# On the Number of Square Classes of a Field of Finite Level

Karim Johannes Becher

Received: May 29, 2001
Revised: October 31, 2001

Communicated by Ulf Rehmann

Abstract. The *level question* is, whether there exists a field $F$ with finite *square class number* $q(F) := |F^\times/F^{\times 2}|$ and finite level $s(F)$ greater than four. While an answer to this question is still not known, one may ask for lower bounds for $q(F)$ when the level is given.

For a nonreal field $F$ of level $s(F) = 2^n$, we consider the filtration of the groups $D_F(2^i)$, $0 \leq i \leq n$, consisting of all the nonzero sums of $2^i$ squares in $F$. Developing further ideas of A. Pfister, P. L. Chang and D. Z. Djoković and by the use of combinatorics, we obtain lower bounds for the invariants $\overline{q}_i := |D_F(2^i)/D_F(2^{i-1})|$, for $1 \leq i \leq n$, in terms of $s(F)$. As a consequence, a field with finite level $\geq 8$ will have at least 512 square classes. Further we give lower bounds on the cardinalities of the Witt ring and of the 2-torsion part of the Brauer group of such a field.

## 1 Introduction

Let $F$ be a field. The *level of $F$,* denoted by $s(F)$, is defined as the least positive integer $m$ such that $-1$ is a sum of $m$ squares in $F$ whenever such an integer exists and $\infty$ otherwise. For fields of positive characteristic this invariant can take only the values 1 and 2, depending just on whether $-1$ is a square in $F$ or not. Fields of level $\infty$, i.e. in which $-1$ is not a sum of squares, are called *real fields* and an equivalent condition to $s(F) = \infty$ is the existence of an ordering on $F$. Fields of finite level are also called *nonreal fields.*
For a long time it has been an open question which values exactly occur as the level of some field. The complete solution to this problem was given by A. Pfister in [10] and it inspired a big part of later advances in the theory of quadratic forms, e.g. the development of the theory of *Pfister forms* and the investigation of isotropy behaviors of quadratic forms under function field extensions.

Pfister proved that the level of a nonreal field is always a power of 2 [10, Satz 4] and further that, if $F$ is any real field (e.g. $\mathbb{Q}$ or $\mathbb{R}$) and $n \geq 0$, then the function field of the projective quadric $X_0^2 + \cdots + X_{2^n}^2 = 0$ over $F$ has level $2^n$ [10, Satz 5]. These were the first examples of nonreal fields of level greater than 4 and, actually, still no examples of an essentially different kind are known.
In general it remains a difficult problem to determine the level of a given field of characteristic zero. For an overview on what is known about levels of common types of fields we refer to [8, Chap. XI, Section 2]. In the same book T. Y. Lam also mentions the following question [8, p. 333]:

1.1. LEVEL QUESTION. *Does there exist a field $F$ such that $4 < s(F) < \infty$ and such that ${F^\times}/{F^\times}^2$ is finite?*

Here and in the sequel we denote by $F^\times$ the multiplicative group of $F$ and by ${F^\times}^2$ the subgroup of nonzero squares in $F$. The quotient $F^\times/{F^\times}^2$ is called the *square class group of $F$*. We call $q(F) := |F^\times/{F^\times}^2|$ the *square class number of $F$*. Another subgroup of $F^\times$ of importance is the group of nonzero sums of squares in $F$, denoted as $\sum {F^\times}^2$.
Further, for any $m \in \mathbb{N}$ we denote by $D_F(m)$ the set of elements of $F^\times$ which can be written as a sum of $m$ squares over $F$. Pfister has shown that $D_F(m)$ is a group whenever $m$ is a power of 2 [10, Satz 9]. We thus have the following group filtration for $\sum {F^\times}^2$:

$$ {F^\times}^2 \subsetneq D_F(2) \subsetneq D_F(4) \subsetneq \cdots \subsetneq D_F(2^{i-1}) \subsetneq D_F(2^i) \subsetneq \cdots \subset \textstyle\sum {F^\times}^2. \quad (1.2) $$

If $F$ is nonreal of level $2^n$ then we actually have $D_F(2^n+1) = \sum {F^\times}^2 = F^\times$.
For $i \geq 1$ we define $\bar{q}_i(F) := |D_F(2^i)/D_F(2^{i-1})|$. Note that the quotients $F^\times/{F^\times}^2$ and $D_F(2^i)/D_F(2^{i-1})$ are 2-elementary abelian groups. So $q(F)$ and $\bar{q}_i(F)$ are each either a power of 2 or $\infty$.
From (1.2) we see that the inequality

$$ q(F) \geq \bar{q}_1(F) \cdots \bar{q}_n(F) \quad (1.3) $$

holds for any $n \geq 1$. We will use this in particular when $s(F) = 2^n$.

While an answer to the level question is still not known, one may look for lower bounds on $|F^\times/{F^\times}^2|$ in terms of $s(F)$.
One approach is to search for lower bounds on the invariants $\bar{q}_i(F)$ and to use then (1.3) to obtain a bound for $q(F)$. Following this idea, A. Pfister obtained in [11, Satz 18.d] the following estimate for a field $F$ of level $2^n$:

$$ q(F) \geq 2^{\frac{n(n+1)}{2}}. \quad (1.4) $$

His proof (see also [8, p. 325]) actually shows for $1 \leq i \leq n$ that

$$ \bar{q}_i(F) \geq 2^{n+1-i}. \quad (1.5) $$

Our standard examples of fields of level $1, 2$ and $4$, respectively, are the field of complex numbers $\mathbb{C}$, the finite field $\mathbb{F}_3$ and $\mathbb{Q}_2$, the field of dyadic numbers. These examples show that (1.4) is best possible for $n \leq 2$. For higher $n$, however, P. L. Chang has improved the bound using combinatorics. In [1] he shows that $q(F) \geq 128$ for a field $F$ of level eight and further that $q(F) \geq 16 \cdot \frac{2^s}{s^2}$ for any nonreal field $F$ of level $s \geq 16$. His approach has been refined by D. Ž. Djoković in [2], leading to the following estimate:

$$q(F) \;\geq\; 2 \cdot \sum_{i=1}^{s/2} \frac{1}{s+2-i} \binom{s+1}{i} > \frac{2^s}{s} \; . \tag{1.6}$$

Their method does not provide any information about the invariants $\bar{q}_i(F)$.

The aim of the present work is to extend this method and to get lower bounds for the invariants $\bar{q}_i(F)$ with respect to $s(F)$ which improve (1.5). The combinatorial aspect is postponed to the two appendices where a certain coloring problem for (hyper-)graphs is considered.

We use common notations and results from quadratic form theory; the standard references are [8] and [12]. (Note that the uncomfortable case of characteristic 2 is implicitly excluded whenever we deal with a field of level greater than 1.) For isometry of quadratic forms we use the symbol $\cong$ . For a quadratic form $\varphi$ over $F$ we denote by $D_F(\varphi)$ the set of nonzero elements of $F$ represented by $\varphi$. We sometimes say just "*form*" or "*quadratic form*" to mean "*non-degenerate quadratic form*".

A diagonalized quadratic form over $F$ with coefficients $a_1, \ldots, a_m \in F^\times$ is denoted by $\langle a_1, \ldots, a_m \rangle$. An *m-fold Pfister form* is a quadratic form of the shape $\langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_m \rangle$ and shortly written as $\langle\langle a_1, \ldots, a_m \rangle\rangle$; its dimension is $2^m$. A *neighbor* of an $m$-fold Pfister form $\pi$ is a quadratic form $\varphi$ which is similar to a subform of $\pi$ and of dimension greater than $2^{n-1}$. We know that in this situation $\varphi$ is isotropic if and only of $\pi$ is hyperbolic.

By $W(F)$ we denote the Witt ring of $F$, further by $\mathrm{Br}(F)$ the Brauer group and by $\mathrm{Br}_2(F)$ its 2-torsion part. In (3.1), (5.4) and (5.5) we shall use Milnor $K$-theory. For definitions and properties of the Milnor ring $k_*F$ and its homgenous components $k_mF$ ($m \geq 0$) we refer to [9] and [3]. However, we use the notation $\{a_1, \ldots, a_m\}$ instead of $\ell(a_1) \cdots \ell(a_m)$ for a symbol in $k_mF$. We recall that this symbol is zero in $k_mF$ if and only if the corresponding $m$-fold Pfister form $\langle\langle -a_1, \ldots, -a_m \rangle\rangle$ over $F$ is hyperbolic (see [3, Main Theorem 3.2]). In particular, $s(F) = 2^n$ is equivalent to $\{-1\}^n \neq 0$ and $\{-1\}^{n+1} = 0$ in $k_*F$. Everywhere else in the text, $\{x_1, \ldots, x_n\}$ stands simply for the set of elements $x_1, \ldots, x_1$.

## Acknowledgments

This work is part of the author's Ph.D. thesis research, carried out at the *Université de Franche-Comté* under the supervision of Eva Bayer-Fluckiger and Detlev W. Hoffmann. The author expresses his gratitude to his supervisors as well as to Ján Mináč and to David B. Leep for encouragement and stimulating discussions on the subject.

## 2 Sums of squares in fields

Let $F$ be a field. For an element $x \in F$ we define its *length (over $F$)* to be the least positive integer $m$ such that $x$ can be written as a sum of $m$ nonzero squares over $F$ if such an integer exists and $\infty$ otherwise (i.e. if $x$ is not a nontrivial sum of nonzero squares over $F$). We denote this value in $\mathbb{N} \cup \{\infty\}$ by $\ell_F(x)$, or just by $\ell(x)$ whenever the context makes clear over which field $F$ we are working. Obviously $\ell_F(x)$ depends on $x$ only up to multiplication by a nonzero square in $F$; in other words, $\ell_F(x)$ is an invariant of the square class $xF^{\times 2}$ whenever $x \neq 0$.
For $m \geq 1$, $D_F(m)$ is by definition the set $\{x \in F^\times \mid \ell(x) \leq m\}$. Our investigation into lengths of field elements is based on the following famous result [10, Satz 2]:

2.1. Theorem (Pfister). *For any $i \geq 0$, $D_F(2^i)$ is a subgroup of $F^\times$.*

A simple proof within the theory of Pfister forms can be found in [12, 4.4.1. Lemma]. As a consequence of this theorem one gets an inequality linking the lengths of two elements to the length of their product. We include a proof of this result, which is [10, Satz 3].

2.2. Lemma. *For any $x, y \in F$ we have the inequalities $\ell(x+y) \leq \ell(x) + \ell(y)$ and $\ell(xy) \leq \ell(x) + \ell(y) - 1$.*

*Proof:* The first inequality is obvious from the definition of the length.
The second inequality is trivial if $xy$ is zero or if $x$ or $y$ is not a sum of squares. So we may suppose that both $x$ and $y$ are nonzero sums of squares in $F$. Let then $r$ be the least nonnegative integer such that $x, y \in D_F(2^r)$. We will prove $\ell(xy) < \ell(x) + \ell(y)$ by induction on $r$. If $r = 0$ then $x, y$ and $xy$ are squares in $F$ and the inequality is clear. Suppose now that $r > 0$. Since $D_F(2^r)$ is a group we know that $\ell(xy) \leq 2^r$. So the inequality is clear if $2^r < \ell(x) + \ell(y)$. Otherwise, we may suppose that $\ell(y) \leq 2^{r-1}$. By the choice of $r$ we then have $2^{r-1} < \ell(x) \leq 2^r$ and may therefore write $x = a + z$ with $a, z \in F^\times$ such that $\ell(a) = 2^{r-1}$ and $\ell(z) = \ell(x) - 2^{r-1} \leq 2^{r-1}$. By the induction hypothesis we have $\ell(zy) < \ell(y) + \ell(z)$. As $D_F(2^{r-1})$ is a group we have $\ell(ay) \leq 2^{r-1}$. Since $xy = ay + zy$, using the first inequality of the statement we obtain finally $\ell(xy) \leq \ell(ay) + \ell(zy) < 2^{r-1} + \ell(y) + \ell(z) = \ell(x) + \ell(y)$. □

According to the definition we gave in the introduction, the *level* of $F$ is the length of $-1$ in $F$. We may also conclude that $\ell_F(0) = s(F) + 1$. Therefore, from any of the inequalities of the lemma we obtain immediately:

2.3. Corollary. *For any $x \in F$ we have $\ell(x) + \ell(-x) \geq s(F) + 1$.* □

2.4. Corollary. *Let $a_1, \dots, a_m \in F^\times$. If the quadratic form $\langle a_1, \dots, a_m \rangle$ over $F$ represents the element $x \in F$ nontrivially then $\ell(a_1) + \cdots + \ell(a_m) \geq \ell(x)$.*

*Proof:* If the form $\langle a_1, \ldots, a_m \rangle$ represents $x \in F$ nontrivially, this means that there are $x_1, \ldots, x_m \in F$, not all zero, such that $a_1 x_1^2 + \cdots + a_m x_m^2 = x$. We may suppose that $x_i$ is nonzero for $1 \le i \le m'$ and zero for $m' < i \le m$. From the first inequality of the lemma we obtain $\ell(x) \le \ell(a_1 x_1^2) + \cdots + \ell(a_{m'} x_{m'}^2) = \ell(a_1) + \cdots + \ell(a_{m'})$. □

For $i \ge 0$, we say that the elements $a_1, \ldots, a_m \in F^\times$ are *independent modulo* $D_F(2^i)$ if in $F^\times / D_F(2^i)$, considered as an $\mathbb{F}_2$-vectorspace, the classes represented by $a_1, \ldots, a_m$ are $\mathbb{F}_2$-linear independent.

2.5. PROPOSITION. *For $i \ge 2$, let $a, b \in D_F(3 \cdot 2^{i-2}) \setminus D_F(2^{i-1})$ and $c \in D_F(2^i)$ such that $\ell(a+b+c) > 2^{i+1}$. Then the elements $a, b$ and $c$ of $D_F(2^i)$ are independent modulo $D_F(2^{i-1})$.*

*Proof:* We have to show that $a, b, c, ab, ac, bc, abc \notin D_F(2^{i-1})$. For $a$ and $b$ this is already given. We put $x := a + b + c$. Each of the quadratic forms $\langle a, b, c \rangle$, $\langle a, b, abc \rangle$, $\langle 1, ab, ac \rangle$ and $\langle ac, bc, 1 \rangle$ over $F$ represents one of the elements $x$, $abx$, $ax$ and $cx$ and neither of these elements lies in the group $D_F(2^{i+1})$. We obtain from (2.4) that each of the numbers $\ell(a) + \ell(b) + \ell(c)$, $\ell(a) + \ell(b) + \ell(abc)$, $1 + \ell(ab) + \ell(ac)$ and $\ell(ac) + \ell(bc) + 1$ is greater than $2^{i+1}$. Since $\ell(a) + \ell(b) \le 3 \cdot 2^{i-1}$ and $ab, ac, bc \in D_F(2^i)$ we obtain $\ell(c), \ell(abc) \ge 2^{i-1}$ and further $\ell(ab) = \ell(ac) = \ell(bc) = 2^i$. □

For the rest of this section we fix a sum of squares

$$x = x_1^2 + \cdots + x_l^2 \tag{2.6}$$

with $x_1, \ldots, x_l \in F^\times$, $x \in F$ and $l = \ell_F(x)$. For a subset $I \subset \{1, \ldots, l\}$ we denote $x_I := \sum_{i \in I} x_i^2$. If $I$ is not empty then we have $\ell(x_I) = |I|$.
For a real number $z$ we denote by $\lceil z \rceil$ the least integer $\ge z$.

2.7. THEOREM. *Let $I$ and $J$ be nonempty proper subsets of $\{1, \ldots, l\}$. Let $r$ be a nonnegative integer such that $x_I x_J \in D_F(2^r)$. Then the following hold:*

(*i*) $\left\lceil \frac{|I|}{2^r} \right\rceil = \left\lceil \frac{|J|}{2^r} \right\rceil$, *in particular* $\big|\,|I| - |J|\,\big| < 2^r$,

(*ii*) $|I \setminus J|\ ,\ |J \setminus I| \ \le\ 2\ \ell(x_I x_J) - 1 \ <\ 2^{r+1}$,

(*iii*) $|I \cup J| - |I \cap J| \ \le\ 2^{r+1} + \ell(x_I x_J) - 1 \ \le\ 3 \cdot 2^r - 1$.

*Proof:* The hypothesis implies that $x_I$ and $x_J$ are nonzero elements of $F$. We set $m := \ell(x_I x_J)$ and $a := \frac{x_J}{x_I}$. Then $\ell(a) = m \le 2^r$.
If $\nu$ is an integer such that $|I| \le \nu 2^r$ then we can write $x_I$ as a sum of $\le \nu$ elements of $D_F(2^r)$. As $D_F(2^r)$ is a group, $x_J = a x_I$ can also be written as a sum of $\le \nu$ elements of $D_F(2^r)$ which means that $|J| = \ell(x_J) \le \nu 2^r$. By symmetry we obtain for any $\nu \in \mathbb{N}$ that $|I| \le \nu 2^r$ if and only if $|J| \le \nu 2^r$. This shows (*i*).

We compute $x_{I\cup J} = x_{I\setminus J} + x_J = (1+a)x_{I\setminus J} + ax_{I\cap J}$ and then substitute $y := (1+a)x_{I\setminus J}$ and $z := ax_{I\cap J}$ to have $x_{I\cup J} = y + z$.
If $y \neq 0$ then we have $\ell(y) \leq m + |I \setminus J|$ by (2.2), but also $\ell(y) \leq 2^{r+1}$ since $D_F(2^{r+1})$ is a group. If $z \neq 0$ then (2.2) yields $\ell(z) \leq m + |I \cap J| - 1$. Therefore, if at least one of $y$ and $z$ is nonzero then we obtain the inequalities $\ell(y+z) \leq |I| + 2m - 1$ and $\ell(y+z) \leq 2^{r+1} + m + |I \cap J| - 1$. Both inequalities remain valid in the case $y = z = 0$, since then necessarily $a = -1$, whence $\ell(y+z) = \ell(0) = m+1$. As $|I \cup J| = \ell(y+z)$ we obtain (*ii*) by symmetry from the first and (*iii*) from the second inequality. □

For $m = 1$ this leads to an observation made in the proof of [1, Theorem 1]:

2.8. COROLLARY (CHANG). *Let $I$ and $J$ be as in the theorem. If $x_I$ and $x_J$ lie in the same square class then both sets have the same cardinality and differ by at most one element.* □

2.9. COROLLARY. *Let $I$ and $J$ be as in the theorem with $|I| = |J| = 2^i$, $i \geq 2$. If $x_I$ and $x_J$ represent the same class modulo $D_F(2^{i-1})$ then $|I \cap J| \geq 2^{i-2} + 1$.*

*Proof:* If $x_I$ and $x_J$ lie in the same class modulo $D_F(2^{i-1})$ then $\ell(x_I x_J) \leq 2^{i-1}$. Applying part (*iii*) of the theorem for $r = i - 1$ we obtain $|I \cup J| - |I \cap J| \leq 3 \cdot 2^{i-1} - 1$. But our hypothesis here gives $|I \cup J| = 2 \cdot 2^i - |I \cap J|$. This together implies $|I \cap J| > 2^{i-2}$. □

## 3 THE INVARIANTS $\bar{q}_i$

For a nonreal field $F$ of level $2^n$ we are going to study the invariants $\bar{q}_i(F) = |D_F(2^i)/D_F(2^{i-1})|$ for $1 \leq i \leq n$. In particular, we are interested to know whether Pfister's bounds (1.5) can be improved.

First we note that the bound $\bar{q}_n(F) \geq 2$, obtained from (1.5) for $i = n$, just takes into account that $-1$ represents a nontrivial class in the group $D_F(2^n)/D_F(2^{n-1})$. In spite of the simple argument, this bound is optimal for every $n \geq 1$. More precisely, for any $n \geq 1$ there is a field $F$ of level $2^n$ such that $F^\times = D_F(2^{n-1}) \cup -D_F(2^{n-1})$. The construction of such an example will be included in a forthcoming paper of the author.
We now turn to consider $\bar{q}_{n-1}(F)$. For $i = n-1$, (1.5) gives $\bar{q}_{n-1}(F) \geq 4$. The example $F = \mathbb{Q}_2$ shows that this bound is optimal for $n = 2$.

3.1. THEOREM. *Let $F$ be a field of level $2^n$ with $n \geq 3$. Then $\bar{q}_{n-1}(F) \geq 16$.*

*Proof:* Since $\ell(0) = 2^n + 1$ and $n \geq 3$, we may choose elements $a_1, a_2, a_3 \in F^\times$ such that $a_1 + a_2 + a_3 = 0$ and $2^{n-2} + 1 \leq \ell(a_i) \leq 3 \cdot 2^{n-3}$ for $i = 1, 2, 3$. Then by (2.5), $a_1, a_2$ and $a_3$ are independent modulo $D_F(2^{n-2})$. Let $H$ be the subgroup of $D_F(2^{n-1})$ generated by $D_F(2^{n-2})$ and the elements $a_1, a_2$ and $a_3$. Since $|H/D_F(2^{n-2})| = 8$ it remains to show that $H \neq D_F(2^{n-1})$.

To this aim, we will calculate in the Milnor ring $k_*F$. For $i = 1, 2, 3$ we fix the symbols $\beta_i := \{a_1a_2a_3, a_i\}$ and $\gamma_i := \{-a_1a_2a_3, -a_i\}$ in $k_2F$. Let $\varepsilon$ denote the element $\{-1\}$ in $k_1F$. Since $s(F) = 2^n$ we have $\varepsilon^n \neq 0$. As $a_1, a_2, a_3 \in D_F(2^{n-1})$ we observe that $\beta_1+\beta_2+\beta_3 = \{-1, a_1a_2a_3\}$ is annihilated by $\varepsilon^{n-2}$ and that $\varepsilon^{n-2}(\beta_i+\gamma_i) = \varepsilon^{n-2}(\{a_1a_2a_3, -1\}+\{-1, a_i\}+\{-1,-1\}) = \varepsilon^n$ for $i = 1, 2, 3$.

If $\varepsilon^{n-2}\beta_i \neq 0$ in $k_nF$ for some $i$ then by the above relations we may suppose that $\varepsilon^{n-2}\beta_i \neq 0$ for $i = 1, 2$ and $\varepsilon^{n-2}\beta_3 \neq \varepsilon^n$, i.e. $\varepsilon^{n-2}\gamma_3 \neq 0$. Using that $a_1 + a_2 + a_3 = 0$ we compute $\{-a_2, -a_3\} = \{a_1, -a_2a_3\} = \beta_1$ and equally $\{-a_1, -a_3\} = \beta_2$. Since none of $\beta_1$, $\beta_2$ and $\gamma_3$ is annihilated by $\varepsilon^{n-2}$, the symbols $\varepsilon^{n-2}\{-a_2, -a_3\}$, $\varepsilon^{n-2}\{-a_1, -a_3\}$ and $\varepsilon^{n-2}\{-a_1a_2, -a_3\}$ in $k_nF$ are all nonzero. Therefore the Pfister forms $2^{n-2}\times\langle\langle a_2, a_3\rangle\rangle$, $2^{n-2}\times\langle\langle a_1, a_3\rangle\rangle$ and $2^{n-2}\times\langle\langle a_1a_2, a_3\rangle\rangle$ are anisotropic. Further, $2^{n-2}\times\langle\langle 1, a_3\rangle\rangle \cong 2^n\times\langle 1\rangle$ is anisotropic since $s(F) = 2^n$. This shows that $-1, -a_1, -a_2, -a_1a_2 \notin D_F(2^{n-2}\times\langle\langle a_3\rangle\rangle)$. As the group $D_F(2^{n-2}\times\langle\langle a_3\rangle\rangle)$ contains the subgroup $D_F(2^{n-2})$ and the element $a_3$ we conclude that $D_F(2^{n-2}\times\langle\langle a_3\rangle\rangle) \cap -H = \emptyset$. On the other hand, since $\ell(-a_3) \leq \ell(a_1)+\ell(a_2) \leq 3\cdot 2^{n-2}$ we can write $-a_3 = x+y$ with $x \in D_F(2^{n-1})$, $y \in D_F(2^{n-2})$ and obtain $-x = y + a_3 \in D_F(2^{n-2}\times\langle\langle a_3\rangle\rangle) \cap -D_F(2^{n-1})$.

Now we study the case where $\varepsilon^{n-2}\beta_i = 0$ for $i = 1, 2, 3$. As $\varepsilon^{n-2}\beta_i = \varepsilon^{n-2}\{-a_1a_2a_3, a_i\}$, this means that the Pfister form $2^{n-2} \times \langle\langle a_1a_2a_3, -a_i\rangle\rangle$ is hyperbolic for $i = 1, 2, 3$. We conclude that $H \subset D_F(2^{n-2} \times \langle\langle a_1a_2a_3\rangle\rangle)$. As the Pfister form $2^{n-1} \times \langle\langle a_1a_2a_3\rangle\rangle \cong 2^n \times \langle 1\rangle$ is anisotropic we have $-1 \notin D_F(2^{n-2} \times \langle\langle a_1a_2a_3\rangle\rangle)$ and therefore $D_F(2^{n-2} \times \langle\langle a_1a_2a_3\rangle\rangle) \cap -H = \emptyset$. Since $-a_1a_2a_3 = a_1^2a_2 + a_2^2a_1$ we have $\ell(-a_1a_2a_3) \leq \ell(a_2) + \ell(a_1) \leq 3 \cdot 2^{n-2}$ and may therefore write $-a_1a_2a_3 = x+y$ with $x \in D_F(2^{n-1})$ and $y \in D_F(2^{n-2})$ to obtain this time $-x = y + a_1a_2a_3 \in D_F(2^{n-2} \times \langle\langle a_1a_2a_3\rangle\rangle) \cap -D_F(2^{n-1})$.

In both cases we have found an element $x \in D_F(2^{n-1}) \setminus H$. $\square$

While the lower bound on $\bar{q}_{n-1}$ of the last theorem is based upon several algebraic arguments, the improvement (with respect to (1.5)) for the lower bounds on $\bar{q}_i(F)$ for $2 \leq i \leq n-2$ which we present now, is obtained by combinatorial reasoning, developed in appendix A.

For integers $0 \leq k \leq l$ we denote by $\mathcal{P}_k^l$ the set of subsets of $\{1, \ldots, l\}$ with exactly $k$ elements.

3.2. Theorem. *Let $F$ be a field of level $2^n$. Then*

$$\bar{q}_i(F) \;\geq\; \begin{cases} 2^7 & \textit{for} \quad i = n-2 \geq 3\,, \\ 2^{(n-i)(2^{n-i}+1)+1} & \textit{for} \quad \frac{n+1}{2} < i \leq n-3\,, \\ 2^{(n-i)(2^{i-2}+1)+1} & \textit{for} \quad 2 \leq i \leq \frac{n+1}{2}\,. \end{cases}$$

*Proof:* We fix elements $x_1, \ldots, x_{2^n} \in F^\times$ such that $x_1^2 + \cdots + x_{2^n}^2 = -1$. For a subset $J \subset \{1, \ldots, 2^n\}$ we denote $x_J := \sum_{j\in J} x_j^2$.

Let $2 \leq i \leq \frac{n+1}{2}$. We consider the map $f : \mathcal{P}_{2^i}^{2^n} \longrightarrow D_F(2^i)/D_F(2^{i-1})$ which sends a $2^i$-subset $J \subset \{1, \ldots, 2^n\}$ to the class $x_J D_F(2^{i-1})$. By (2.9), if $J_1, J_2 \in$

$\mathcal{P}^{2^n}_{2^i}$ are such that $f(J_1) = f(J_2)$ then $|J_1 \cap J_2| \geq 2^{i-2}+1$. Therefore (A.8) in appendix A shows $|D_F(2^i)/D_F(2^{i-1})| \geq |Im(f)| > 2^r$ for $r := (n-i)(2^{i-2}+1)$. Since $D_F(2^i)/D_F(2^{i-1})$ is a 2-elementary abelian group it must then have at least $2^{r+1}$ elements. This establishes the third case in the statement.

In the remaining cases we cannot apply (A.8) directly for $i$ and $m := n$. In the case $\frac{n+1}{2} < i \leq n-3$ we have $n \geq 8$ and $i \geq 5$ and define $n' := 2(n-i+1)$ and $i' := n-i+2 = \frac{n'}{2}+1$. In the case $i = n-2$ and $n \geq 5$ we set instead $n' := 5$ and $i' := 3 = \frac{n'+1}{2}$. Note that in both cases $n'-i' = n-i$.

For $1 \leq \nu \leq 2^{n'}$ let $J_\nu := \{(\nu-1)\cdot 2^{n-n'}+1, \ldots, \nu \cdot 2^{n-n'}\}$ and $y_\nu := x_{J_\nu}$. This yields $y_1 + \cdots + y_{2^{n'}} = -1$ and $\ell(y_\nu) = |J_\nu| = 2^{n-n'}$ for $1 \leq \nu \leq 2^{n'}$. Now we consider the map $f' : \mathcal{P}^{2^{n'}}_{2^{i'}} \longrightarrow D_F(2^i)/D_F(2^{i-1})$ which sends a $2^{i'}$-subset $N \subset \{1, \ldots, 2^{n'}\}$ to the class $(\sum_{\nu \in N} y_\nu) D_F(2^{i-1})$.

Suppose that $f'(N_1) = f'(N_2)$ for $N_1, N_2 \in \mathcal{P}^{2^{n'}}_{2^{i'}}$. For $k = 1, 2$ let $I_k := \bigcup_{\nu \in N_k} J_\nu \in \mathcal{P}^{2^n}_{2^i}$. Since by hypothesis $\sum_{\nu \in N_1} y_\nu = x_{I_1}$ and $\sum_{\nu \in N_2} y_\nu = x_{I_2}$ lie in the same class of $D_F(2^i)/D_F(2^{i-1})$, (2.9) shows that $|I_1 \cap I_2| \geq 2^{i-2}+1$ and it follows that $|N_1 \cap N_2| \geq 2^{i-2-(n-n')}+1 = 2^{i'-2}+1$.

Having established this intersection property of $f'$, we obtain from (A.8) that $|D_F(2^i)/D_F(2^{i-1})| \geq |Im(f')| > 2^{r'}$ holds for $r' := (n'-i')(2^{i'-2}+1)$. As before, we conclude that $|D_F(2^i)/D_F(2^{i-1})| \geq 2^{r'+1}$. This finishes the proof since $r' = 6$ in case $i = n-2$ and $r' = (n-i)(2^{n-i}+1)$ otherwise. □

## 4 Nonreal fields with $\bar{q}_1$ equal to the level

From (1.5) we know that $\bar{q}_1(F) \geq s(F)$ holds for any nonreal field $F$. This bound is optimal for fields of level 1, 2 and 4 as the standard examples show (see introduction). For nonreal fields of higher level, however, there is still no known example where $\bar{q}_1(F) < \infty$.

We show that $\bar{q}_1(F) = s(F) < \infty$ is a rather strong condition, with several consequences on the quadratic form structure of $F$. In particular, for $s(F) \geq 8$ it implies that $\bar{q}_2(F) \geq \frac{s(F)^2}{2}$ (4.9).

Let $\xi$ be an element of length $l \geq 3$ of $F$. We fix a representation of $\xi$ as a sum of $l$ squares

$$\xi = x_1^2 + \cdots + x_l^2 \tag{4.1}$$

with $x_1, \ldots, x_l \in F^\times$. Let $f : \mathcal{P}^l_2 \to D_F(2)/F^{\times 2}$ be the function which sends a (nonordered) pair of distinct $i, j \leq l$ to the square class of $x_i^2 + x_j^2$. Considering the elements of $D_F(2)/F^{\times 2}$ as a set of colors, we can interprete $f$ as an edge-coloring of a complete graph in $l$ vertices $v_1, \ldots, v_l$. We denote this graph together with its edge-coloring $f$ by $\mathcal{G}$. If in this graph two edges $[v_i, v_j]$ and $[v_{i'}, v_{j'}]$ are of the same color (with $\{i,j\}, \{i',j'\} \in \mathcal{P}^l_2$) this means that $x_i^2 + x_j^2$

and $x_{i'}^2 + x_{j'}^2$ lie in the same square class of $F$, which by (2.8) implies that the sets $\{i, j\}$ and $\{i', j'\}$ intersect. In other words, two edges of the same color in $\mathcal{G}$ need to have a vertex in common, i.e. $\mathcal{G}$ is a *CC-graph* in the terminology of appendix B.

We get from (B.1) that at least $l - 2$ colors appear in $\mathcal{G}$. Furthermore, since $x_1^2 + \cdots + x_l^2$ is of length $l$, no sum $x_i^2 + x_j^2$ with $i \neq j$ can be a square. This gives a proof of [13, Theorem 1]:

4.2. Proposition (Tort). *In* (4.1), *the partial sums* $x_i^2 + x_j^2$ *with* $1 \leq i < j \leq l$ *represent at least* $l - 2$ *different nontrivial classes of* $D_F(2)/F^{\times 2}$. □

Let now $F$ be a nonreal field of level $s = 2^n$. We then can choose $\xi := 0$, which is of length $s + 1$ over $F$, and write (4.1) as

$$0 = x_1^2 + \cdots + x_{s+1}^2 \,. \tag{4.3}$$

By the above proposition the partial sums $x_i^2 + x_j^2$ (with $1 \leq i < j \leq s + 1$) represent at least $s - 1$ nontrivial classes of $D_F(2)/F^{\times 2}$. This shows:

4.4. Corollary. *Let* $F$ *be a nonreal field of level* $s$. *Then* $\bar{q}_1(F) \geq s$. *Moreover, if* $\bar{q}_1(F) = s$ *then, given any representation* (4.3) *of zero as a sum of* $s+1$ *nonzero squares over* $F$, *every nontrivial class of* $D_F(2)/F^{\times 2}$ *is represented by a partial sum* $x_i^2 + x_j^2$ *with* $1 \leq i < j \leq s+1$. □

Given a subgroup $G \subset F^\times/F^{\times 2}$ of finite order $2^m$ we may choose an irredundant set of representatives $a_1, \ldots, a_{2^m} \subset F^\times$ of the square classes in $G$ and define the quadratic form $\pi_G := \langle a_1, \ldots, a_{2^m} \rangle$. Up to isometry, this form does only depend on $G$ and not on the particular choice of the $a_i$. If we choose the $a_i$ such that $a_1, \ldots a_m$ are independent modulo $F^{\times 2}$ then $\pi_G$ is equal to $\langle\langle a_1, \ldots, a_m \rangle\rangle$, hence $\pi_G$ is an $m$-fold Pfister form. If $\bar{q}_1(F)$ is finite we write $\pi_{D(2)}$ for $\pi_G$ with $G := D_F(2)/F^{\times 2}$.

4.5. Proposition. *Let* $F$ *be a nonreal field with* $s(F) > 1$ *and* $\bar{q}_1(F) < \infty$. *Then* $\pi_{D(2)}$ *is hyperbolic.*

*Proof:* Let $s := s(F)$. Given a representation (4.3) of zero as sum of $s + 1$ squares over $F$ we define $a_i := x_{2i-1}^2 + x_{2i}^2$ for $1 \leq i \leq s/2$. By (2.8) the $a_i$ lie in distinct nontrivial square classes. Since $a_1 + \cdots + a_{s/2} + x_{s+1}^2 = 0$ the form $\langle 1, a_1, \ldots, a_{s/2} \rangle$ is isotropic. On the other hand, this is a subform of the Pfister form $\pi_{D(2)}$, which then must be hyperbolic. □

4.6. Lemma. *Let* $H$ *be a subgroup of* $F^\times$ *containing* $F^{\times 2}$ *such that* $H/F^{\times 2}$ *is of order* $2^m$ *with* $m \geq 2$. *If* $a, b, c, d \in H$, *lie in distinct square classes then there are* $a_3, \ldots, a_m \in H$ *such that* $\pi_H = \langle a, b, c, d \rangle \otimes \langle\langle a_3, \ldots, a_m \rangle\rangle$.

*Proof:* It is easy to verify that, given four distinct elements $t, u, v, w$ in a 2-elementary abelian group $G$ there exists a subgroup $K$ of index 4 in $G$ such that $t, u, v, w$ represent the four classes of $G/K$.
We apply this fact to the square classes $aF^{\times 2}, bF^{\times 2}, cF^{\times 2}$ and $dF^{\times 2}$ in $G := H/F^{\times 2}$. A subgroup $K$ with the stated property must have order $2^{m-2}$. We choose elements $a_3, \ldots, a_m \in F^\times$ such that their square classes form an $\mathbb{F}_2$-basis of $K$. The rest is clear. □

4.7. PROPOSITION. *Let $F$ be a field with $\bar{q}_1(F) = s(F) = 2^n$, $n \geq 2$, and let $a, b, c, d$ be elements of $D_F(2)$ which lie in distinct square classes.*

- *(a) If $a \notin F^{\times 2}$ then $D_F(\langle 1, 1\rangle) \cap D_F(\langle 1, a\rangle) = \{1, a\}F^{\times 2}$.*
- *(b) If $x \in D_F(\langle 1, a\rangle) \cap D_F(\langle 1, b\rangle) \cap D_F(\langle 1, c\rangle)$ then $\ell(-x) = 2^n$.*
- *(c) $D_F(\langle a, b\rangle) \cap D_F(\langle c, d\rangle) = \emptyset$.*
- *(d) If $n \geq 3$ then $D_F(\langle a, b\rangle) \cap D_F(\langle a, c\rangle) \cap D_F(\langle b, c\rangle) = \emptyset$.*
- *(e) If $x \in D_F(\langle 1, a\rangle) \cap D_F(\langle 1, b\rangle)$ then $\ell(cx) = 4$ or $\ell(-x) \geq 2^n - 1$.*

*Proof:* (a) Given $a$ and $b$ lying in distinct nontrivial classes of $D_F(2)/F^{\times 2}$ we may choose $a_3, \ldots, a_{2^{n-1}} \in D_F(2)$ such that $\varphi := \langle 1, a, b, a_3 \ldots, a_{2^{n-1}}\rangle$ is a neighbor of the Pfister form $\pi_{D(2)}$ which is hyperbolic by the last proposition. So $\varphi$ is isotropic. Now $b \in D_F(\langle 1, a\rangle)$ would imply that $\varphi$ is isometric to $\langle 1, 1, ab, a_3 \ldots, a_{2^{n-1}}\rangle$ which is a subform of $2^n \times \langle 1\rangle$. This is impossible since the latter form is anisotropic by the hypothesis that $s(F) = 2^n$.
(b) Let $x \in D_F(\langle 1, a\rangle) \cap D_F(\langle 1, b\rangle) \cap D_F(\langle 1, c\rangle)$ where $a, b, c \in D_F(2)$ are distinct modulo squares. Then clearly $\ell(x) \leq 3$ and we have also $x \in D_F(\langle 1, abc\rangle)$ (with $-a, -b$ and $-c$ also $-abc$ lies in $D_F(\langle 1, -x\rangle)$). It follows from (a) that $\ell(x) \neq 2$. If $x$ is a square then $\ell(-x) = \ell(-1) = 2^n$. Otherwise we must have $\ell(x) = 3$. Then none of $a, b, c, abc$ can be a square. Further $\ell(-x) \geq 2^n - 2$ by (2.3). Thus (4.2) shows that, in a representation of $-x$ as sum of $\ell(-x)$ squares over $F$, the partial sums of length two lie in at least $2^n - 4$ distinct nontrivial square classes. As $|D_F(2)/F^{\times 2}| = 2^n$ by hypothesis, at least one of these square classes must also be represented by one of $a, b, c$ or $abc$. Without loss of generality we may suppose that $-x = y + at^2$ with $\ell(y) = \ell(-x) - 2$. Writing $x = u^2 + av^2$ yields $0 = x - x = y + u^2 + a(t^2 + v^2)$. Thus $2^n + 1 \leq \ell(y) + 3$ and $2^n \leq \ell(y) + 2 = \ell(-x)$. Then $-x = (-1) \cdot x \in D_F(2^n)$ implies $\ell(-x) = 2^n$.
(c) By the above lemma there are $a_3, \ldots, a_n \in D_F(2)$ such that $\pi_{D(2)}$ is equal to $\langle a, b, c, d\rangle \otimes \langle\langle a_3, \ldots, a_n\rangle\rangle$.
Suppose now that there exists an $x \in D_F(\langle a, b\rangle) \cap D_F(\langle c, d\rangle)$. Then $\langle a, b, c, d\rangle \cong \langle x, abx, x, cdx\rangle$, which is similar to $\langle 1, 1, 1, abcd\rangle$. Hence $\pi_{D(2)}$ is similar to $\langle 1, 1, 1, abcd\rangle \otimes \langle\langle a_3, \ldots, a_n\rangle\rangle \cong 2^{n-1} \times \langle 1\rangle \perp \langle\langle abcd, a_3, \ldots, a_n\rangle\rangle$. It follows that the form $(2^{n-1} + 1) \times \langle 1\rangle$ is a Pfister neighbor of $\pi_{D(2)}$, hence isotropic since $\pi_{D(2)}$ is hyperbolic. This is a contradiction to $s(F) = 2^n$.

(d) After multiplying by $a$ in the statement we may suppose that $a = 1$. Suppose that there exists $x \in D_F(\langle 1, b\rangle) \cap D_F(\langle 1, c\rangle) \cap D_F(\langle b, c\rangle)$. It follows $-b, -c \in D_F(\langle 1, -x\rangle)$, thus $bc \in D_F(\langle 1, -x\rangle) \cap D_F(\langle 1, 1\rangle) \subset D_F(\langle 1, x\rangle)$. Therefore we have $\langle 1, b, c, bc\rangle \cong \langle 1, x, bcx, bc\rangle \cong \langle bc, bcx, bcx, bc\rangle$, whence $\langle 1, b, c, bc\rangle \cong \langle 1, 1, x, x\rangle$. Next we choose $a_3, \ldots, a_n \in D_F(2)$ such that $\pi_{D(2)} \cong \langle 1, b, c, bc\rangle \otimes \langle\langle a_3, \ldots, a_n\rangle\rangle$ and obtain $\pi_{D(2)} \cong \langle\langle 1, x, a_3, \ldots, a_n\rangle\rangle \cong 2^{n-1} \times \langle\langle x\rangle\rangle \cong 2^n \times \langle 1\rangle$, since $a_3, \ldots, a_n \in D_F(2)$, $n \geq 3$ and $x \in D_F(4)$. This is contradictory since $\pi_{D(2)}$ is hyperbolic but $s(F) = 2^n$.
(e) Let $x \in D_F(\langle 1, a\rangle) \cap D_F(\langle 1, b\rangle)$. Then, certainly, $x$ and $cx$ belong to $D_F(4)$. If $\ell(cx) \leq 2$ then $\ell(x) \leq 2$ and (2.3) yields $\ell(-x) \geq 2^n - 1$. Suppose now $\ell(cx) = 3$ and write $cx = e + t^2$ with $t \in F^\times$ and $e \in D_F(2)$. We have $cx \in D_F(\langle c, ac\rangle) \cap D_F(\langle c, bc\rangle) \cap D_F(\langle 1, e\rangle)$. Since $1, c, ac$ and $bc$ represent distinct square classes, we conclude with (c) that $e$ and $c$ lie in the same square class. Therefore $x \in D_F(\langle 1, a\rangle) \cap D_F(\langle 1, b\rangle) \cap D_F(\langle 1, c\rangle)$, which by (b) implies $\ell(-x) = 2^n$. □

4.8. Theorem. *Let $F$ be a nonreal field of level $s$, equal to $\bar{q}_1(F)$. Any representation* (4.3) *of zero as a nontrivial sum of $s+1$ squares over $F$ may be reordered in such way that the following holds: for $\{i, j\}, \{i', j'\} \in \mathcal{P}_2^{s+1}$ the partial sums $x_i^2 + x_j^2$ and $x_{i'}^2 + x_{j'}^2$ lie in the same square class if and only if* $\max\{i, j, 3\} = \max\{i', j', 3\}$.

*Proof:* Let $\mathcal{G}$ be a complete graph in $s+1$ vertices $v_1, \ldots, v_{s+1}$ and with the edge-coloring given by $f : \mathcal{P}_2^{s+1} \to D_F(2)/F^{\times 2}, \{i, j\} \mapsto (x_i^2 + x_j^2)F^{\times 2}$ (see at the beginning of this section). We know from (4.4) that exactly $s - 1$ colors appear in $\mathcal{G}$. Further, $\mathcal{G}$ does not contain any triangle with three different colors; indeed, such a triangle would correspond to a partial sum of three squares $x := x_i^2 + x_j^2 + x_k^2$ with $1 \leq i < j < k \leq s+1$ where $a := x_i^2 + x_j^2$, $b := x_i^2 + x_k^2$ and $c := x_j^2 + x_k^2$ lie in three distinct square classes which is impossible by part (b) of the last proposition since $\ell(-x) = s - 2$. Therefore by (B.3), $\mathcal{G}$ is a total CC-graph.
Since $\mathcal{G}$ has precisely $(s+1) - 2$ colors we obtain from the definition of a total CC-graph in appendix B and the subsequent remarks: the vertices in $\mathcal{G}$ (and at the same time the $x_i$) may be renumbered in such way that for $\{i, j\} \in \mathcal{P}_2^{s+1}$ the color of the edge between $v_i$ and $v_j$ (i.e. the square class of $x_i^2 + x_j^2$) depends precisely on $\max\{i, j, 3\}$. □

4.9. Corollary. *Let $F$ be a nonreal field of level $s = \bar{q}_1(F) \geq 8$. Then* $\bar{q}_2(F) \geq \frac{s^2}{2}$.

*Proof:* Let $0 = x_1^2 + \cdots + x_{s+1}^2$ be a representation of zero as a nontrivial sum of $s+1$ squares over $F$. By the theorem we may, after reordering the indices, suppose that for $\{i, j\} \in \mathcal{P}_2^{s+1}$ the square class of $x_i^2 + x_j^2$ depends precisely on $\max\{i, j, 3\}$.

Defining $a_i := x_{i+1}^2 + x_{i+2}^2$ for $1 \le i \le s-1$, we get a system of representatives $a_1, \ldots, a_{s-1}$ of the $s-1$ nontrivial classes of $D_F(2)/F^{\times 2}$. Further we set $c_{jk} := x_1^2 + x_{j+2}^2 + x_{k+2}^2$ for $1 \le j < k \le s-1$.
Suppose now that $b\,c_{jk} = c_{j'k'}$ for $b \in D_F(2)$ and $1 \le j' < k' \le s-1$. Then $c_{j'k'} \in D_F(\langle 1, a_{j'}\rangle) \cap D_F(\langle 1, a_{k'}\rangle) \cap D_F(\langle b, b\,a_j\rangle) \cap D_F(\langle b, b\,a_k\rangle)$. In view of (b), (c) and (d) of the proposition this is only possible if $b \in F^{\times 2}$, $j = j'$ and $k = k'$.
This shows that the elements $c_{jk}$ for $1 \le j < k \le s-1$ represent distinct nontrivial classes of $D_F(4)/D_F(2)$. Therefore $\bar{q}_2(F) > \binom{s-1}{2}$. Since $s$ is a power of 2, at least 8, and $\bar{q}_2(F)$ is a power of 2 or infinite we obtain $\bar{q}_2(F) \ge \frac{s^2}{2}$. □

## 5 Lower bounds for the square class number

We start this section with Djoković's proof of his bound (1.6), rephrased in the terminology of appendix A.

5.1. Theorem (Djoković). *If $F$ is a nonreal field of level $s \ge 8$ then*

$$q(F) \;\ge\; 2\cdot |D_F(s/2)/F^{\times 2}| \;\ge\; 2\cdot \sum_{i=1}^{s/2} \frac{1}{s+2-i}\binom{s+1}{i}\,.$$

*Proof:* The first inequality is clear since $|F^\times/D_F(s/2)| \ge 2$.
Next we consider a representation $0 = x_1^2 + \cdots + x_{s+1}^2$ of zero as a sum of $s+1$ nonzero squares over $F$. We denote by $\mathcal{P}$ the set of nonempty subsets of $\{1, \ldots, s+1\}$ of cardinality not greater than $s/2$. We define $f : \mathcal{P} \to D_F(s/2)/F^{\times 2}, J \mapsto (\sum_{j\in J} x_j^2)F^{\times 2}$. For $1 \le k \le s/2$ we write $f_k$ for the restriction of $f$ to $\mathcal{P}_k^{s+1}$. By (2.8), for $k \ne k'$ the images of $f_k$ and $f_{k'}$ are disjoint. Also by (2.8), $f_k$ is $(k-1)$-connected for any $k \le s/2$ and therefore $|Im(f_k)| \ge \frac{1}{(s+1)-k+1}\binom{s+1}{k}$ by (A.4, c). All together we obtain

$$|D_F(s/2)/F^{\times 2}| \;\ge\; \sum_{k=1}^{s/2} |Im(f_k)| \;\ge\; \sum_{k=1}^{s/2} \frac{1}{s-k+2}\binom{s+1}{k}$$

which shows the second inequality. □

5.2. Remark. For an integer $s \ge 8$, let $\sum(s)$ denote the term on the right hand side in the inequality of the above theorem. Djoković showed by an elementary counting argument that $\sum(s) > \frac{2^s}{s}$ [2]. As was pointed out by David B. Leep, the argument may be improved to obtain the bound $\sum(s) > \frac{2^{s+1}}{s}$ for every even $s \ge 8$. Under the hypothesis of the last theorem one has thus $q(F) > \frac{2^{s+1}}{s}$; further, since $s = s(F)$ is a power of 2 and $q(F)$ is also a power of 2 or infinite, it follows that $q(F) \ge \frac{2^{s+2}}{s}$.
Our calculations have shown that, at least for $s$ a power of 2 in the range between 8 and $2^{13}$, actually one has $\frac{2^{s+1}}{s} < \sum(s) \le \frac{2^{s+2}}{s}$.

However, for level 8 and 16 we get stronger bounds on $q(F)$.

5.3. Theorem. *Let $F$ be a field. If $s(F) = 8$ then $q(F) \geq 512$. If $s(F) = 16$ then $q(F) \geq 2^{15}$.*

*Proof:* Under the hypothesis $s(F) = 8$ we have $\bar{q}_3(F) \geq 2$, $\bar{q}_2(F) \geq 16$ (3.1) and $\bar{q}_1(F) \geq 8$ (1.5). Moreover, by (4.9) one of the last two inequalities must be proper. From $|F^\times/F^{\times 2}| \geq \bar{q}_1(F) \cdot \bar{q}_2(F) \cdot \bar{q}_3(F)$ we get therefore $q(F) \geq 512$, since $F^\times/F^{\times 2}$ is an elementary abelian 2-group.
For $s(F) = 16$ we have by the previous sections $\bar{q}_4(F) \geq 2$, $\bar{q}_3(F) \geq 16$, $\bar{q}_2(F) \geq 32$ and $\bar{q}_1(F) \geq 16$ and one of the last two inequalities must be proper. As $|F^\times/F^{\times 2}| \geq \bar{q}_1(F) \cdots \bar{q}_4(F)$ this leads to $q(F) \geq 2^{15}$. □

For $s(F) = 2^n$ with $n \geq 5$ the analogous arguments are not sufficient to improve Djoković's result. For $s(F) = 32$, for example, we may get in this way $q(F) \geq 2^{25}$ while (5.1) yields $q(F) \geq 2^{29}$.

5.4. Theorem. *Let $F$ be a field of level $2^n$ with $n \geq 3$. Then $|k_{n-1}F| \geq 128$. More precisely, the subgroup $\{-1\}^{n-2}k_1F$ of $k_{n-1}F$ is of index at least 4 and order at least 32.*

*Proof:* Again, we use the notation $\varepsilon := \{-1\} \in k_1F$. The homomorphism $F^\times \to \{-1\}^{n-2}k_1F$ which maps $x \in F^\times$ to the symbol $\varepsilon^{n-2} \cdot \{x\}$, has kernel $D_F(2^{n-2})$. Since $\bar{q}_n(F) \geq 2$ and $\bar{q}_{n-1}(F) \geq 16$ by (3.1), we have $|F^\times/D_F(2^{n-2})| \geq \bar{q}_n(F) \cdot \bar{q}_{n-1}(F) \geq 32$. Therefore $\{-1\}^{n-2}k_1F$ has at least 32 elements.
To show that the index of this group in $k_{n-1}F$ is at least 4 we just need to find $\alpha, \beta, \gamma \in k_{n-1}F \setminus \{-1\}^{n-2}k_1F$ such that $\alpha + \beta + \gamma \in \{-1\}^{n-2}k_1F$.
By the hypothesis there are $a, b, c \in D_F(3 \cdot 2^{n-3}) \setminus D_F(2^{n-2})$ such that $a + b + c = 0$. In $k_2F$ we compute $\{-a, -b\} + \{-a, -c\} + \{-b, -c\} = \{-a, bc\} + \{a, -bc\} = \{-1, abc\}$. Therefore we are finished if we show that none of the symbols $\varepsilon^{n-3}\{-a, -b\}, \varepsilon^{n-3}\{-a, -c\}$ and $\varepsilon^{n-3}\{-b, -c\}$ in $k_{n-1}F$ lies actually in $\{-1\}^{n-2}k_1F$.
If this is not true we may by case symmetry suppose that $\varepsilon^{n-3}\{-a, -b\} = \varepsilon^{n-2}\{-x\}$ for some $x \in F^\times$. Then the $(n-1)$-fold Pfister forms $2^{n-3} \times \langle\langle a, b\rangle\rangle$ and $2^{n-2} \times \langle\langle x\rangle\rangle$ over $F$ are isometric, i.e. the quadratic form $\varphi := 2^{n-3} \times \langle 1, x, x, -a, -b, -ab\rangle$ over $F$ is hyperbolic. It follows that any subform of $\varphi$ of dimension greater than $\frac{1}{2}\dim(\varphi) = 3 \cdot 2^{n-3}$ is isotropic. In particular, the form $2^{n-2} \times \langle -ax\rangle \perp 2^{n-3} \times \langle 1\rangle \perp \langle b\rangle$, similar to a subform of $\varphi$, must be isotropic. It follows that $ax \in D_F(2^{n-2}) \cdot D_F(2^{n-3} \times \langle 1\rangle \perp \langle b\rangle) \subset D_F(2^{n-1})$ whence $x \in D_F(2^{n-1})$. On the other hand, $\varphi \cong 2^{n-3} \times \langle 1, x, x, c, abc, -ab\rangle$ shows that $2^{n-2} \times \langle x\rangle \perp 2^{n-3} \times \langle 1\rangle \perp \langle c\rangle$ is isotropic. This in turn implies that $-x \in D_F(2^{n-2}) \cdot D_F(2^{n-3} \times \langle 1\rangle \perp \langle c\rangle) \subset D_F(2^{n-1})$. Together this leads to $-1 \in D_F(2^{n-1})$ which contradicts $s(F) = 2^n$. □

5.5. Corollary. *Let $F$ be a nonreal field with $s(F) \geq 8$. Then $|\mathrm{Br}_2(F)| \geq 128$ and $|W(F)| \geq 2^{18}$.*

*Proof:* If $s(F) = 8$ then the theorem shows $|k_2F| \geq 128$. But this is also true if $s(F) = 2^n > 8$ since then already the subgroup $\{-1\}k_1F$, isomorphic to $F^\times/D_F(2)$, has order at least $\bar{q}_n(F) \cdot \bar{q}_{n-1}(F) \cdot \bar{q}_{n-2}(F)$ which is sufficiently large by the results of section 3. By Merkuriev's theorem, $\mathrm{Br}_2(F)$ is isomorphic to $k_2F$, so in particular we have $|\mathrm{Br}_2(F)| \geq 128$. (In fact, the arguments to estimate the size of $k_2F$ work similarly for $\mathrm{Br}_2(F)$, so it is not necessary to invoke Merkuriev's theorem here.)
Let $I$ denote the fundamental ideal of $W(F)$ and let $\bar{I}^i := I^i/I^{i+1}$ for $i \geq 0$. For $i = 0, 1, 2$ it follows from [9] that $\bar{I}^i \cong k_iF$. Thus $|\bar{I}^0| = 2$, $|\bar{I}^1| = q(F) \geq 512$ and $|\bar{I}^2| \geq 128$. Moreover, $s(F) \geq 8$ implies $|\bar{I}^3| \geq 2$. Therefore $|W(F)| \geq |\bar{I}^0| \cdot |\bar{I}^1| \cdot |\bar{I}^2| \cdot |\bar{I}^3| \geq 2^{18}$. □

## A Hypergraphs with connected colorings

In this appendix $t, k$ and $n$ denote nonnegative integers with $t \leq k \leq n$. We briefly say *$k$-set* for a set of cardinality $k$. A *$k$-hypergraph* is a system $\mathcal{H} = (V, \mathcal{E})$ where $V$ is a set whose elements are called *vertices* and $\mathcal{E}$ a collection of distinct $k$-subsets of $V$ called *edges.* A graph in the usual sense is then just a 2-hypergraph.
Let $\mathcal{H} = (V, \mathcal{E})$ be a $k$-hypergraph. Its number of vertices $|V|$ is called the *order of $\mathcal{H}$* . We say that $\mathcal{H}$ is *complete* if each $k$-subset of $V$ is actually an edge, i.e. if $\mathcal{E} = \{E \subset V \mid |E| = k\}$. By an *edge-coloring* of $\mathcal{H}$ we mean a function $f : \mathcal{E} \to C$. We consider the elements of $C$ as *colors* and for $E \in \mathcal{E}$ we call $f(E)$ the *color of $E$*. For $t > 0$ we say that the edge-coloring $f$ is *$t$-connected* if any two edges of the same color meet in at least $t$ vertices, i.e. if for any $E, E' \in \mathcal{E}$ with $f(E) = f(E')$ we have $|E \cap E'| \geq t$.

A.1. Problem. *Let $t, k, n$ be nonnegative integers with $t \leq k \leq n$. Let $\mathcal{H} = (V, \mathcal{E})$ be a complete $k$-hypergraph of order $n$. What is the least integer $m$ such that there exists a $t$-connected edge-coloring $f : \mathcal{E} \to C$ on $\mathcal{H}$ with $|C| = m$ ?*

The integer $m$ which meets the condition in the problem depends only on the values of $t$, $k$ and $n$ and will be denoted by $M(t, k, n)$. We recall our notation $\mathcal{P}_k^n$ for the set of all $k$-subsets of $\{1, \ldots, n\}$. A complete $k$-hypergraph of order $n$ is then given by $\mathcal{K}_k^n := (\{1, \ldots, n\}, \mathcal{P}_k^n)$. So $M(t, k, n)$ is just the least integer $m$ such that there exists a function $f : \mathcal{P}_k^n \to C$ where $|C| = m$ and such that $f(X) = f(X')$ implies $|X \cap X'| \geq t$ for any $X, X' \in \mathcal{P}_k^n$. To study $M(t, k, n)$ as a function in $t, k$ and $n$ we use the theory of *intersecting families* in combinatorics.
Let $\mathcal{F}$ be a family of sets. We write $\bigcup \mathcal{F}$ (resp. $\bigcap \mathcal{F}$) for the union (resp. the intersection) of all sets belonging to $\mathcal{F}$. If $|U \cap V| \geq t$ holds for every $U, V \in \mathcal{F}$ then we say that the family $\mathcal{F}$ is *$t$-intersecting* (just *intersecting* for $t = 1$). A

coloring $f : \mathcal{E} \to C$ of a $k$-hypergraph $\mathcal{H} = (V, \mathcal{E})$ is thus $t$-connected if and only if $f^{-1}(\{c\})$ is a $t$-intersecting family for every $c \in C$.
The crucial result on intersecting families is the Erdös-Ko-Rado theorem [4] which we state in the slightly stronger version of [14]:

A.2. Theorem (Erdös-Ko-Rado). *Let $n \geq (k-t+1)(t+1)$. If $\mathcal{F}$ is a $t$-intersecting family of $k$-subsets of an $n$-set then $|\mathcal{F}| \leq \binom{n-t}{k-t}$.*

This theorem gives the optimal bound. Indeed, if $N$ is an $n$-set and $T$ a $t$-subset then $\mathcal{F} := \{U \subset N \mid |U| = k, T \subset U\}$ is a $t$-intersecting family with precisely $\binom{n-t}{k-t}$ elements. However, under the additional condition $|\bigcap \mathcal{F}| < t$, better bounds on $|\mathcal{F}|$ can be given. In the case $t = 1$ this is the following main result of [6]. (A short proof of this can be found in [5] where the case $t > 1$ is also treated.)

A.3. Theorem (Hilton-Milner). *Let $\mathcal{F}$ be a family of pairwise intersecting $k$-subsets of an $n$-set such that $\bigcap \mathcal{F} = \emptyset$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$.*

Now we begin with the investigation $M(t,k,n)$ as a function in $t, k$ and $n$ with $0 < t \leq k \leq n$. We first treat the easy cases when $t$ and $k$ take extremal values. Part (c) is implicitly shown in [2].

A.4. Proposition. *(a) $M(t,k,n) = 1$ is equivalent to $n \leq 2k-t$.*

*(b) $M(t,k,n) = \binom{n}{k}$ is equivalent to $k = t$.*

*(c) $M(k-1,k,n) = M(n-k-1, n-k, n) \geq \frac{1}{n-k+1}\binom{n}{k}$ for $1 \leq k \leq n/2$.*

*Proof:* (a) $M(t,k,n)$ is equal to 1 if and only if $\mathcal{P}_k^n$ is $t$-intersecting; this is the case if and only if $n \leq 2k-t$.
(b) Each condition holds if and only if any nonempty $t$-intersecting family of $k$-subsets of $\{1,\dots,n\}$ consists of just one $k$-set.
(c) It is quite obvious that a family $\mathcal{F} \subset \mathcal{P}_k^n$ is $(k-1)$-intersecting if and only if the family of complement sets $\{\{1,\dots,n\} \setminus U \mid U \in \mathcal{F}\}$ is $(n-k-1)$-intersecting. So $f : \mathcal{P}_k^n \to C$ is $(k-1)$-connected if and only if $f' : \mathcal{P}_{n-k}^n \to C, V \mapsto f(\{1,\dots,n\} \setminus V)$ is $(n-k-1)$-connected. This shows in particular $M(k-1,k,n) = M(n-k-1, n-k, n)$.
For a $(k-1)$-intersecting family $\mathcal{F} \subset \mathcal{P}_k^n$ it is easy to check that either $|\bigcap \mathcal{F}| \geq k-1$ or $|\bigcup \mathcal{F}| \leq k+1$. In the first case we conclude $|\mathcal{F}| \leq n-k+1$ and in the second case $|\mathcal{F}| \leq k+1 \leq n-k+1$. If now $f : \mathcal{P}_k^n \to C$ is $(k-1)$-connected then $\mathcal{P}_k^n$ is covered by the $(k-1)$-intersecting families $f^{-1}(\{c\})$ for $c \in C$, which implies that $\binom{n}{k} = |\mathcal{P}_k^n| \leq (n-k+1) \cdot |C|$. □

A.5. Examples. (1) The function $f : \mathcal{P}_k^n \to \mathcal{P}_t^{n-k+t}$ which associates to $X \in \mathcal{P}_k^n$ the set of the $t$ smallest numbers in $X$ is a $t$-connected edge-coloring of $\mathcal{K}_k^n$.

(2) If $n \geq 2k-1$ then a 1-connected edge-coloring of $\mathcal{K}_k^n$ is given by

$$f : \mathcal{P}_k^n \longrightarrow \{1, \ldots, n-2k+2\}, \ X \longmapsto \max\,(X \cup \{2k-1\}) - 2k + 2\,.$$

(3) Let $t < k < n$. If $f : \mathcal{P}_k^n \to C$ be a $t$-connected edge-coloring of $\mathcal{K}_k^n$ and $g : \mathcal{P}_{k+1}^n \to C'$ is a $(t+1)$-connected edge-coloring of $\mathcal{K}_{k+1}^n$, where $C$ and $C'$ are disjoint sets, then a $(t+1)$-connected edge-coloring of $\mathcal{K}_{k+1}^{n+1}$ is defined by

$$h : \mathcal{P}_{k+1}^{n+1} \longrightarrow C \cup C', \ X \longmapsto \begin{cases} f\,(X \setminus \{n+1\}) & \text{if } n+1 \in X, \\ g\,(X) & \text{otherwise.} \end{cases}$$

From these examples we conclude:

A.6. Proposition. *(a)* $M(t,k,n) \leq \binom{n-k+t}{t}$.

*(b) If* $n \geq 2k-1$ *then* $M(1,k,n) \leq n-2k+2$.

*(c) If* $t<k<n$ *then* $M(t+1,k+1,n+1) \leq M(t,k,n) + M(t+1,k+1,n)$.

□

For lower bounds on $M(t,k,n)$ we first consider the case $t \geq 2$.

A.7. Theorem. *Let* $2 \leq t < k$. *Then for* $n \geq (k-t+1)(t+1)$ *we have*

$$M(t,k,n) \ \geq \ \prod_{i=0}^{t-1} \frac{n-i}{k-i} \ > \left(\frac{n}{k}\right)^t.$$

*Proof:* Let $f : \mathcal{P}_k^n \to C$ be a $t$-connected edge-coloring of $\mathcal{K}_k^n$ with $n \geq (k-t+1)(t+1)$. For each $c \in C$ we have then by the Erdös-Ko-Rado theorem $|f^{-1}(\{c\})| \leq \binom{n-t}{k-t}$. As $\mathcal{P}_k^n = \bigcup_{c \in C} f^{-1}(\{c\})$ we get $\binom{n}{k} \leq |C| \cdot \binom{n-t}{k-t}$. Therefore $|C| \geq \frac{n}{k} \cdot \frac{n-1}{k-1} \cdots \frac{n-t+1}{k-t+1}$ and an easy computation shows the second inequality. □

For the purposes of section 3 we state the following particular case:

A.8. Corollary. *Let* $i$ *and* $m$ *be positive integers satisfying either* $2 \leq i \leq \frac{m}{2}$ *or* $3 \leq i = \frac{m+1}{2}$ *or* $5 \leq i = \frac{m}{2}+1$. *Then* $M(2^{i-2}+1, 2^i, 2^m) > 2^{(m-i)(2^{i-2}+1)}$. □

Now we come to the case $t = 1$.

A.9. Lemma. *For* $k > 1$ *we define the polynomial*

$$F_k(X) := \prod_{i=0}^{k-1}(X-i) - k\,(X-2k+1)\left(\prod_{i=1}^{k-1}(X-i) - \prod_{i=1}^{k-1}(X-k-i) + (k-1)!\right).$$

*If* $k \leq n$ *and* $f : \mathcal{P}_k^n \to C$ *is such that* $\bigcap f^{-1}(\{c\}) = \emptyset$ *for every* $c \in C$ *then either* $|C| \geq n-2k+2$ *or* $F_k(n) \leq 0$.

*Proof:* Suppose that $f$ has the stated property. Then the Hilton-Milner theorem implies $\binom{n}{k} \leq |C| \cdot [\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1]$. On the other hand, $(k!)^{-1} \cdot F_k(n) = \binom{n}{k} - (n-2k+1) \cdot [\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1]$. Thus $F_k(n) > 0$ implies $|C| > (n-2k+1)$. $\square$

A.10. Remark. The polynomial $F_k$ defined in the lemma is monic of degree $k$. In particular, we have $F_k(n) > 0$ for all $n$ sufficiently large. Computation for small values of $k$ yields: $F_2(X) = X^2 - 7X + 18$, $F_3(X) = X^3 - 21X^2 + 140X - 240$ and $F_4(X) = X^4 - 54X^3 + 731X^2 - 3534X + 5880$. Thus we have $F_2(n) > 0$ for any $n \in \mathbb{N}$, $F_3(n) > 0$ for $n \geq 3$ and $F_4(n) > 0$ for $n \geq 37$ whereas $F_4(36) < 0$.

A.11. Theorem. *For any $k \geq 1$ there is a constant $c_k \geq 2k-2$ such that for all $n \in \mathbb{N}$ sufficiently large we have*

$$M(1,k,n) = n - c_k.$$

*For $k \leq 3$ we have, more precisely, $M(1,k,n) = n - 2k + 2$ for $n \geq 2k-1$.*

*Proof:* For $k = 1$ there is nothing to show since $M(1,1,n) = n$. For $k \geq 2$ let $F_k(X)$ be defined as in the lemma. By the above remark we may choose the least integer $n_k \geq 2k-1$ such that $F_k(n) > 0$ for all $n \geq n_k - 1$. In particular we have $n_2 = 3$ and $n_3 = 5$. Let $c_k := n_k - M(1,k,n_k)$. Then (A.6, b) implies $c_k \geq 2k-2$ and we check that equality holds for $k = 2, 3$.
We want to prove by induction that $M(1,k,n) = n - c_k$ for $n \geq n_k$. For $n = n_k$ this is trivial statement. Suppose it is true for $n - 1 \geq n_k$. Let $f : \mathcal{P}_k^n \to C$ be a 1-connected edge-coloring of $\mathcal{K}_k^n$. If $\bigcap f^{-1}(\{c\}) = \emptyset$ for each $c \in C$ then by the lemma we have $|C| \geq n - 2k + 2 \geq n - c_k$. On the other hand, if there is $c \in C$ such that the intersection $\bigcap f^{-1}(\{c\})$ is not empty then we may suppose that it contains the element $n$. Then the restriction $f' : \mathcal{P}_k^{n-1} \to C \setminus \{c\}$ of $f$ to $\mathcal{P}_k^{n-1}$ is a 1-connected edge-coloring of $\mathcal{K}_k^{n-1}$. By the induction hypothesis we have $|C \setminus \{c\}| \geq M(1,k,n-1) = (n-1) - c_k$ and thus $|C| \geq n - c_k$. This implies $M(1,k,n) \geq n - c_k$. But (A.6, c) shows $M(1,k,n) \leq M(1,k,n-1) + M(0,k-1,n-1) = n - c_k$ since $M(0,k-1,n-1) = 1$. Hence $M(1,k,n) \geq n - c_k$ which finishes the induction step. $\square$

A.12. Question. *Does $M(1,k,n) = n - 2k + 2$ hold for all $n \geq 2k-1$, even if $k > 3$ ?*

## B CC-Graphs

In this appendix we study connected edge-colorings for usual complete graphs. Here we are not only interested in the minimal number of colors but also in the distribution of the colors in the graph.

Let $\mathcal{G}$ denote a complete graph of order $n$ with vertices $v_1, \dots, v_n$ and colored edges. The distribution of colors in $\mathcal{G}$ can be equivalently represented by an edge-coloring of $\mathcal{K}_2^n$ (see appendix A), i.e. by a function $f : \mathcal{P}_2^n \to C$, where $C$ stands for the set of colors in $\mathcal{G}$ and $f$ associates to $\{i, j\} \in \mathcal{P}_2^n$ the color of the edge between the vertices $v_i$ and $v_j$.

A set of all the edges of a certain color shall be called a *color-component.* If such a color-component consists of $r \geq 3$ edges all together having a vertex $x$ in common we call it an *$r$-star* and $x$ its *center.* By a *triangle* in $\mathcal{G}$ we mean a complete subgraph of order 3 of $\mathcal{G}$. A *triangle* is said to be *monochrome* (resp. *three-colored*) if the three edges are of the same color (resp. of three different colors). A second complete colored graph $\mathcal{G}'$ of order $n$ is said to be *equivalent to* $\mathcal{G}$ if there is a bijection between the sets of vertices of $\mathcal{G}$ and $\mathcal{G}'$ such that the induced bijection on the sets of edges preserves the color-components (in both directions).

We call $\mathcal{G}$ *color-connected* or a *CC-graph* if in $\mathcal{G}$ any two edges of the same color are adjacent. This is equivalent to the edge-coloring $f$ being 1-connected. The only possible color-components in $\mathcal{G}$ are then single edges, pairs of edges with a vertex in common, stars and monochrome triangles.

Theorem (A.11) says that $M(1, 2, n) = n - 2$ for $n \geq 3$. This corresponds to a result of [13]. We rephrase it as follows and give a direct proof.

B.1. PROPOSITION (TORT). *A CC-graph of order $n \geq 3$ has at least $n - 2$ colors.*

*Proof:* For $n = 3$ the statement is trivial. If $n > 3$ and $\mathcal{G}$ has less than $n$ colors then one of its color-components must be a star. Deleting the center of this star yields a CC-graph $\mathcal{G}'$ of order $n - 1$ with less colors. By induction hypothesis $\mathcal{G}'$ has at least $n - 3$ and therefore $\mathcal{G}$ at least $n - 2$ colors. □

For any $n \geq 3$ the complete graph $\mathcal{K}_2^n$, whose vertices are the integers $1, \dots, n$, together with the 1-connected coloring $f_n : \mathcal{P}_2^n \to \{1, \dots, n-2\}$, $\{i, j\} \mapsto \max\{i, j, 3\} - 2$ defines a particular CC-graph $\mathcal{G}_n$ of order $n$ with $n-2$ colors (compare with example (A.5, 2)). The color-components of $\mathcal{G}_n$ are one monochrome triangle and one $i$-star for each $3 \leq i \leq n-1$. For $3 \leq n \leq 5$, every CC-graph with $n-2$ colors is equivalent to $\mathcal{G}_n$. This is not true for $n = 6$, since there is a CC-graph of order 6 with color-components a triangle and three 4-stars.

B.2. PROPOSITION. *Let $\mathcal{G}$ be a CC-graph with $n \geq 3$ vertices and $n-2$ colors. Then $\mathcal{G}$ has as color-components one monochrome triangle and $n - 3$ stars. Moreover, each vertex of $\mathcal{G}$ lies either on the monochrome triangle or is the center of exactly one star.*

*Proof:* Let $\mathcal{G}'$ be the complete subgraph spanned by all vertices of $\mathcal{G}$ which are not the center of a star in $\mathcal{G}$. We want to show that $\mathcal{G}'$ is a monochrome triangle. Then the vertices of $\mathcal{G}$ outside of $\mathcal{G}'$ will be the centers of $n - 3$ stars and as $\mathcal{G}$ has just $n - 2$ colors the entire statement follows.

Let $n'$ be the order of $\mathcal{G}'$. The $n - n'$ vertices of $\mathcal{G}$ outside of $\mathcal{G}'$ are all centers of stars whose colors do not appear in $\mathcal{G}'$. As a consequence, $\mathcal{G}'$ has at least $n - n'$ colors less than $\mathcal{G}$. Then by (B.1), $\mathcal{G}'$ has exactly $n' - 2$ colors. Since $\mathcal{G}'$ is a graph without stars each color appears at most three times, counting the edges yields $3(n' - 2) \geq \frac{n'(n'-1)}{2}$ whence $n' \leq 5$. As $\mathcal{G}'$ has $n' - 2$ colors and contains no star, we have $n' = 3$ and $\mathcal{G}'$ is a monochrome triangle. □

A CC-graph $\mathcal{G}$ will be called *total* if there is a permutation $\sigma \in \mathcal{S}_n$ such that for any $\{i, j\} \in \mathcal{P}_2^n$ the color of the edge between $v_i$ and $v_j$ depends only on $\max\{\sigma(i), \sigma(j)\}$. After renumbering the vertices $\mathcal{G}$ we may then suppose that the permutation $\sigma$ is the identity on $\{1, \dots, n\}$.
Let $\mathcal{G}$ be a total CC-graph of order $n$ with vertices $v_1, \dots, v_n$ enumerated in such a way that the color of any edge linking $v_i$ and $v_j$ depends only on $\max\{i, j\}$. Then $\mathcal{G}$ has at most $n-1$ different colors. From (B.1) it follows that the number of colors in $\mathcal{G}$ is either $n - 2$ or $n - 1$. Further, by (B.2) the number of colors is $n - 2$ if and only if $v_1, v_2$ and $v_3$ form a monochrome triangle and then the color of the edge between $v_i$ and $v_j$ depends precisely on $\max\{i, j, 3\}$. In both cases the enumeration of the vertices is unique up to changing the first three respectively the first two indices. Moreover, $\mathcal{G}$ contains exactly $n - 3$ stars. More precisely, for each $4 \leq i \leq n$ there is exactly one $(i-1)$-star in $\mathcal{G}$ whose center is $v_i$. It is clear from the definition that a complete subgraph of a total CC-graph is also a total CC-graph.

B.3. Proposition. *A CC-graph $\mathcal{G}$ is total if and only if it contains no three-colored triangle.*

*Proof:* The necessity of the condition follows from the definition of a total CC-graph. Suppose now that $\mathcal{G}$ is a CC-graph with $n$ vertices with no three-colored triangle. We show by induction on $n$ that $\mathcal{G}$ is total. For $n \leq 3$ this is evident. If $n \geq 4$ then any complete subgraph with 4 vertices contains a star since otherwise it would contain a three-colored triangle. So we can choose an $r$-star in $\mathcal{G}$ where $r$ is as large as possible. For the ease of imagination say, it is of red color. We may suppose that $v_n$ is the center of this star. Let $\mathcal{G}'$ be the complete subgraph of $\mathcal{G}$ with all the vertices of $\mathcal{G}$ except $v_n$. Then $\mathcal{G}'$ is also a CC-graph with $n-1$ vertices and contains no three-colored triangle. So, by the induction hypothesis, $\mathcal{G}'$ is total, i.e. its vertices can be enumerated as $v_1, \dots, v_{n-1}$ in such a way that the color of an edge connecting vertices $v_i$ and $v_j$ depends just on $\max\{i, j\}$. This would still be true for the enumeration of the vertices $v_1, \dots, v_n$ of $\mathcal{G}$, if $v_n$ is connected with each of the $v_1, \dots, v_{n-1}$ by an edge of red color. So we just have to show that $r = n - 1$. Suppose that $r < n - 1$. Then certainly $n > 4$ since $r \geq 3$ by the definition of an $r$-star. But $v_{n-1}$ is the center of an $n-2$-star in $\mathcal{G}'$, say of blue color. By the maximality of $r$ we see that the edge between $v_{n-1}$ and $v_n$ cannot be blue and that $r = n-2$. So there must be exactly one vertex $v_k$ with $1 \leq k \leq n - 1$ which is connected with $v_n$ with an edge of color different from red. It cannot be of blue color either so say that its color is green. Now we see that there is a triangle of colors red, blue and green contained in $\mathcal{G}$, formed by $v_k, v_{n-1}, v_n$ if $k < n-1$ and by $v_1, v_{n-1}, v_n$ if $k = n-1$, which gives the desired contradiction. □

## REFERENCES

[1] P. L. Chang. On the number of square classes of a field with finite Stufe. *J. Number Theory*, 6:360–368, 1974.

[2] Dragomir Ž. Djoković. Level of a field and the number of square classes. *Math. Z.*, 135:267–269, 1973/74.

[3] Richard Elman and T. Y. Lam. Pfister forms and $K$-theory of fields. *J. Algebra*, 23:181–213, 1972.

[4] P. Erdős, Chao Ko, and R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)*, 12:313–320, 1961.

[5] P. Frankl and Z. Füredi. Nontrivial intersecting families. *J. Combin. Theory Ser. A*, 41(1):150–153, 1986.

[6] A. J. W. Hilton and E. C. Milner. Some intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)*, 18:369–384, 1967.

[7] Irving Kaplansky. Quadratic forms. *J. Math. Soc. Japan*, 5:200–207, 1953.

[8] T. Y. Lam. *The algebraic theory of quadratic forms.* W. A. Benjamin, Inc., Reading, Mass., 1973. Mathematics Lecture Note Series.

[9] John Milnor. Algebraic $K$-theory and quadratic forms. *Invent. Math.*, 9:318–344, 1969/1970.

[10] Albrecht Pfister. Zur Darstellung von $-1$ als Summe von Quadraten in einem Körper. *J. London Math. Soc.*, 40:159–165, 1965.

[11] Albrecht Pfister. Quadratische Formen in beliebigen Körpern. *Invent. Math.*, 1:116–132, 1966.

[12] Winfried Scharlau. *Quadratic and Hermitian forms.* Springer-Verlag, Berlin, 1985.

[13] Jean-René Tort. Sur deux invariants d'un corps de niveau fini. *Bull. Soc. Math. France*, 106(2):161–168, 1978.

[14] Richard M. Wilson. The exact bound in the Erdős-Ko-Rado theorem. *Combinatorica*, 4(2-3):247–257, 1984.

Karim Johannes Becher
Laboratoire de Mathématiques
U.F.R. Sciences et Techniques
Université de Franche-Comté
16, Route de Gray
25030 Besançon Cedex
France
becher@math.univ-fcomte.fr