Drawing Instruments: Theories and Practices from History to Didactics

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Abstract. Linkages and other drawing instruments constitute one of the most effective fields of experience at secondary and university level to approach the theoretical dimension of mathematics. The main thesis of this paper is the following: By exploring, with suitable tasks and under the teacher's guidance, the field of experience of linkages and other drawing instruments, secondary and university students can 1) relive the making of theories in a paradigmatic case of the historical phenomenology of geometry; 2) generate 'new' (for the learners) pieces of mathematical knowledge by taking active part in the production of statements and the construction of proofs in a reference theory 3) assimilate strategies for exploration and representative tools (such as metaphors, gestures, drawings, and argumentations) that nurture the creative process of statement production and proof construction. This thesis will be defended by referring to research studies already published or in progress.

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1. Introduction.

In recent years several efforts have been made at the international level to clarify the objects, the aims, the research questions, the methodologies, the findings and the criteria to evaluate the results of research in didactics of mathematics (or mathematics education, according to the name preferred in some countries). I may quote the volume edited for the 20 years of work at the IDM, Bielefeld University, and Professor Hans-Georg Steiner' s 65th birthday [BSSW] ; the ICMI Study held in 1994 in Washington DC about 'What is Research in Mathematics Education and what are its Results' [KS]; the Working Group 25 in ICME 8 [Mal]; the International Handbook edited by Bishop [Bi]. Didactics of mathematics as a scientific discipline is fairly young compared to other sciences, yet is deeply rooted in the perennial effort of mathematicians to advance human understanding of mathematics and to transmit mathematics knowledge to future generations. It has become clear that analytical tools are needed from different disciplines (such as epistemology, history, psychology) to obtain results that can increase the knowledge of the teaching and learning processes in the classroom, produce effective innovation in schools and understand why some designed innovation works or does not work, and, at a larger level, influence the development of school systems.

Analytical tools from history and epistemology are necessary to tackle one issue which is perhaps crucial: the nature of mathematics knowledge. One of the distinctive features of mathematics is theoretical organisation. This has created a very specific mathematician's style, with a very impressive form, that alternates definitions and theorems. Yet, when a mathematician reads a theorem and, in particular, its proof, it is not the form that commands most attention, but rather the process by means of which mathematical ideas have been generated or have been illuminated by the proof in a new way. If we look at the 'confessions' of working mathematicians [T], we have an idea of a continuous (not always individual) process: the major discontinuity seems to happen in the final phase of written communication in Journals, where the leading ideas, the intuitions, the associations, the metaphors or the explorations of special cases are hidden by the formidable and conventional mathematician's style. Unfortunately the curriculum revolution of the sixties gave too much importance to the product (i. e. the form) and put in shadow the process (i. e. the construction of reasoning and arguments). But it was realised soon that teaching beginners the formalities of proof might be very difficult (and, perhaps, meaningless). Instead of scrutinising the reason for failure, what happens now is that, in some countries, proving processes are being eliminated from mathematics curriculum, not taking into account that giving up proofs for a sheer acquisition of isolated facts and notions hides the theoretical organisation of mathematics (for a detailed discussion of these issues see [Ha]).

This is the scenario in which a collective project has been set some years ago by a group of Italian researchers [MBBFG], [AMORP1]. The project highlights the permanent value of proof in mathematics and didactics of mathematics and aims to design, implement and analyse effective teaching experiments, that can introduce students to the theoretical dimension of mathematical culture up to the construction of theorems and proofs. As far as the activity of mathematicians is concerned, from a didactic perspective, we are much more interested in the hidden process of conjecture production and proof construction than in the final product: this very process does offer suggestions on the way of organising effective classroom activity. In particular, whenever the process of producing conjectures about something may evolve continuously and smoothly into the process of constructing proofs, the task of producing 'new' theorems is proved to be easier for students. In confirmation of that, we may recall a typical strategy, used by good teachers.

When a difficult and crucial theorem is introduced in the standard lecture format, before giving the proof, the students are presented with examples, counterexamples and reasons for the plausibility of the statement to make them relive the intellectual experience of the prior inventor of the theorem, although they have been deprived of the long process of generating the conjecture by themselves.

The issue of continuity between the production of conjectures and the construction of proofs has been raised from a cognitive perspective in a study carried out in the 8th grade [GBLM], concerning the production of a theorem of geometry about a problem situation in the field of sunshadows. The authors have described the cognitive continuity as a process with the following characteristics. During the production of the conjecture, the student progressively works his/her statement through an intense argumentative activity; during the subsequent statement proving stage, the student links up with this process in a coherent way, organising some of the justifications ('arguments') produced during the construction of the statement according to a logical chain. The construct of cognitive continuity, further developed by Arzarello & al. [AMORP1] to include also the case of advanced learners has proved to be useful to interpret existing teaching experiments and to design new ones.

In recent years several experiments in different fields have been carried out at very different school level, from primary to tertiary education (primary school : [B2], [BBFG]; middle school: [BGM], [GBLM], [BPR1], [BPR2]; secondary school: [B1], [BP], [Mar], [AMORP2] [MB]; tertiary: [AMORP1]). Some characteristics are shared by nearly all the experiments: 1) the selection, on the basis of historic-epistemological analysis, of fields of experience, rich in concrete and semantically pregnant referents (e. g. perspective drawing; sunshadows; Cabri-constructions; gears; linkages and drawing instruments); 2) the design of tasks, which require the students to take part in the whole process of production of conjectures, of construction of proofs and of generation of theoretical organisation; 3) the use of a variety of classroom organisation (e. g. individual problem solving, small group work, classroom discussion orchestrated by the teacher, lectures); 4) the explicit introduction of primary sources from the history of mathematics into the classroom at any school level.

In my own research, I have found that linkages and other drawing instruments might be one of the most effective fields of experience at secondary and university level. In the following I shall give some details on this case, by analysing the activities designed and implemented for approaching mathematical theorems and more generally the theoretical organisation of mathematics.

2. Linkages and Drawing Instruments: An Historical Digression.

In this section, I shall outline the history of linkages and other drawing instruments by using the metaphor of a theatre play. Only planar drawing instruments will be considered; however spatial drawing instruments such as perspectographs have also played a relevant role in specific practices (e. g. painting, architecture) and have given rise to specific theories (such as projective geometry). But this is another story and, maybe, the topic of a different paper (examples in http://www.museo.unimo.it/labmat/)

2. 1. The Prologue : Euclid and the Classical Age. Drawing instruments have been considered in geometry treatises from the time of Euclid, whose first postulates implicitly define the kind of instruments that are allowed for geometrical constructions [He] : '1) Let the following be postulated : to draw a straight line from any point to any point; 2) To produce a finite straight line continuously in a straight line ; 3) To describe a circle with any centre and distance.'

Even if the description is supposed to recall a practical use of instruments, there is no doubt that the intention is theoretical. Actually the instruments are never quoted directly, not even in the large number of constructions that are discussed in the following books. Moreover, the problem is never to find the approximate solution that could be useful for applications: rather a theoretical solution by straight lines and circles is looked for. Other drawing instruments (and curves) were known at the time of Euclid, yet not included in the set of accepted theoretical tools (e. g. the conchoid of Nicomedes, [He]). They were rather used to solve practical problems. For instance, by means of the conchoid it is possible to find two mean proportionals between two straight lines and, hence, to construct a cube which is in any given ratio to a given cube. This allows to find a set of weights in given proportion to calibrate catapults.

2.2. The First Act : Descartes and Seventeen Century Geometers. Descartes, like most scientists of his age, was deeply involved in the study of mechanisms for either practical or theoretical purposes. A famous example of the former kind (i. e. the machine to cut hyperbolic lenses) is described in the 'Dioptrique'. The latter issue forms the core of the 'Géométrie', where two methods of representing curves are clearly stated: the representation by a continuous motion and the representation by an equation [Bos]. Descartes deals with the following question: 'Which are the curved lines that can be accepted in geometry? (p. 315)' and gives an answer (or, better, two answers) different from the one of classical geometers : 1) '[...] we can imagine them as described by a continuous motion, or by several motions following each other, the last of which are completely regulated by those which precede. For in this way one can always have an exact knowledge of their measure (Géométrie p. 316)'; 2) '[...] those which admit some precise and exact measure, necessarily have some relation to all points of a straight line, which can be expressed by some equation, the same equation for all points (Géométrie, p. 319)' The goal of Descartes was related to the very foundations of geometry: if a curve (e. g. a conic or a conchoid) is to be accepted as a tool to solve geometrical problems, one must be sure that, under certain conditions, the intersection points of two such curves exist. Hence, pointwise generation is not sufficient and the continuum problem is called into play: by the standards of the seventeenth century mathematicians, it is solved by referring to one of the most primitive intuitions about the continuum, i. e. the movement of an object. Descartes did not confront the question whether the two given criteria - i. e. the mechanical and the algebraic - are equivalent or not. This problem actually requires constructing more advanced algebraic tools and, what is more important, changing the status of drawing instruments from tools for solving geometric problems to objects of a theory. The importance of the generation of curves by movement is proved by the flourish of innumerable treatises of 'organic' geometry (i. e. geometry developed by instruments), thanks to leading mathematicians, such as Cavalieri , L'Hospital, Newton, or van Schooten. They designed and studied dozens of different drawing instruments for algebraic curves (incidentally, in the same age when the very concept of algebraic curve started to be worked out).

2.3. The Second Act : Kempe and the Nineteenth Century Geometers. In the nineteenth century there was a shift from studying individual drawing instruments to developing a theory of drawing instruments, in the special case of linkages. On the one side, geometers started to study which curves could be drawn by any n-bar linkage; on the other side they asked which linkages could be used to draw any curve. The curve that resisted longest the attack of geometers was the simplest one, i. e. the straight line. After the approximate 3-bar solution offered by Watt in 1784 (that is still used in nearly every beam-engine), only in 1864 Peaucellier presented a 7-bar linkage, that embodies a rigorous solution based on the properties of circular inversion [K2]. The general problem of drawing any algebraic curve of any degree was temporarily solved by Kempe, a few years later (1876), with the paper entitled 'On a General Method of Describing Plane Curves of the nth Degree by Linkwork' [K1]. The structure of Kempe's proof is quite interesting. Starting from the equation F(x,y)=0 of any plane algebraic curve and from a particular point P of the curve, the polynomial is expanded into a linear combination of cosines of suitable angles. For each element of the sum, an elementary linkage is provided. By combining such linkages, a new linkwork is obtained, that has the effect of 'drawing' the given curve in the neighbourhood of P. Rather than an actual linkwork, the theorem gives an algorithm to construct a (virtual) linkwork, that depends on the equation of the curve.

2.4. The Third Act : Modern Revival of Curve Drawing Devices. The study of linkages is reconsidered in today's mathematics from two different, yet related, perspectives. The problem of drawing curves is reread as the problem of forcing a point of a robot to execute a given trajectory [Ba], [HJW]. The study of abstract linkages and their realisation is related to the study of algebraic varieties and of immersed submanifolds of Euclidean space [GN], [KM]. According to Kapovich & Millson, a major role in the revival of this field of research has been played by Thurston, who has given lectures on this topic since the late seventies. The new theory is completely algebraized and, at a first glance, has nothing to share with the problems that have been described in the previous acts. Yet, the very theorem of Kempe, combined with the work of today's mathematicians, has lead to proving general realizability theorems for vector-valued polynomial mappings, real-algebraic sets and compact smooth manifolds by moduli spaces of planar linkages. Kempe's proof has been carefully scrutinised, revealing some weakness related for instance to the presence of some 'degenerate' configurations of linkages appearing during the movement. However, the structure of the proof, based on the recourse to elementary linkages as building blocks, is still the original one. Hence Kempe's theorem might be considered an hinge: on the one side it closes Descartes' implicit problem to relate motion of instruments and equations and on the other side it opens the way to the modern theory of abstract linkages.

2.5. The History Goes On: The Critical Impact of Drawing Instruments. The historical analysis sketched in the previous 'acts' suggests that in the domain of geometry the relationship between theoretical and practical issues has always been very rich and complex. On the one side, drawing instruments, intimately connected with the development of algebraic tools, are theoretical products of the continuous modelling effort, that aims at rationalising the perception and the production of shapes. On the other side, drawing instruments are physical objects of the world to be modelled: to understand their functioning means to be able to design instruments which fulfil a desired action. Theories and practices might have been developed for some time independently, but in each age they happen to nurture each other: a double arrow describes the dialectic relationship between them, that is constructed anew repeatedly with shifts of meaning. In the teaching of mathematics such general complex ideas are to be translated into ordered activities for the classroom. If the ideas are interconnected as in a loop, as in this case, an apparently obvious solution is supposed to be to cut the loop somewhere, so that the double arrow becomes a single arrow from theories to practices (i. e. practices are applications of theories) or from practices to theories (i. e. practices are motivations for theories). These are the most common options. It is far beyond the scope of this paper to discuss them in detail. I intend to defend a different option: to put drawing instruments in the centre and to use them as mediators for both theories and practices. This idea is not new: drawing instruments were part of the education of gentlemen in arts such as the military art or the art of navigating since the 17th century [Tu]; they were used in prestigious Institutes of Mathematics (e. g. Goettingen [Mu]) to educate generations of leading mathematicians; drawing instruments are even on show in Scientific Museums for the popularisation of mathematics. In each of these uses, the visibility of theoretical aspects is surely different, because, when concrete referents come into play, the risk is always that the attention is captured by isolated facts and that the argument, if any, is not detached from everyday styles of reasoning [S]. For instance, the very possibility of making 'infinitely many' experiments by dynamic exploration might help, on the one side, the production of conjectures, but, on the other side, might render things self-evident and destroy the need of constructing proofs. If the theoretical aspects of mathematics are central in didactics of mathematics, as we have argued in the introduction, a careful didactic treatment of concrete referents is always needed. Whether an object is considered from a practical or from a theoretical perspective depends on the habits of the students, acquired through a slow process, on the types of exploration tasks and on the issues raised by the teacher in the classroom interaction. This is true for drawing instruments too, for both the material copies and the virtual copies of ancient instruments produced by computer (such as the simulations produced by means of software with graphic interface - such as Cabri - or by means of Java) and for the computer itself considered as the most flexible drawing instrument. In this part of the study, the function of analytical tools from the psychology of mathematics education appears to be relevant.

3. Drawing Instruments in the Classroom.

3. 1. Exploring Linkages. This example concerns the study of one of the pantographs (i. e. the pantograph of Sylvester), which were designed in the 19th century to realise elementary geometric transformations and to give the elemen-

tary blocks of Kempe's theorem. The study was originally carried out with 11th graders [B1],[BP], but we have collected later items of anedoctal evidence that confirm the emergence of similar processes (with the same slowness) when similar tasks are given to undergraduate students, to graduate students or to teachers of mathematics or mathematics educators. Hence, what follows is supposed to apply to both novices and expert explorers. The pantograph of Sylvester is an 8-bar linkage (see the figures).

The students had already been given an introductory lecture concerning the early history of drawing instruments in Euclid's age. They were given a specimen of the pantograph and a set of eight tasks to guide the exploration in small group work. Two tasks are especially relevant for our discussion [B1]: '1) Represent the linkage with a schematic figure and describe it to somebody who has to build a similar one on the basis of your description alone; 2) Are there any geometric properties that are related to all the configurations of the linkage? State a conjecture and try to prove your statement'.

The first task aimed at encouraging students' manipulation of the linkage. It has to be said that the tradition of abstract and symbolic work has often the effect of inhibiting recourse to manipulation in mathematics lessons. In this case, on the contrary, the students had to measure bars and angles and to try to connect these empirical data with the pieces of geometrical knowledge that were part of their past experience. Actually the small group debated for a long time whether the imaginary addressee had to build an 'equal' linkage (i. e. with the same measure) or a 'similar' one (i. e. capable for working in the same way). In the first case, it would have been enough to write down the length of each bar and to give the instructions for assembling the linkage. When they decided for the second solution, they had to cope with the problem of identifying the structural features of the linkage (i. e. the presence of a parallelogram and of two similar isosceles triangles) from the empirical evidence offered by perception and by measuring. The process of solving the second task resulted in three interlaced phases: (1) producing the conjecture; (2) arguing about the conjecture; (3) constructing a proof.

Producing the conjecture was difficult and slow. The linkage actually realises a rotation as for every configuration, a) OP=OP'; b) POP' = PAB = BCP'. Yet the rotation is approached at as a correspondence between two points that has no transparent relationships with the motion of the linkage. The teacher had a helping attitude, but the whole exploring process was carried out by the students, who at the end agreed with the proposal of one of them, who had 'seen' suddenly the invariant during the exploration. The suggestion was checked experimentally in different configurations and then accepted by the whole group.

Arguing about the conjecture and constructing the proof were actually interlaced processes. The students were helped by the large amount of exploration they had made before. For instance the observation of an intermediate limit case (figure 2b when two sides of the parallelogram and two sides of the triangles are aligned) was considered empirical evidence that the triangles POP', PAB and BCP' are similar. While trying to defend the conjecture by arguments, the students mixed continuously experimental data (obtained by direct manipulation of the mechanism) and statements deduced logically from already accepted statements. Whilst the verbal proof was eventually complete, the process of polishing the entire reasoning in order to give it the form of a logical chain and to write it down was slow and not complete, as the students' text shows:

'Thesis: POP' is constant (see figure 2 for notation).

The angle POP' is constant as the triangles POP' obtained by means of the deformations of the mechanism are always similar, whatever the position of P and P'. In fact OP=OP', because the triangles OCP' and OAP are congruent, as CP'=OA,CO=AP and OCP'=OAP (BCO=OAB and P'CB=BAP). The above triangles are also similar to a third triangle PBP', because, as the triangles BCP' and BAP are similar, it follows that BP' : BP = CP' : CO and the angle P'BP = OCP' as (setting CP'B = CBP' = a and CBA = b) we have PBP' = 360 - (2a+b); OCP' = 360 - (2a+b).

This is true because prolonging the line BC from the side of C the angle supplementary to BCP' is equal to 2a and the angle supplementary to BCO is equal to b as two contiguous angles of a parallelogram are always supplementary'.

Surely this written text in neither complete nor well ordered, according to the mathematician's style: the order of the steps recalls the sequence of production of statements, as observed during the small group work, rather than the logical chain that could have been used by an expert. Nevertheless it was easily transformed later with the teacher's help into the accepted format with reference to elementary euclidean geometry; yet, what is important, the time given to laboriously produce their own proof ensured that the final product in the mathematician's style, where the genesis of the proof was eventually hidden, retained meaning for the students.

3.2. Theoretical Framing of Drawing Instruments and Linkages. The small group study was only one step in a long term teaching experiment. The study was done in the frame of Euclid's elementary geometry. From a cultural perspective, students must be introduced to the different theories which have been invented later, with their own goals and objects, and to the different practices which have been developed, from beam-engines to robotics, otherwise we would have relapsed into the standard linear teaching path from concrete referents to a geometrical study framed by Euclid's geometry, where practices are only the starting point, i. e. motivations for theories.

The students who had realised the study of the pantograph, together with their schoolfellows who had studied other pantographs according to the same tasks, took part in lessons where each small group presented the results of the guided study. The teacher related the different pantographs to each other, generating an embryo of a theory of linkages, where the same proof could be applied, with small adaptations, to different instruments [BP]. The shifts in meaning from considering an individual linkage to developing a theory of linkages was introduced by means of guided reading of some historical sources, like the ones quoted in the theatre play of the section 2; historical sources were assimilated by students, producing explorations and proofs according to the inquiry style of each age.

This is only a prototype of teaching experiments which are made every year with secondary school students (by Marcello Pergola) and with university students (by the author). The difference between secondary school and university students concerns the length of the play: the second act is within the reach of secondary school students, whilst university students can understand the whole play.

3.3. Some Issues to be Deepened. In the above sections, a complex teaching experiment has been outlined. Different classroom organisations have been shown with different roles for the teacher: lectures, small group works, whole class discussions. In small group work phases of joint activity between the teacher and the students were accomplished. In the theoretical framing the teacher acted, by his own words or by quoting historical sources, as a cultural mediator. The study of the teacher's role is a crucial problem of didactics of mathematics, whose discussion is far beyond the scope of this paper: it is related to the possibility of reproducing the teaching experiments in different classrooms. For a partial account about this issue, the interested reader could refer to [MB] for the analysis of the teacher's role in a classroom discussion when the object is the theoretical meaning of geometric construction. Further investigations are planned.

In the theoretical framing episode students coped with a cultural problem, i. e. the construction of a balanced image of mathematics, where theories and practices are strictly intertwined yet not confused. In the direct and guided manipulation of instruments, students experienced, at an appropriate slow pace, the continuous and smooth transition from physical experience (gestures and manipulation) to the production of their own conjectures and to the construction of a proof. In this process, they used different linguistic tools to express their ideas, from the metaphors taken from everyday language to the fixation of the procedures according to the speech genre of elementary geometry. The study of student processes is a crucial problem of the psychology of mathematics education. Finer grain analyses are an unavoidable part of each of the research studies quoted in the introduction and of ongoing research.

4. Some Implications for Teaching.

The case of linkages and other drawing instruments gives only one among several examples of teaching experiments about the theoretical organisation of mathematics and the approach to theorems. Systematic experiments in this field have been carried out mainly at secondary and university levels, but the activity with drawing instruments has proven to be effective with younger students too, because the difference between a practical and a theoretical use of instruments might be approached (yet is seldom emphasised) also in primary school. For instance, in an experiment carried out in primary school [BBFG], pupils have become aware that they can use a compass in two very different ways: 1) to imitate a round shape (practical use); 2) to construct (if possible) a triangle with sides of given length (theoretical use). In the former case the focus is on a careful use of compass that assures the precision of the drawing. In the latter case the focus is on the definition of the circle: even a free-hand rough sketch could be effective as the compass is meant as a mental instrument.

What implications for curricula could the quoted experiments have? To give an answer, we can contrast our approach to geometry with the traditional one in a very special case: the case of conics. When this topic is considered, it is usually introduced according to some standard steps: 1) A short introduction, concerning the space generation of conics as conic sections, limited to explaining the origin of the name. 2) A metric definitions of conics as loci determined by the focal properties; in this case a particular drawing instrument for obtaining the so-called gardener ellipse is described. 3) The canonical equations; then every problem is considered in this analytic setting. From a cultural perspective, this path conveys a one-sided image of mathematics, i. e. the physical generation of conics (as conic sections or as drawings by instruments) is nothing but a rough introduction to the very important things, that are, on the contrary, metric definitions and equations. What is even more disappointing is the cognitive counterpart: by this approach (even if it is completed by a careful study of quadratic forms, as in the case of university students of mathematics), students do nor learn how to relate their spatial intuitions (on which heuristics might be based) with the plane synthetic or analytic study [BM].

In this paper I have proposed an alternative approach with two different, yet related, arguments. The cultural argument: for centuries curves have been considered as trajectories determined by linkages and other drawing instruments; only later, the mechanical study has been complemented by the algebraic study, arousing theories which retain the links with the spatial referents and which has proven to be relevant for the development of today mathematics The cognitive argument: the very manipulation of drawing instruments provides students with heuristics and representative tools (such as metaphors, gestures, drawings and arguments) that foster the production of conjectures and the construction of related proofs within a reference theory, with a slow and laborious process that recalls the one of professional mathematicians. Reliving the making of theories and producing one's own theorems is a way to appreciate and assimilate the theoretical dimension of mathematics.

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