Aspects of the Nature and State of Research in Mathematics Education

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Abstract. This paper offers an outline and a characterisation of the didactics of mathematics, alias the science of mathematics education, as a scientific and scholarly discipline. It further presents a number of major, rather aggregate findings in the discipline, including *the astonishing complexity of mathematical learning*, *the key role of domain specificity*, *obstacles produced by the process-object duality*, *students' alienation from proof and proving*, and *the marvels and pitfalls of information technology in mathematics education*.

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1 Introduction

During the last three decades or so mathematics education has become established as an academic discipline on the international scene. This discipline is given slightly different names in different quarters, such as *mathematics education research*, *science of mathematics education*, and *the didactics of mathematics*. In the following I shall use the names interchangeably.

What are the issues and research questions of the didactics of mathematics, what are its methodologies, and what sorts of results or findings does it offer? In this paper attempts will be made to characterise this discipline, in particular as regards its nature and state, and to present and discuss some of its major findings. I shall begin by offering a definition of the field.

2 Characterising the field

A definition

Subject *The didactics of mathematics, alias the science of mathematics education, is the scientific and scholarly field of research and development which aims at* identifying, characterising*, and* understanding *phenomena and processes actually or potentially involved in the* teaching *and* learning of mathematics at any educational level.

ENDEAVOUR *As particularly regards 'understanding' of such phenomena and processes, attempts to uncover and clarify* causal relationships *and* mechanisms *are in focus.*

APPROACHES *In pursuing these tasks, the didactics of mathematics addresses* all matters *that are pertinent to the teaching and learning of mathematics, irrespective of which scientific, psychological, ideological, ethical, political, social, societal, or other spheres this may involve. Similarly, the field makes use of considerations, methods, and results from* other fields and disciplines *whenever this is deemed relevant.*

ACTIVITIES *The didactics of mathematics comprises different kinds of activities, ranging from theoretical or empirical* fundamental research*, over* applied research *and* development*, to* systematic, reflective practice.

It is important to realise a peculiar but essential aspect of the didactics of mathematics: its *dual nature*. As is the case with any academic field, the didactics of mathematics addresses what we may call *descriptive/explanatory* issues, in which the generic questions are 'what *is* (the case)?' and '*why* is this so?'. Objective, neutral answers are sought to such questions by means of empirical and theoretical data collection and analysis without any intrinsic involvement of values (norms). However, by its nature mathematics education implies the fundamental presence of values and norms. So, in addition to its descriptive/explanatory dimension, the didactics of mathematics also has to contain a *normative* dimension, in which the generic questions are 'what *ought to be* the case?' and '*why* should this be so?'. Both dimensions are essential constituents of the science of mathematics education, but they should not be confused with one another.

In a brief outline of the main areas of investigation the two primary ones are *the teaching of mathematics*, and *the learning of mathematics*. A closely related area of investigation is the *outcomes* (results and consequences) of the teaching and the learning of mathematics, respectively.

We may depict, as in Figure 1, these areas in a 'ground floor'. The investigation of these areas leads to derived needs to investigate certain auxiliary areas related to the primary ones but not in themselves of primary didactic concern, such as aspects of mathematics as a discipline, of cognitive psychology, or of curriculum design. As is the case with any new or established scientific field, the didactics of mathematics reflects on its own nature, issues, methods, and results (e.g., Grouws, 1992; Biehler et al., 1994; Bishop et al., 1996; Sierpinska & Kilpatrick, 1998). Theoretical or empirical studies in which the field as such is made subject of investigation form part of the field itself, although at a meta-level, which we depict as an 'upper floor' plane. We may think of it as being transparent so as to allow for contemplation of the ground floor from above. Finally, let us imagine a vertical plane cutting both floors as a common wall. The two half-spaces thus created may be thought of as representing the descriptive/explanatory and the normative dimensions, respectively. If we imagine the vertical wall to be transparent as well, it is possible to look into each dimension from the perspective of the other.

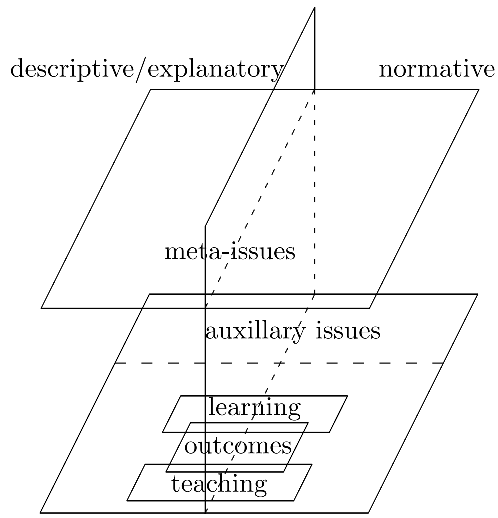

Figure 1: Survey map

Let us sum up the ultimate goals of the didactics of mathematics as follows: We want to be able to specify and characterise *satisfactory learning* of mathematics, including the mathematical competencies we should like to see different categories of individuals possessing. We want to be able to devise and implement *effective mathematics teaching* that can serve to bring about satisfactory learning. We finally want to construct and implement valid and reliable *ways to detect and assess*, without destructive side effects, the results of learning and teaching of mathematics.

For all this to be possible we have to be able to identify and understand the role of mathematics in science and society; what learning of mathematics is, what its conditions are, how it may take place, how it may be hindered, how it can be detected, and how it can be influenced, all with respect to different categories of individuals. We further have to understand what takes place in existing forms mathematics teaching, both as regards the individual student, groups of students, and entire classrooms. We have to invent and investigate new modes of teaching. We have to investigate the relationships between teaching modes and learning processes and outcomes, and the influence of teachers' backgrounds, education, and beliefs on their teaching. We have to examine the properties and effects of established and experimental modes of assessment in mathematics education, with particular regard to the ability to provide valid insight into what students know, understand, and can do.

Traditionally, fields of research within the sciences produce either *empirical findings* of facts', through some form of data collection assisted by theoretical considerations, or they produce *theorems*, i.e. statements derived by means of logical deduction from a collection of 'axioms'. If we go beyond the predominant paradigms in the sciences and look at the humanities and the social sciences, other aspects have to be added to the ones just considered. In philosophical disciplines, the proposal and analysis of distinctions and concepts, and concept clusters, introduced to specify and represent matters from the real world, serve to create a platform for discussion of these matters in a clear and systematic way. Such disciplines often produce *notions*, *distinctions*, *terms*, amalgamated into *concepts*, or extensive hierarchical networks of concepts connected by formal or material reasoning, called *theories*. Disciplines dealing with human beings, as individuals, as members of different social and cultural groups, and as citizens, or with communities and societies at large, primarily produce *interpretations* and *models*, i.e. hypotheses of individual or social forces and mechanisms that may account for phenomena and structures observed in the domain under consideration. Sometimes sets of interpretations are organised and assembled into systems of interpretation, also called 'theories' which we shall refer to as *interpretative theories*. Finally, there are disciplines within all scientific spheres that produce *designs* (and eventually *constructions*) for which the ultimate test is their functioning in the realm in which they are put into practice. However, as designs and constructions are often required to have certain properties before installation, design disciplines are scientific only to the extent they can provide well-founded reasons to believe that their designs possess certain such properties to a satisfactory degree.

The didactics of mathematics contains instances and provides findings of all the categories of disciplines mentioned, but to strongly varying degrees. There are empirical findings as well as 'theorems' (but, in the honour of truth, these are derived within mathematics itself). There are terms, concepts and theories for analysis of a philosophical nature, and there are models, interpretations and interpretative theories of a pscyhological, sociological or historical nature. Finally there are multitudes of designs and constructions of curricula, teaching approaches, instructional sequences, learning environments, and materials.

Some researchers in mathematics education are hesitant to use the term 'finding', in order to avoid too narrow expectations of what the field has to offer. They prefer to see the didactics of mathematics as providing generic tools for analysing teaching/learning situations. Others emphasise that the field offers illuminating case studies which are not necessarily generalisable beyond the cases themselves, but are nevertheless stimulating for thought and practice. However, as long as we keep in mind that the notion of finding is a broad one, I don't see any severe problems in using this term in the didactics of mathematics.

A major portion of recent research has focused on students' *learning processes and products* as manifested on the individual, small group, and classroom levels, and as conditioned by a variety of factors such as mathematics as a discipline; curricula; teaching; tasks and activities; materials and resources, including text books and information technology; assessment; students' beliefs and attitudes; educational environment, including classroom communication and discourse; social relationships amongst students and between students and teacher(s); teachers' education, backgrounds, and beliefs; and so forth. The typical findings take the shape of models, interpretations, and interpretative theories, but often also of solid empirical facts. We know a lot about the possible mathematical learning processes of students and about how these may take place within different areas of mathematics and under different circumstances and conditions, as we know a lot about factors that may hinder or simply prevent successful learning.

We have further come to know a great deal about what happens in *actual mathematics teaching* in classrooms at different levels and in different places (Cobb & Bauersfeld, 1995). However, we are still left with hosts of unanswered questions as to how to design, organise, and carry out teaching-learning environments and situations that to a reasonable degree of certainty lead to satisfactory learning outcomes for various categories of students. This is not to say that we don't know anything in this respect, but as yet our knowledge is more punctual and scattered than is the case with our insights into students' mathematical learning. Based on our growing insight into mathematical learning processes and teaching situations, we know more and more about what *is not* effective teaching vis-à-vis various groups of recipients. Moreover, the didactic literature displays numerous examples of experimental teaching designs and practices that are judged highly successful, without this success being easily analysed and documented in scientific terms.

3 Examples of major findings

In this section, we shall consider a few significant findings, of a pretty high level of aggregation, which can serve to illustrate the range and scope of the field. As it is not possible here to provide full documentation of the findings selected, a few recent references, mainly of survey or review type, have to suffice.

The astonishing complexity of mathematical learning *An individual student's mathematical learning often takes place in immensely complex ways, along numerous strongly winding and frequently interrupted paths, across many different sorts of terrain. Some elements are shared by large classes of students, whereas others are peculiar to the individual.*

Students' misconceptions and errors tend to occur in systematic ways in regular and persistent patterns, which can often be explained by the action of an underlying tacit rationality put to operation on a basis which is distorted or insufficient.

The learning processes and products of the student are strongly influenced by a number of crucial factors, including the epistemological characteristics of mathematics and the student's beliefs about them; the social and cultural situations and contexts of learning; primitive, relatively stable implicit intuitions and models that interact, in a tacit way, with new learning tasks; the modes and instruments by which learning is assessed; similarities and discrepancies between different 'linguistic registers'.

This over-arching finding is an agglomeration of several separate findings, each of which results from extensive bodies of research. The roles of epistemological issues and obstacles in the acquisition of mathematical knowledge have been studied, for instance, by Sierpinska and others (for an overview, see Sierpinska & Lerman, 1996). Social, cultural, and contextual factors in mathematical learning have been investigated from many perspectives, e.g. Bishop, 1988, and Cobb & Bauersfeld, 1995. Pehkonen (e.g. Pehkonen & Törner, 1996), among others, have investigated students' (and teachers') belief's. Fischbein and his collaborators have studied the influence of tacit models on mathematical activity (Fischbein, 1989). The influence of assessment on the learning of mathematics has been subject of several theoretical and empirical studies (e.g. Niss 1993). The same is true with the role of language and communication (see Ellerton & Clarkson, 1996, for an overview).

The studies behind these findings teach us to be cautious when dealing with students' learning of mathematics. Neither processes nor outcomes of mathematical learning are in general logically ordered. For instance, research has shown that many students who are able to correctly solve an equation such as $7x-3 = 13x+15$ are unable to subsequently correctly decide whether $x = 10$ is a solution. The explanation normally given to this phenomenon is that *solving* equations resides in one domain, strongly governed by rules and procedures with no particular attention being paid to the objects involved, whereas examining whether or not a given element solves the equation requires an understanding of what a *solution* means. So, the two facets of the solution of equations, intimitely linked in the mind of the mature knower, need not even both exist in the mind of the novice mathematical learner, let alone be intertwined.

The key role of domain specificity *For a student engaged in learning mathematics, the specific nature, content and range of a mathematical concept that he or she is acquiring or building up are, to a large part, determined by the set of specific domains in which that concept has been concretely exemplified and embedded for that particular student.*

The finding at issue is closely related to the finding that students' *concept images* are not identical with the *concept definitions* they are exposed to (for overviews, see Vinner, 1991, and Tall, 1992). The concept images are generated by previous notions and experiences as well as by the examples against which the concept definitions have been tested.

The range and depth of the instances of this finding have far-reaching bearings on the teaching and learning of mathematics. Thus, not only are most 'usual' students unable to grasp an abstract concept, given by a definition, in and of itself unless it is elucidated by multiple examples (which is well known), but, more importantly, the scope of the notion that a student forms is often barred by the very examples studied to support that notion. For example, even if students who are learning calculus or analysis are presented with full theoretical definitions, say of $\epsilon - \delta$ type, of function, limit, continuity, derivative, and differentiability, their actual notions and concept images will be shaped, and limited, by the examples, problems, and tasks on which they are actually set to work. If these are drawn exclusively from objects given as standard expressions of familiar, well-behaved objects, the majority of students will gradually tie their notions more and more closely to the specimens actually studied. Thus, the general concept image becomes equipped with properties resulting from an over-generalisation of properties held by the special cases but not implied by the general concept. Remarkably enough, this does not prevent many of the very same students from correctly remembering and citing general theoretical definitions. These definitions seem to just be parked in mental compartments detached from the ones activated in the study of the cases. In other words, if average students are to understand the range of a mathematical concept, they have to experience this range by exploring a large variety of manifestations of the concept in various domains.

The danger of forming too restricted images of general concepts seems to be particularly manifest in domains — such as arithmetic, calculus, linear algebra, statistics — that lend themselves to an algorithmic 'calculus', in a general sense. In such domains, algorithmic manipulations — procedures — tend to attract the main part of students' attention so as to create a 'concept filter': Only those instances (and aspects) of a general concept that are relevant in the context of the 'calculus' are preserved in students' minds. In severe cases an over-emphasis in instruction on procedures may even prevent students from developing further understanding of the concepts they experience through manipulations only.

The present finding shows that it is a non-trivial matter of teaching and learning to establish mathematical concepts with students so as to be both sufficiently general and sufficiently concrete. Research further suggests that for this to happen, several different *representations* (e.g. numerical, verbal, symbolic, graphical, diagrammatical) of concepts and phenomena are essential, as are the links and transitions between these representations.

There is a large and important category of mathematical concepts of which the acquisition becomes particularly complex and difficult, namely concepts generated by *encapsulating* specific processes into objects. Well-known examples of this are the concept of function as an *object*, encapsulating the mechanisms that *produce the values* of the function into an entity, and the concept of derivative, encapsulating the processes of differentiating a function pointwise, and of amalgamating the outcomes into a new function. This *process-object duality*, so characterstic of many mathematical concepts, is referred to in the research literature by different terms, such as 'tool-object' (Douady, 1991), 'reification' (Sfard, 1991), 'procept', a hybrid of process and concept, (Tall, 1991, Chapter 15). It constitutes the following finding:

Obstacles produced by the process-object duality *The process-object duality of mathematical concepts that are constituted as objects by encapsulation/reification of specific processes, typically gives rise to serious learning obstacles for students. They often experience considerable problems in leaving the process level and entering the object level.*

For example, many students conceive of an equation as signifying a prompt to perform certain operations, without holding any conception of an equation as such, distinct from the operations to be performed. To them, an equation simply does not constitute a mathematical entity, such as a statement or a predicate.

Undoubtedly, the notions of mathematical proof and proving are some of the most crucial, demanding, complex, and controversial, in all of mathematics education. Deep scientific, philosophical, psychological, and educational issues are involved in these notions. Hence it is no wonder that they have been made subject of discussion and study in didactic research to a substantial extent over the years (for a recent discussion, see Hanna & Jahnke, 1996). Here, we shall confine ourselves to indicating but one finding pertinent to proof and proving.

STUDENTS' ALIENATION FROM PROOF AND PROVING *There is a wide gap between students' conceptions of mathematical proof and proving and those held by the mathematics community. Typically, students experience great problems in understanding what a proof is supposed to be, and what its purposes and functions are, as they have substantial problems in proving statements themselves. Research further shows that many students who are able to correctly reproduce a (valid) proof, do not see the proof to have, in itself, any bearing on the truth of the proposition being proved.*

The fact that proof and proving represent such great demands and challenges to the learning of mathematics implied that proof and proving have received, in the '80's and '90's, a reduced emphasis in much mathematics teaching. However, there seems to be a growing recognition that there is a need to revitalise them as central components in mathematics education. Also there is growing evidence that it is possible design and stage teaching-learning environments and situations so as to successfully meet parts of the demands and challenges posed by proof and proving.

The last finding to be discussed here, briefly, is to do with the role and impact of information technology on the teaching and learning of mathematics. This is perhaps the single most debated issue in mathematics education during the last two decades, and one which has given rise to large amounts of research (for recent overviews, see Balacheff & Kaput, 1996; and Heid, 1997). The following finding sums up the state-of-the-art:

THE MARVELS AND THE PITFALLS OF INFORMATION TECHNOLOGY IN MATHEMATICS EDUCATION *Information technology has opened avenues to new ways of teaching and learning which may help to greatly expand and deepen students' mathematical experiences, insights, and abilities. However, this does not happen automatically but requires the use of technology to be embedded, with reflection and care, as one element amongst others into the overall design and implementation of teaching-learning environments and situations. The more students can do in and with information technology in mathematics, the greater is the need for their understanding and critical analysis of what they are doing.*

One pitfall of information technology indicated in the research literature is that the technological system itself can form a barrier and an obstacle to learning, either by simply becoming yet another topic in the curriculum, or by distracting students' attention to the system and away from the learning of mathematics. Once again, for this to be avoided it is essential that information technology be assigned a role and place in the entire teaching- learning landscape on the basis of an overall reflective and analytic strategy.

In other words, it is not a simple matter to make information technology assume a role in mathematics education which serves to extend and amplify students' general mathematical capacities rather than replacing their intellects. There is ample research evidence for the claim that when it is no longer our task to train the 'human calculator', some of the traditional drill does becomes obsolete. However, we have yet to see research pointing out exactly what and how much procedural ability is needed for understanding the processes and products generated by the information technology.

4 Conclusion

In a short paper it is not possible to do justice to then entire field of the didactics of mathematics. Instead of the few findings put forward here, hosts of other findings could equally well have been selected in their place.

Important findings concerning the demands and potentials of *problem solving* and *applications and modelling*; the problems and potentials of *assessment*; the values and efficiency of *collaborative learning* and *innovative teaching approaches and forms of study*, such as project work; the significance of carefully balanced, innovative *multifaceted curricula*, elucidating historical, philosophical, societal, and applicational aspects of mathematics; the impact of *social, cultural and gender factors* on mathematics education; and many others, have not, regrettably, been given their due shares in this presentation. The same is true with the findings contributed by impressive bodies of research on the teaching and learning of specific mathematical *topics*, such as arithmetic, abstract and linear algebra, calculus/analysis, geometry, discrete mathematics, and probability and statistics, and with the findings represented by the instrumental interpretative theories. Also the extensive and elaborate examples of didactical engineering (design and construction) contributed by a number of research and development centres in different countries have been left out of this survey.

Nevertheless, the findings which we have been able to present suffice to teach us two lessons which we might want to call *super-findings*. If we want to teach mathematics to students other than the rather few who can succeed without being taught, or the even fewer who cannot learn mathematics irrespective of how they are taught, two matters have to be kept in mind at all times:

1. We have to be infinitely careful not to jump to conclusions and make false inferences about the processes and outcomes of students' learning of mathematics.

2. If there is something we want our students to know, understand, or be able to do, we have to make it object of explicit and carefully designed teaching. There is no such thing as guaranteed transfer of knowledge, insight and ability from one context or domain to another, it has to be cultivated.

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References

Balacheff, N. and Kaput, J. (1996). 'Computer-Based Learning Environments in Mathematics'. In Bishop et al., 1996, chapter 13, pages 469–501.

Biehler, R., Scholz, R., Strässer, R., and Winkelmann, B., eds. (1994). *Didactics of Mathematics as a Scientific Discipline*. Dordrecht: Kluwer Academic Publishers.

Bishop, A. (1988). *Mathematical Enculturation. A Cultural Perspective on Mathematics Education*. Dordrecht: Kluwer Academic Publishers.

Bishop, A., Clements, K., Keitel, C., Kilpatrick, J., and Laborde, C., eds. (1996). *International Handbook of Mathematics Education*, volume 1–2. Dordrecht: Kluwer Academic Publishers.

Cobb, P. and Bauersfeld, H. (1995). *The Emergence of Mathematical Meaning: Interaction in Classroom Cultures*. Hillsdale (NJ): Lawrence Erlbaum.

Douady, R. (1991). 'Tool, Object, Setting, Window'. In Bishop, A., Mellin-Olsen, S., and van Dormolen, J., eds., *Mathematical Knowledge: Its Growth Through Teaching*, pages 109–130. Kluwer Academic Publishers, Dordrecht.

Ellerton, N. and Clarkson, P. (1996). 'Language Factors in Mathematics Teaching and Learning'. In Bishop et al., 1996, chapter 26, pages 987–1033.

Fischbein (1989). 'Tacit Models and Mathematics Reasoning'. *For the Learning of Mathematics*, 9(2), 9–14.

Grouws, D., ed. (1992). *Handbook of Research on Mathematics Teaching and Learning*. New York (NY): Macmillan Publishing Company.

Hanna, G. and Jahnke, H. (1996). 'Proof and Proving'. In Bishop et al., 1996, chapter 23, pages 877–908.

Heid, K. (1997). 'The Technological Revolution and the Reform of School Mathematics'. *American Journal of Education*, 106(1), 5–61.

Niss, M., ed. (1993). *Investigations into Assessment in Mathematics Education*. Dordrecht: Kluwer Academic Publishers.

Pehkonen, E. and Törner, G. (1996). 'Mathematical beliefs and different aspects of their meaning'. *Zentralblatt für Didaktik der Mathematik*, 28(4), 101–108.

Sfard, A. (1991). 'On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the Same Coin'. *Educational Studies in Mathematics*, 22(1), 1–36.

Sierpinska, A. and Kilpatrick, J., eds. (1998). *Mathematics Education as a Research Domain*, volume 1–2. Dordrecht: Kluwer Academic Publishers.

Sierpinska, A. and Lerman, S. (1996). 'Epistemologies of Mathematics and of Mathematics Education'. In Bishop et al., 1996, chapter 22, pages 827–876.

Tall, D., ed. (1991). *Advanced Mathematical Thinking*. Dordrecht: Kluwer Academic Publishers.

Tall, D. (1992). 'The Transition to Advanced Mathematical Thinking: Functions, Limits, Infinity, and Proof'. In Grouws, 1992, pages 495–511.

Vinner, S. (1991). 'The Role of Definitions in Teaching and Learning'. In Tall, 1991, chapter 5, pages 65–81.

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