handy), cf. [28], to shrinking obstacles, cf. [25], see also [2], to confidence intervals on principal eigenvalues, cf. [33], see also [39]. A version of the method in the discrete setting can be found in Antal [1]. Recently L. Erdös applied in [6] a version of the method to the study o f the Lifschitz tail effect for the density of states of the magnetic Schrödinger operator with Poissonian obstacles.

IV. Lyapunov norms

The technique of Lyapunov norms has been very helpful in the investigation of "off-diagonal" properties of the path measures (0.2), (0.3), in particular in the derivation of large deviation principles governing the location of Z_t. The Lyapunov norms describe the principal exponential decay of the normalizing constants in (0.4), (0.5). In a one-dimensional setting, in the co ntext of wave propagation in random media, they can be traced back to the work of Gärtner and Freidlin, cf. Chapter 7 of Frei dlin [7].

At the heart of the method lies the fact that the functions $e_\lambda(x,y,\omega)$ satisfy an almost supermultiplicative property and still contain much information about Brownian motion in a Poissonian potential. An important role is played by certain sha pe theorems (analogous to shape theorems of first passage percolation, cf.

Kesten [11]), which construct two families of norms on $\mathbb{R}^d$, $\beta_\lambda(\cdot) \le \alpha_\lambda(\cdot)$, $\lambda \ge 0$, the annealed and quenched Lyapunov coeffici ents:

$$\text{(4.1)} \quad \mathbb{P}\text{-a.s. for } M > 0, \ \lim_{x\to\infty} \sup_{0\le\lambda\le M} \frac{1}{|x|}\, |-\log e_\lambda(0,x,\omega) - \alpha_\lambda(x)| = 0\ ,$$

$$\text{(4.2)} \quad \text{for } M > 0, \ \lim_{x\to\infty} \sup_{0\le\lambda\le M} \frac{1}{|x|}\, |-\log\ tog(\mathbb{E}[e_\lambda(0,x,\omega)]) - \beta_\lambda(x)| = 0\ .$$

These shape theorems are quite robust and one can replace in (4.1), (4.2), $e_\lambda(0,x,\omega)$ by the λ-Green function $g_\lambda(0,x,\omega)$, or $e_\lambda(x,0,\omega)$, or $\exp\{-d_\lambda(0,x,\omega)\}$, with d_λ certain natural random distance functions (in general nongeodesic) constructed with the e_λ, cf. [36]. The Lyapunov coefficients enter several large d eviation theorems, cf. [29], [30], [31], as well as the random variational problem of the pinning effect. For instance when $arphi(t) \to \infty$,

(4.3)

$\mathbb{P}$-a.s. under $Q_{t,\omega}$, $Z_t/\varphi(t)$ satisfies a large deviation principle at rate $\varphi(t)$, with rate function:

i) $\alpha_0(x)$, if $\varphi(t) = t(\log t)^{-2/d}$, cf. [31],

ii) $\alpha_0(x)$, if $t(\log t)^{-2/d} << \varphi << t$, cf. [29],

iii) $I(x) = \sup_{\lambda\ge 0}\,(\alpha_\lambda(x) - \lambda)$, if $\varphi(t) = t$, cf. [29].

Similar results hold under the annealed measure Q_t, when $d \ge 2$, with $t^{\frac{d}{d+2}}$ in place of $t(\log t)^{-2/d}$ and $\beta_\lambda(\cdot)$ in place of $\alpha_\lambda(\cdot)$, (the one-dimensional case is singular, cf. Povel [22]). In the discrete setting (4.3) iii) has been proved by Zerner in [40]. In fact the above strategy also applies in the context of random walks in random environments, cf. Zerner [41]. This is especially interesting since there are few mathematical results on this model.

The understanding of crossing Brownian motion in a Poissonian potential, see (0.4), (0.5), is so far rather primitive. However recently for rotationally invariant truncated Poissonian potentials, Wüthrich has been able to relate in [37], the fluctuatio n properties of $-\log e_\lambda(0,x,\omega)$ to transversal fluctuations of the path under the path measure (0.4). In a slightly d ifferent situation ("point to line" model), he was also able to obtain a result about the superdiffusive nature of transversa l fluctuations, cf. [38]. This is qualitatively similar to what happens in first passage percolation, cf. Licea-Newman-Piza [15], Newman-Piza [18].

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