Travelling Water-Waves, as a Paradigm for Bifurcations in Reversible Infinite Dimensional "Dynamical" Systems

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Abstract. We first show a typical bifurcation study for a finite dimensional reversible system, near a symmetric equilibrium taken at 0. We state the results on known small bounded solutions: periodic, quasiperiodic, homoclinic to 0, and homoclinics to periodic solutions. The main tool for such a study is center manifold reduction and normal form theory, in presence of reversibility. This allows to prove persistence of large class of reversible (symmetric) solutions under higher order terms, not considered in the normal form. We then present water-wave problems, where we look for 2D travelling waves in a potential flow. In case of finite depth layers, the problem of finding small bounded solutions, is shown to be reducible to a finite dimensional center manifold, on which the system reduces to a reversible ODE. Bounded solutions of this ODE lead to various kinds of travelling waves which are discussed.

If the bottom layer has infinite depth, which appears to be the most physically realistic case, concerning the validity of results in the parameter set, the mathematical problem is more difficult. We don't know how to reduce it to a finite dimensional one, due to the occurence of a continuous spectrum (of the linearized operator) crossing the imaginary axis. We give some hints, on how to attack this difficulty, specially for periodic and homoclinic solutions which have now a *polynomial decay* at infinity .

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1 Bifurcations of reversible systems near a symmetric equilibrium

1.1 Basic tools

Let us first consider a finite dimensional vector field of the form

$$\frac{dU}{dx} = F(U) \tag{1}$$

where $U(x)$ lies in $\mathbb{R}^n$, we say that system (1) is *reversible* if there exists a linear symmetry S, satisfying $S^2 = \mathbb{I}$, such that $SF = -F \circ S$. This implies, in particular, that if $x \longmapsto U(x)$ is solution of (1), then $x \longmapsto SU(-x)$ is also solution. Assume in addition that $F(0) = 0$ and that F is $C^k, k \geq 2$, and define the derivative at the origin: $L = DF(0)$. It is clear that $SL = -LS$, which implies that the set of eigenvalues of L is symmetric with respect to both axis in $\mathbb{C}$. In what follows, we are specially interested in solutions of (1) which *stay in a neighborhood of* 0 *for* $x \in \mathbb{R}$. The main tool for understanding such solutions is a center manifold reduction theorem [19] (see [28] for a complete and pedagogic proof):

Theorem 1 (Center manifold theorem) *Assume that the spectrum of L is composed with a part σ_0 on the imaginary axis and another part σ_h lying at a positive distance from the imaginary axis. Let us denote respectively by E_0 and E_h the subspaces invariant under L, corresponding to this splitting of the set of eigenvalues of L. Then, there exists a function $\Psi \in C^k(E_0, E_h), \Psi(0) = 0, D\Psi(0) = 0$, and a neighborhood $\mathcal{U}$ of 0 in $\mathbb{R}^n$, such that the manifold*

$$M_0 = \{X + \Psi(X) | X \in E_0\} \subset \mathcal{U}$$

has the following properties

(i) M_0 is locally invariant under (1);

(ii) M_0 contains all solutions of (1) staying in $\mathcal{U}$ for all $x \in \mathbb{R}$;

(iii) Ψ commutes with the symmetry $S : S\Psi = \Psi \circ S_0$ (we denote by S_0 the restriction of S on the space E_0).

The part (iii) of the above theorem is not in [19] but results easily from the proof of the theorem, as it is also true for any linear unitary operator commuting with F.

We are in fact interested in *Bifurcations of solutions* lying in a neighborhood of 0, i.e. in a structural change of these solutions when some parameter varies. To fix ideas, we now consider systems of the form

$$\frac{dU}{dx} = F(\mu, U), \quad F(0,0) = 0 \tag{2}$$

where μ is a real parameter, F being smooth with respect to both arguments, and $F(0, \cdot)$ satisfies the same assumptions as F in (1). Then, it is nearly straightforward that there is a neighborhood of 0 in $\mathbb{R} \times \mathbb{R}^n$ for (μ, U) for which a family of center manifolds M_μ exist, of the form

$$U = X + \Psi(\mu, U), X \in E_0, \Psi(0,0) = 0, D_X\Psi(0,0) = 0. \tag{3}$$

The interest of this result rests in particular in the "uniform" validity for μ in a neighborhood of 0. Indeed, in case 0 stays an equilibrium of (2) when μ varies, the eigenvalues of $D_U F(\mu, \cdot)$ may escape from the imaginary axis, and we might be tempted to apply the classical invariant manifold theorem for hyperbolic situations. This would lead to a domain of validity much smaller than the one given by the present theorem. Of course we "pay" this by the non uniqueness of such center manifolds, and the fact that the more regularity we wish, the smaller is the existence domain for M_μ.

The reduced system on M_μ is written

$$\frac{dX}{dx} = f(\mu, X) \text{ in } E_0 \tag{4}$$

and is *still reversible*, provided that the representation S_0 of S on E_0 is not trivial. Moreover, $f(0,0) = 0$ and $D_X f(0,0) = L_0 = D_U F(0,0)|_{E_0}$ has all its eigenvalues of zero real part.

Now, a very powerfull tool for studying the reduced system (4) is *normal form* theory. This technique consists in making near the origin, a change of variables close to identity and polynomial in X, which modifies the form of (4) in simplifying its Taylor expansion up to a fixed order (the degree of the polynomial). We then expect to recognize more easily relevant solutions of our system on a "simplified" f. Normal form theory goes back to Poincaré and Birkhoff and was more recently developed in particular by V.Arnold [2], Belitskii [4], Cushman & Sanders [6] and Elphick et al [9]. In the context of a system like (4) where all eigenvalues of L_0 lie on the imaginary axis, we use the following global characterization result of [9], (see also [12]):

Theorem 2 (Normal form theorem) *For any $p \leq k$, there is a neighborhood $\widetilde{\mathcal{U}}_p$ of 0 in $\mathbb{R} \times E_0$ and there are polynomials $\Phi(\mu, \cdot)$ and $N(\mu, \cdot)$ $E_0 \to E_0$, of degree p, with coefficients smooth in μ, such that $\Phi(0,0) = N(0,0) = 0, D_X \Phi(0,0) = D_X N(0,0) = 0$ and such that for $(\mu, X) \in \widetilde{\mathcal{U}}_p$ the change of variable*

$$X = \widetilde{X} + \Phi(\mu, \widetilde{X})$$

transforms (4) into the following system which has the same regularity in $(\mu, \widetilde{X})$:

$$\frac{d\widetilde{X}}{dx} = L_0 \widetilde{X} + N(\mu, \widetilde{X}) + R(\mu, \widetilde{X}) \tag{5}$$

where N is characterized by

$$N(\mu, e^{L_0^* x} X) = e^{L_0^* x} N(\mu, X), \forall x \in \mathbb{R}, \forall X \in E_0, \forall \mu \text{ near } 0,$$

and $R(\mu, \widetilde{X}) = o(||X||^p)$. In addition, (5) inherits the symmetries of (2).

This theorem provides an additional symmetry to nonlinear "simplified" terms, this symmetry only resulting from the linearized operator! The proof which includes the parameter dependance of the polynomial coefficients and the optimal estimate on the rest R, is quite technical (see hint in [9], and [12]).

1.2 Study of some reversible normal forms

Let us restrict our attention to systems such that 0 stays solution of (2) for $\mu \neq 0$. This eliminates some cases which are not of interest here. Now, because of reversibility, we know that the eigenvalues of $D_U F(\mu, 0)$ are symmetric with respect to both axis, hence theorem 1 indicates that bifurcation situations may occur at least when some eigenvalues meet (by pairs) the imaginary axis. The simplest case is when L_0 has only a double 0 eigenvalue on the imaginary axis. This leads to a 2 dimensional center manifold for the study of small bounded solutions of (4). We give below some details only on the next most important cases, i.e. when

(i) L_0 has only a double 0 and a pair of simple pure imaginary eigenvalues on the imaginary axis;

(ii) L_0 has only a pair of double pure imaginary eigenvalues on the imaginary axis.

Notice that case (ii) was introduced by Y.Rocard (see chapter I.14 of [27]) when he presents the instability "par confusion de fréquences propres", which occurs in the phenomenon of the fluttering of a wing (submitted to the aerodynamic forcing of a big wind), particularly dangerous for planes and for long suspended bridges.

1.2.1 Case (i)

Here the center manifold is four dimensional. Let us denote by $\pm iq$ the pair of simple eigenvalues and decompose $X = A\xi_0 + B\xi_1 + C\zeta + \overline{C\zeta}$, where (A, B) are real amplitudes, C a complex one and $L_0\xi_0 = 0, L_0\xi_1 = \xi_0, L_0\zeta = iq\zeta$.

Then we need to know how the reversiblity symmetry S_0 acts on $(A, B, C, \overline{C})$. There are two theoretical possibilities, depending on whether $S\xi_0 = \xi_0$ or $-\xi_0$. In most physical problems we have the first case, so $S_0 : (A, B, C, \overline{C}) \to (A, -B, \overline{C}, C)$ and after parameter dependent rescaling, the normal form, truncated at quadratic order, reads

$$\begin{cases} \frac{dA}{dx} = B \\ \frac{dB}{dx} = \mu A + A^2 + c|C|^2, \\ \frac{dC}{dx} = iC(q + d_1\mu + d_2 A), \end{cases} \tag{6}$$

where $c = \pm 1$ and (real) coefficients d_j can be explicitely computed (see [14] for a proof of (6) and the computation of principal part of coefficients on a specific physical problem). This system is *integrable*, with the two first integrals

$$K = |C|^2, H = B^2 - (2/3)A^3 - \mu A^2 - 2cKA, \tag{7}$$

and we show at figure 1 the various graphs of functions

$$f_{\mu,K,H}(A) = (2/3)A^3 + \mu A^2 + 2cKA + H$$

depending on (K, H), for $\mu > 0$. In this case, we have, in addition to the trivial equilibrium, another "conjugate" equilibrium, and several types of periodic solutions, quasi-periodic solutions (interior of the triangular region in (K, H) plane, and homoclinic solutions, one homoclinic to 0, and all others homoclinic to one of the periodic solutions.

We represent on figure 2, in the (A, B) plane all bounded solutions for $cK < \mu^2/4$. Notice that the homoclinic solution to A_+ corresponds here to a solution

Figure 1: case (i). Graphs of $f_{\mu,K,H}(A)$ for $\mu > 0$.

homoclinic to a periodic solution since $K \neq 0$. Notice that $A_+ \sim -cK/\mu$ when $|K| \lll |\mu|$, meaning that oscillations at ∞ are very small in this case. For $K = 0$ this corresponds to a *solution homoclinic to 0*, even though the stable and unstable manifolds of 0 are only one dimensional (in the 4 dim space!). We shall see in next section that this solution does not exist in general for the full system (5), even though one may compute its expansion in powers of the bifurcation parameter μ up to any order.

Figure 2: case (i). Bounded solutions of (6) for various values of H in the (A, B) plane, for $\mu > 0, cK < \mu^2/4$. $A_\pm = 1/2(-\mu \pm \sqrt{\mu^2 - 4cK})$.

An analogous study holds for $\mu < 0$ using $f_{\mu,H,K}(A) = -f_{-\mu,-H,K}(-A))$.

1.2.2 Case (ii)

Here the center manifold is again four dimensional. Let us denote by $\pm iq$ the pair of double eigenvalues at criticallity, and define by (A, B) the complex amplitudes corresponding respectively to the eigenmode and to the generalized eigenmode. This case is often denoted by "1:1 reversible resonance". We can always assume that the reversibility symmetry S_0 acts as: $(A, B) \longmapsto (\overline{A}, -\overline{B})$. The normal form, at any order, reads (see a proof in [12]):

$$\begin{aligned} \frac{dA}{dx} &= iqA + B + iAP[\mu, |A|^2, i/2(A\overline{B} - \overline{A}B)], \\ \frac{dB}{dx} &= iqB + iBP[\mu, |A|^2, i/2(A\overline{B} - \overline{A}B)] + AQ[\mu, |A|^2, i/2(A\overline{B} - \overline{A}B)], \end{aligned} \tag{8}$$

where $P(\mu,\cdot,\cdot)$ and $Q(\mu,\cdot,\cdot)$ are real polynomials. Let us define more precisely the coefficients of Q, for the cubic normal form [N of degree 3 in (5)]: $Q(\mu,u,v) = \mu + q_2 u + q_3 v$, where q_2 may be taken as ± 1, after a parameter dependent rescaling. This means that for $\mu > 0$ the eigenvalues are at a distance $\sqrt{\mu}$ from the imaginary axis, while, for $\mu < 0$, they sit on the imaginary axis. The explicit computation of the principal parts of coefficients of polynomials P and Q is made for instance in [7] on a specific physical example. The vector field (8) is integrable, with the two following first integrals:

$$K = i/2(A\overline{B} - \overline{A}B), \qquad H = |B|^2 - \int_0^{|A|^2} Q[\mu, u, K]du.$$

It is then possible to describe all small bounded solutions of (8), and to discuss the various types of solutions in the (K,H) plane, for $\mu > 0$, or $\mu < 0$ (see [17]). We obtain families of periodic and quasi-periodic solutions and, for $\mu > 0, q_2 < 0$ a "circle" of solutions homoclinic to 0, for $H = K = 0$, due to the $SO(2)$ invariance of the normal form, while for $\mu < 0, q_2 > 0$ we have a family (curve in the (H,K) plane) of "circles" of solutions homoclinic to periodic solutions (as in case (i)) (the amplitude is here minimum at $x = 0$).

1.3 Typical persistence results

In section 1.2, we investigated the normal forms, i.e. equation (5) with no remaining term R, and we obtained various type of solutions that we would like to be persistent for the complete problem (5). The problem consists now in proving persistence results. In summary, the *persistence of periodic solutions* of the normal form can in general be performed, through an adaptation of the Lyapunov-Schmidt technique (see [14],[22]). The *persistence of quasi-periodic solutions* is much more delicate, and can only be performed in a subset of the (H,K) plane, where these solutions exist for the normal form. Typically, it is proved for case (i), that for any fixed μ, quasi-periodic solutions exist on a subset of the interior of the triangular region of figure 1, which is locally the cartesian product of a curve with a Cantor set (see a complete proof in [16] for case (ii), and see [14] for case (i), both applied to specific examples in fluid mechanics). The persistence of solutions homoclinic to periodic solutions, provided that they are not too small, needs some technicality, see for instance [14] for case (i) and [17] for case (ii). The same results holds for solutions homoclinic to 0 in case (ii). In fact one can prove the persistence of two symmetric (reversible) solutions (instead of a a full circle of solutions), using a transversality argument (intersection of the stable manifold of a periodic orbit (or of the fixed point in case (ii) for $\mu < 0$) with the subspace of symmetric points), after controlling the size of the perturbation due to R, which applies for $x \in [0, +\infty)$.

Now, for the normal form of case (i), there is a family of orbits homoclinic to periodic solutions whose amplitudes can be chosen arbitrarily small. It can be proved (see Lombardi [22] for a complete proof) that there are two families of reversible solutions homoclinic to a periodic solution whose size may be chosen arbitrary, until a (non zero) *exponentially small size* $\mu^{-1}e^{-c/\sqrt{\mu}}$ (smaller than any power of the bifurcation parameter μ). The method used by Lombardi consists in a complete justification of a matching asymptotic expansion method of the solution which is extended in a strip of the complex plane, where the singularity in the complex plane originates from one of the homoclinic solution of the truncated normal form (6). Moreover, despite of the fact that a solution homoclinic to 0 exists for the normal form (6), this is not true in general for the full system (5) (see [23]), even though one can compute an asymptotic expansion up to any order of such an homoclinic (non existing) "solution"! This non obvious result says in particular that we cannot avoid small oscillations at infinity in this case.

2 Application to the water wave problem

Let us consider the case of one layer (thickness h) of an inviscid fluid, the flow is assumed potential, under the influence of gravity g and surface tension T acting at the free surface (see left of figure 3). We are interested in steady waves of permanent form, i.e. travelling waves with constant velocity c. Formulating the problem in a moving reference frame, our solutions are steady in time, and we intend to consider the unbounded horizontal coordinate ξ as a "time". Let us denote by ρ the fluid density, then we choose c as the velocity scale and $l = T/\rho c^2$ as the length scale. The important dimensionless parameters occuring in the equations are $\lambda = ghc^{-2}$, $b = T(\rho hc^2)^{-1} = l/h$.

Figure 3: Left: geometric configuration of the water-wave problem. Right: positions of the 4 critical eigenvalues of L_μ in function of $\mu = (b, \lambda)$.

A nice formulation of this problem uses a change of coordinates introduced by Levi-Civita [21]. He uses the coordinates (x,y) defined by the complex potential $w(\xi+i\eta)=x+iy$ and unknown are α and β defined by $w'(\xi+i\eta)=e^{-i(\alpha+i\beta)}$ (complex velocity). The free surface is given by $y=0$, the rigid bottom by $y=-1/b$. The physical free surface is given by $\eta=Z(\xi)=\widetilde{Z}(x)=\int_{-1/b}^{0}(e^{-\beta}\cos\alpha-1)dy$. In our formulation, the unknown is $[U(x)](y)=(\alpha_0(x),\alpha(x,y),\beta(x,y))^t$ and the system has the form

$$\frac{dU}{dx}=F(\mu,U)=\left\{\begin{array}{l}\sinh\beta_0+\lambda be^{-\beta_0}\int_{-1/b}^{0}(e^{-\beta}\cos\alpha-1)dy\\ \left.\begin{array}{l}\frac{\partial\beta}{\partial y},\\ -\frac{\partial\alpha}{\partial y}\end{array}\right\}-1/b<y<0.\end{array}\right. \tag{9}$$

where $\mu=(b,\lambda)$, and equation (9) has to be understood in the space $\mathbb{H}=\mathbb{R}\times\{L^1(-1/b,0)\}^2$, and $U(x)$ lies in $\mathbb{D}=\mathbb{R}\times\{W^{1,1}(-1/b,0)\}^2\cap\{\alpha_0=\alpha|_{y=0}, \alpha|_{y=-1/b}=0\}$, where we denote by β_0 the trace $\beta|_{y=0}$ and by $W^{1,1}(-1/b,0)$ the space of integrable functions with an integrable first derivative on the interval $(-1/b,0)$. A solution of our water-wave problem is any $U\in\mathcal{C}^0(\mathbb{D})\cap\mathcal{C}^1(\mathbb{H})$ which is solution of (9), where (e.g.) $\mathcal{C}^0$ means continuous and bounded for $x\in\mathbb{R}$.

It is clear that $U=0$ is a particular solution of (9), which corresponds to the flat free surface state. A very important property of (9) is its *reversibility*: indeed let us define the symmetry S: $SU=(-\alpha_0,-\alpha,\beta)^t$, then it is easy to see that the linear operator S *anticomutes with* $F(\mu,\cdot)$. This reflects the invariance under reflexion symmetry $\xi\to-\xi$ of our original problem.

Remark 3 *There is a large class of water-wave problems which can be treated in a similar way: one may consider several layers of non miscible perfect fluids, and consider cases with or without surface (or interface) tension (see [11] for these formulations).*

Since we are interested in solutions near 0, it is natural to study the problem obtained after linearization near 0. We then define the linear operator $L_\mu=D_UF(\mu,0)$, unbounded and closed in $\mathbb{H}$. In all problems, for layers with finite depth , it can be shown that the spectrum of L_μ which is *symmetric with respect to both axis* of the complex plane because of reversibility, is only composed of isolated eigenvalues of finite multiplicities, accumulating only at infinity. More precisely, denoting by ik these eigenvalues (not necessary pure imaginary), then one has the classical "dispersion relation" for solving the eigenvalues, under the form of a complex equation $f(\mu,k)=0$. For problem (9), we obtain the following dispersion relation:

$$(\lambda b+k^2)k^{-1}\sinh k/b-\cosh k/b=0,\qquad \text{for } k\neq 0. \tag{10}$$

There is no more than 4 eigenvalues on (or close to) the imaginary axis, the rest of them being located in a sector $(ik\in\mathbb{C};|k_r|<p|k_i|+r)$ of the complex plane (see right side of figure 3). There is a codimension 2 case when $(b,\lambda)=(1/3,1)$, where 0 is a quadruple eigenvalue. The roots of the dispersion equation give the

poles of the resolvent operator $(ik\mathbb{I} - L_\mu)^{-1}$. In addition, we obtain an estimate of the form

$$||(ik\mathbb{I} - L_\mu)^{-1}||_{\mathcal{L}(\mathbb{H})} \leq c/|k|, \tag{11}$$

where $c > 0$ is fixed and for large enough $|k|$, and where $\mathcal{L}(\mathbb{H})$ is the space of bounded linear operators in $\mathbb{H}$. The choice of the basic space $\mathbb{H}$ should be appropriate for finding the good estimate (11) of the resolvent, this is a little delicate for problems with several layers and no surface tension (see [11]). This estimate is essential in our method of reduction to a center manifold.

For the study of the nonlinear problem (9) the idea is now to use a *center manifold* reduction like in section 1, which leads to *an ordinary differential equation* of dimension at most 4 in the present problem. Let us assume that, for μ near μ_0, the eigenvalues of L_μ are contained either in a small vertical strip of width tending towards 0 for $\mu \to \mu_0$, or at a distance of order 1 from the imaginary axis, then the estimate (11) allows us to find such a center manifold as in finite dimensional case (see [20], [24], [29]). Roughly speaking, all "small" bounded continuous solutions taking values in $\mathbb{D}$, of the system (9) for values of μ near μ_0, lie on an invariant manifold M_μ which is smooth (however loosing the C^∞ regularity) and which exists in a neighborhood of 0 independent of μ (depending on the required smoothness). The dimension of M_μ is equal to the sum of dimensions of invariants subspaces belonging to pure imaginary eigenvalues occuring for the critical value μ_0 of the parameter. In addition, the trace of system (9) on M_μ is also *reversible under the restriction* S_0 of the symmetry S. It results, in particular that the study we made at sections 1.2 and 1.3 applies here (after a suitable choice of the bifurcation parameter). The situation near the set $\lambda = 1, b < 1/3$ was studied first in [1] and [25] in an uncomplete way. Here case (i) applies (not too close to the codimension 2 point, since we would need to use another normal form there (see [10] for such a study). Denoting by $\mu = \lambda - 1$, it is shown in [14] that we are in situation of figure 1, with $c > 0$. The study of unavoidable exponentially small oscillations at infinity was first studied directly on the water wave problem in [3] and [26], and as a general property for a large class of problems in [22]. The generic non existence of solitary waves in this case follows from [23]. Now, the study made for case (ii) applies near the curve Γ of figure 3 (right). For problem (9) it is shown that coefficient q_2 *is negative*. For other water wave problems with more than one layer, this coefficient may change of sign, which leads to new types of solutions near this singular case. In the present problem, we then have for (b, λ) slightly above the curve Γ, the *bifurcation of two reversible solitary waves, with exponentially damping oscillations at infinity* [13].

Remark 4 *Such reversible bifurcations in function of 2 parameters also appear in various physical problems. A very nice example is in the study of localized stuctures for long (assumed infinitely long) rubber rods subject to end tension and moment! The basic state is the straight rod. The study of eigenvalues of the linearized operator lead to a picture analogue to figure 3 (right). In particular, the two homoclinic orbits above, become four because of an extra symmetry of the problem, and are physically important in the study of buckling of such rods (see [5]).*

3 Physical relevance - infinite depth problem

A common point for the various water wave problems, is that when the bottom layer thickness grows ($b \to 0$ in (10)), there is an *accumulation of eigenvalues on the whole real axis*, and at the limit, as we choose a space $\mathbb{D}$ where we replace $1/b$ by ∞ and suppress the boundary condition at $y = -1/b$, all real eigenvalues disappears, leaving the place to the *entire real axis forming the essential spectrum*: for σ real $\neq 0$ the operator $(\sigma\mathbb{I} - L_\mu)$ is not Fredholm [18]: it is one-to-one, but its range is not closed and its closure has a non zero finite codimension (see [15],[11]).

At this point we should emphasize that the physical relevance of the center manifold reduction for the finite depth problem is linked with the distance of the rest of (non critical) eigenvalues from the imaginary axis. So, the validity of the bifurcation analysis is becoming empty when the thickness of the layer increases. To fix ideas, let us give some physical numerical values for air-water free surface waves. The point $(b, \lambda) = (1/3, 1)$ then corresponds to a thickness $h = 0.48$ *cm*, and a velocity of waves $c = 21.6$ *cm/s*. This means that a layer with thickness more than few centimeters leads to a spectrum with real eigenvalues very close to 0, so the analysis which might be done (as in previous section) near the curve Γ (right of figure 3) for λb near $1/4$ on the upper branch) would be valid only in a very tiny neighborhood of this curve, and *this analysis would have no physical interest*. We need to study the *worse limiting case, which is here the infinite depth case*, and physical cases are in fact considered as regular perturbations of this limiting case. We shall see below that this has dramatic consequences on the mathematical analysis!

For the limiting problem the dispersion relation (10) has at most 4 roots. There is a pair of two pure imaginary double eigenvalues for $\lambda b = 1/4$. The remaining of the spectrum of L_μ is formed by the full real line, hence it crosses the imaginary axis at 0, and we *cannot use the center manifold reduction*. However, we still have the resolvent estimate (11), due to a good choice of space $\mathbb{H}$. In particular this type of results is also valid for problems with several layers, one being of infinite depth, with an additional eigenvalue in 0 (embedded in the essential spectrum), when there is no surface tension at one of the free surfaces [11].

3.1 Normal forms in infinite dimensions

Since we cannot reduce our problems to finite dimensional ODE's, and since we still would like to believe that eigenvalues near the imaginary axis are ruling the bounded solutions, this is a motivation for developing a theory of normal forms in separating the finite dimensional critical space, from the rest (the "hyperbolic" part of the spectrum, including 0). This leads to "partial normal forms", where there are *coupling terms,* specially in the infinite dimensional part of the system (see [15],[8]). For developing this theory, there are some technical difficulties, specially for problems with more than one layer and no surface tension at some free surface. A first difficulty is due to cases where 0 is an eigenvalue embedded in the essential spectrum: for extracting it from the spectrum, we use the *explicit form of the resolvent operator near the real axis*, to explicitely obtain the continuous linear form which can be used for the projection on the eigenspace belonging to 0. A second difficulty is that in space $\mathbb{H}$ the linear operator has not an "easy" (even formal) adjoint. This adjoint and some of its eigenvectors are usually necessary for expressing projections on the critical finite dim space. Fortunately, in our problems, we use the explicit form of the resolvent operator near the (for example double) eigenvalues, to make explicit the projection commuting with the linear operator (see [18]).

3.2 Typical results

Since we have not yet a center manifold reduction process to a finite ODE, the method we use now, needs to give *a priori* the type of solution, we are looking for. This is a major difference with the cases we had before, for finite depth layers. For periodic solutions, we use an adaptation of Lyapunov-Schmidt method, except that the presence of 0 in the spectrum gives some trouble (resonant terms). It appears that we can formulate all these problems, such that there is no such resonant term for *reversible solutions* (symmetric under S). As a result, there are *as many periodic solutions as in the finite depth problem* [11]. For solutions homoclinic to 0 (*solitary waves*), for example in our one layer problem, we first derive the infinite dimensional normal form, then we inverse the infinite dimensional part of the system, using Fourier transform. Indeed, the linearized Fourier transform uses the above resolvent operator, where we eliminated, via a suitable projection, the poles given by eigenvalues sitting on the imaginary axis. The fact that the resolvent operator is not analytic near 0 (there is a jump of the resolvent in crossing the real axis [15]), leads to the fact that this "hyperbolic part" of the solution *decays polynomially at infinity.* Putting this solution into the four dimensional part of the system, we can solve as before except that the decay of solutions is now polynomial (as $1/x^2$), instead of exponential. The principal part of the solution (of order $(\lambda b - 1/4)^{1/2}$) at finite distance still comes from the four dimensional truncated normal form, but its decays faster at infinity than the other part of the solution, which makes this queue part predominant at infinity. This is the main difference with the finite depth case, where the principal part coming from the normal form is valid for all values of x (see [15] for the proofs related with problem (9)).

As a conclusion, let us just say that I present here a specific type of physical problems which motivate some developments of existing mathematical theories. It also gives motivation for finding a new tool, probably very difficult to produce, like a center manifold reduction in cases when a continuous part of the spectrum crosses the imaginary axis. This is another illustration of the fact that progresses in mathematics may come from non academic questions raised naturally from discussions and collaboration with other disciplines.

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