Isotropy and Factorization in Reduced Witt Rings

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Abstract. We consider reduced Witt rings of finite chain length. We show there is a bound, in terms of the chain length and maximal signature, on the dimension of anisotropic, totally indefinite forms. From this we get the ascending chain condition on principal ideals and hence factorization of forms into products of irreducible forms.

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R will denote a (real) reduced Witt ring. A form $q \in R$ is totally indefinite if $|\mathrm{sgn}_\alpha q| < \dim q$ for all orderings α of R. It is well-known that such a form need not be isotropic. However, when R has finite chain length, $\mathrm{cl}(R)$, we show there are restrictions on the possible dimensions of anisotropic, totally indefinite forms. To be specific,

$$\dim q \leq \tfrac{1}{2}\mathrm{cl}(R) \max_\alpha \{|\,\mathrm{sgn}_\alpha q|^2\},$$

unless $R = \mathbb{Z}$ and q is one-dimensional. The proof depends on Marshall's classification of reduced Witt rings of finite chain length.

This bound allows us to show that R, of finite chain length, satisfies the ascending chain condition on principal ideals. One consequence of this result is that chains of basic clopen sets $H(a_1, \dots, a_n)$, for fixed n, stabilize. Another consequence is that non-zero, non-units of R factor into a finite product of irreducible elements (in the sense of Anderson and Valdes-Leon). This had been previously known only for odd dimensional forms in rings with only finitely many orderings.

Conversely, we show, for a wide class of reduced Witt rings R, that the ascending chain condition on principal ideals implies R has finite chain length. The proof relies on Marshall's notion of a sheaf product. We close with examples of factorization into irreducible elements. These illustrate how the factorization of even dimensional forms is less well behaved than the factorization of odd dimensional forms studied in [8].

We set some of the notation. R will be an abstract Witt ring, in the sense of Marshall [11], and reduced. The main case of interest is the Witt ring of a Pythagorean field. X_R, or just X if the ring is understood, denotes the set of orderings (equivalently, signatures) on R. We always assume X is non-empty. For a form $q \in R$ and ordering $\alpha \in X$, the signature of q at α will be denoted by either $\mathrm{sgn}_\alpha q$ or $\hat{q}(\alpha)$.
We let G_R, or just G when R is understood, denote the group of one-dimensional forms of R. When R is the Witt ring of a field, $G = F^*/F^{*2}$. Forms in R are written as $\langle a_1, \dots, a_n\rangle$, with each $a_i \in G$. An n-fold Pfister form is a product $\langle 1, a_1\rangle\langle 1, a_2\rangle \cdots \langle 1, a_n\rangle$, denoted by $\langle\langle a_1, a_2, \dots, a_n\rangle\rangle$. The set of orderings X has a topology with basic clopen sets

$$H(a_1, \dots, a_n) = \{\alpha \in X : a_i >_\alpha 0 \quad \text{for all } i\},$$

where each $a_i \in G$. The *chain length* of R, denoted by $\mathrm{cl}(R)$, is the supremum of the set of integers k for which there is a chain

$$H(a_0) \subsetneq H(a_1) \subsetneq \cdots \subsetneq H(a_k)$$

of length k (each $a_i \in G$).
A subgroup $F \subset G$ is a *fan* if it satisfies : any subgroup $P \supset F$ such that $-1 \notin P$ and P has index 2 in G is an ordering. The index of the fan is $[G : F]$. The set of orderings P that contain F is denoted X/F. Note that $|X/F| = 2^{n-1}$ if F has index 2^n. The *stability index* of R, denoted by $\mathrm{st}(R)$, is the supremum of $\log_2 |X/F|$ over all fans in G.
If R_1 and R_2 are reduced Witt rings then so is the product

$$R_1 \sqcap R_2 = \{(r_1, r_2) : r_1 \in R_1, r_2 \in R_2 \quad \text{and} \quad \dim r_1 \equiv \dim r_2 \pmod 2\}.$$

E will always denote a group of exponent 2. If R is a reduced Witt ring then so is the group ring generated by E, denoted by $R[E]$. E_k will denote the group of exponent 2 and order 2^k. We will always take $t_1, \dots, t_k$ as generators of E_k (except when $k = 1$ when we use just t). For an arbitrary E we use $t_1, t_2, \dots$ as generators. When E is uncountable we are assuming the use of infinite ordinals as indices. Lastly, if $S \subset G$ we write $sp(S)$ for the subgroup generated by S.

1. Isotropy.

Over $\mathbb{R}$ a form q is hyperbolic iff $\mathrm{sgn}\, q = 0$ and isotropic iff $|\mathrm{sgn}\, q| < \dim q$. The first statement holds for any reduced Witt ring but not the second. Our goal is to find a limit on the difference between $|\mathrm{sgn}\, q|$ and $\dim q$ for anisotropic forms. We restrict ourselves to reduced Witt rings with a finite chain length. Recall [12, 4.4.2] ([5] in the field case) that such rings are built up from copies of $\mathbb{Z}$ by finite products and arbitrary group ring extensions. The decomposition is unique except that $\mathbb{Z} \sqcap \mathbb{Z} = \mathbb{Z}[E_1]$.
We introduce some notation. Recall that E_k is generated by $t_1, \dots, t_k$. We fix a listing $x_1, \dots, x_{2^k}$ of the elements of E_k as follows. The list for E_1 is

$1, t_1$. The list for E_{k+1} is the list of E_k followed by t_{k+1} times the list for E_k. We also fix a listing $\alpha_1, \dots, \alpha_{2^n}$ of the orderings on $\mathbb{Z}[E_k]$. For the $k = 1$ we take α_1 to be the ordering with t_1 positive and α_2 to be the ordering with t_1 negative. The list for $\mathbb{Z}[E_{k+1}]$ consists of the orderings on $\mathbb{Z}[E_k]$ extended by taking t_{k+1} positive, followed by the extensions with t_{k+1} negative. Lastly, we define P_k to be the $2^k \times 2^k$-matrix whose (i, j) entry is the sign of x_j at the α_i ordering. Thus $P_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Lemma 1.1. *For each* $k \geq 1$

(1) P_k *is symmetric.*
(2) $P_k^2 = 2^k I$.
(3) *For* $q = \sum n_i x_i \in \mathbb{Z}[E_k]$ *let* $s_i = \hat{q}(\alpha_i)$. *Set* $\bar{n} = (n_1, \dots n_{2^k})^T$, *where* T *denotes the transpose, and* $\bar{s} = (s_1, \dots, s_{2^k})^T$. *Then* $P_k \bar{n} = \bar{s}$.

Proof. We use induction on k to prove (1) and (2). Both are clear for $k = 1$. By our construction,

$$P_{k+1} = \begin{pmatrix} P_k & P_k \\ P_k & -P_k \end{pmatrix}.$$

Thus P_k symmetric implies P_{k+1} is also. And

$$P_{k+1}^2 = \begin{pmatrix} 2P_k^2 & 0 \\ 0 & 2P_k^2 \end{pmatrix} = 2^{k+1} I.$$

Statement (3) is simple to check. □

The reader may notice that each P_k is a Hadamard matrix, indeed the simplest examples of Hadamard matrices, namely Kronecker products of copies of $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Notation. Let $M(q) = \max\{|\hat{q}(\alpha)| : \alpha \in X\}$.

Proposition 1.2. *Let* $R = \mathbb{Z}[E]$, *where* E *is an arbitrary group of exponent two. Suppose* $q \in R$ *is anisotropic. Then* $\dim q \leq M(q)^2$.

Proof. We may assume $q \in \mathbb{Z}[E_k]$ for some k. Write $q = \sum n_i x_i$ where $n_i \in \mathbb{Z}$ and the x_i form the list of the elements of E_k described above. Let $\bar{n}$ and $\bar{s}$ be as in (1.1). Then:

$$\begin{aligned} P_k \bar{n} &= \bar{s} \\ 2^k \bar{n} = P_k^2 \bar{n} &= P_k \bar{s} \\ \sum n_i^2 = \bar{n}^T \bar{n} &= \frac{1}{2^{2k}} \bar{s}^T P_k^T P_k \bar{s} \\ &= \frac{1}{2^k} \bar{s}^T \bar{s} = \frac{1}{2^k} \sum s_i^2. \end{aligned}$$

Now for each i we have $s_i^2 \leq M(q)^2$. So $\sum n_i^2 \leq M(q)^2$. Further, $|n_i| \leq n_i^2$ so $\dim q = \sum |n_i| \leq M(q)^2$. □

Remarks. (1) The bound in (1.2) is sharp infinitely often. Let $\epsilon = (\epsilon_1, \dots, \epsilon_k)$ be a choice of signs, that is, each $\epsilon_i = \pm 1$. Pick a one-to-one correspondence between the 2^k many sign choices and the elements of $sp\{t_{k+1}, \dots, t_{2k}\}$, say $\epsilon \mapsto x_\epsilon$. Then consider

$$q = \sum_\epsilon x_\epsilon \langle\langle \epsilon_1 t_1, \dots, \epsilon_k t_k \rangle\rangle \in \mathbb{Z}[E_{2k}],$$

where the sum is over all possible sign choices. At each ordering of $\mathbb{Z}[E_k]$ exactly one of the Pfister forms has signature 2^k, the others having signature zero. In any extension of this ordering to $\mathbb{Z}[E_{2k}]$ we get $\operatorname{sgn} q = \pm 2^k$. Thus q is anisotropic, $\dim q = 2^{2k}$ and $M(q) = 2^k$. Hence $\dim q = M(q)^2$.
(2) The bound of (1.2) is not sharp for M's that are not 2-powers. For instance, suppose q is anisotropic and $M(q) = 3$. We may assume (see (2.6)) that q has signature 3 or -1 at each ordering. Let $q_0 = (q-1)_{an}$, the anisotropic part. Then $M(q_0) = 2$ and so $\dim q_0 \le 4$. Thus $\dim q \le 5 < M(q)^2$.
The bound of (1.2) can also be improved if k is fixed. For instance, one can show for anisotropic $q \in \mathbb{Z}[E_3]$ that $\dim q \le \frac{5}{2} M(q)$.

Theorem 1.3. *Suppose R is a reduced Witt ring of finite chain length. Let $q \in R$ be anisotropic. Then $\dim q \le \frac{1}{2} cl(R) M(q)^2$, unless $R = \mathbb{Z}$ and q is one-dimensional.*

Proof. The result is clear if $\dim q = 1$ so assume $\dim q \ge 2$. We may thus ignore the exceptional case. We will prove the result for $R = S[E]$, any E, by induction on the chain length of S. Say $\text{cl}(S) = 1$ so that $S = \mathbb{Z}$. If $E = 1$ then $\dim q = M(q) \le \frac{1}{2} M(q)^2$ as $\dim q \ge 2$. If $E \ne 1$ then we are done by (1.2) as $\text{cl}(\mathbb{Z}[E]) = 2$.
In the general case we may assume $S = S_1 \sqcap S_2$, with at least one of S_1 or S_2 not $\mathbb{Z}$. Then both S_1 and S_2 have smaller chain length than S and so we are assuming the result holds for $S_i[E]$, $i = 1, 2$ and any E.
First suppose $E = 1$. Write $q = (a, b)$ with $a \in S_1$ and $b \in S_2$. We may assume that $\dim a \ge \dim b$. Then $\dim q = \dim a$. We have by induction

$$\begin{aligned} \dim q = \dim a &\le \tfrac{1}{2} \text{cl}(S_1) M(a)^2 \\ &\le \tfrac{1}{2} \text{cl}(R) M(a)^2, \quad \text{since } \text{cl}(R) = \text{cl}(S_1) + \text{cl}(S_2) \\ &\le \tfrac{1}{2} \text{cl}(R) M(q)^2, \end{aligned}$$

as $\hat{q}(\alpha) = \hat{a}(\alpha)$ or $\hat{b}(\alpha)$ for every $\alpha \in X$ so that $M(a) \le M(q)$.
Next suppose $E \ne 1$. Since q has only finitely many entries we may assume that $q \in (S_1 \sqcap S_2)[E_k]$, for some k. Write $q = \sum (a_i, b_i) x_i$, where each $a_i \in S_1$ and $b_i \in S_2$ and the x_i's are our listing of the elements of E_k. Set

$$\varphi = (\sum a_i x_i, 0) + r(0, \sum b_i x_i) \in (S_1[E_k] \sqcap S_2[E_k])[E_1].$$

Now

$$\dim q = \sum_i \max\{\dim a_i, \dim b_i\}$$
$$\dim \varphi = \sum \dim a_i + \sum \dim b_i \geq \dim q.$$

We check the signatures. If $\alpha \in X_{S_1}$ and α^ϵ is an extension of α to $R = (S_1 \sqcap S_2)[E_k]$ then $\hat{q}(\alpha^\epsilon) = \sum \hat{a}_i(\alpha)\epsilon_i$ (here $\epsilon_i = \pm 1$ depending on the sign of x_i in the extension). Similarly, if $\beta \in X_{S_2}$ and β^ϵ is an extension to R then $\hat{q}(\beta^\epsilon) = \sum \hat{b}_i(\beta)\epsilon_i$.
We may also view α^ϵ as an extension of α to $S_1[E_k]$ and hence to $S_1[E_k] \sqcap S_2[E_k]$. Let $\alpha^{\epsilon+}$ denote the further extension to $(S_1[E_k] \sqcap S_2[E_k])[E_1]$ with r positive. We also have the other extensions $\alpha^{\epsilon-}, \beta^{\epsilon+}$ and $\beta^{\epsilon-}$. Then:

$$\hat{\varphi}(\alpha^{\epsilon+}) = \sum \hat{a}_i(\alpha)\epsilon_i$$
$$\hat{\varphi}(\alpha^{\epsilon-}) = \sum \hat{a}_i(\alpha)\epsilon_i$$
$$\hat{\varphi}(\beta^{\epsilon+}) = \sum \hat{b}_i(\beta)\epsilon_i$$
$$\hat{\varphi}(\beta^{\epsilon-}) = -\sum \hat{b}_i(\beta)\epsilon_i.$$

Thus $M(\varphi) = M(q)$.
Set $\varphi_1 = \sum a_i x_i \in S_1[E_k]$ and $\varphi_2 = \sum b_i x_i \in S_2[E_k]$. Then by induction we have:

$$\dim \varphi_1 \leq \tfrac{1}{2}\mathrm{cl}(S_1)M(\varphi_1)^2$$
$$\dim \varphi_2 \leq \tfrac{1}{2}\mathrm{cl}(S_2)M(\varphi_2)^2.$$

The previous computation shows that for any ordering γ of $(S_1[E_k] \sqcap S_2[E_k])[E_1]$ that $\hat{\varphi}(\gamma)$ equals $\hat{\varphi}_1(\alpha)$ or $\pm\hat{\varphi}_2(\beta)$ where γ restricts to either α on $S_1[E_k]$ or β on $S_2[E_k]$. Thus $M(\varphi_i) \leq M(\varphi)$ for $i = 1, 2$. We obtain

$$\begin{aligned}\dim \varphi = \dim \varphi_1 + \dim \varphi_2 &\leq \tfrac{1}{2}(\mathrm{cl}(S_1) + \mathrm{cl}(S_2))M(\varphi)^2 \\ &= \tfrac{1}{2}\mathrm{cl}(R)M(\varphi)^2,\end{aligned}$$

using [12, 4.2.1]. Lastly, we have already checked that $\dim q \leq \dim \varphi$ and $M(q) = M(\varphi)$, giving the desired bound. □

Remarks. (1) The bound of (1.3) is sometimes achieved. For example, in

$$R = (\mathbb{Z}[E_2] \sqcap \mathbb{Z}[E_2] \sqcap \mathbb{Z}[E_2])[E_2],$$

where the last E_2 is generated by s_1, s_2, let $\varphi = \langle 1, t_1, t_2, -t_1t_2 \rangle$ and set $q = (\varphi, 0, 0) + s_1(0, \varphi, 0) + s_2(0, 0, \varphi)$. Then q is anisotropic, $\dim q = 12$, $M(q) = 2$ and $\mathrm{cl}(R) = 6$. Thus $\dim q = \frac{1}{2}\mathrm{cl}(R)M(q)^2$.

(2) Bröcker [3] has a result that looks similar to (1.3) but is apparently unrelated. There, in the version of [12, 7.7.3], if q is anisotropic, $\hat{q}(\alpha) = \pm 2^k$ for all α and $Y = \{\alpha : \hat{q}(\alpha) = 2^k\}$ is the union of basic open sets each of stability index at most $k+1$, then $\dim q \leq 2^{2k} = M(q)^2$.
(3) Bonnard [2] also has a result that looks like (1.3), which in fact uses Bröcker's result in the proof. In our notation, her result is: if R has finite stability index s and $q \in R$ is anisotropic then $\dim q \leq 2^{s-1}M(q)$. Her bound is slightly better than this. Chain length and stability index are independent invariants so again there is no apparent connection between (1.3) and Bonnard's result.
Recall that a form q is *weakly isotropic* if mq is isotropic for some $m \in \mathbb{N}$.

Corollary 1.4. *Let R be a real Witt ring (not necessarily reduced) of finite chain length. Let $q \in R$ be a form of dimension at least 2. If $\dim q > \frac{1}{2} cl(R) M(q)^2$ then q is weakly isotropic.*

Proof. Let $q_r = q + R_t \in R_{red}$, the reduced Witt ring. Then q_r is isotropic by (1.3). Hence $q_r \simeq \langle 1, -1\rangle + \varphi_r$, for some form $\varphi_r = \varphi + R_t \in R_{red}$. Then $2^k q \simeq 2^k\langle 1, -1\rangle + 2^k\varphi$, for some k, and so q is weakly isotropic. □

2. Chains of principal ideals.

We use the standard abbreviation ACC for *ascending chain condition.*

Proposition 2.1. *If ACC holds for the principal ideals of R then R has finite chain length.*

Proof. Suppose we have a tower

$$H(a_1) \supseteq H(a_2) \supseteq \cdots \supseteq H(a_n) \supseteq \cdots .$$

Set $q_n = \langle 1, 1, a_n\rangle$. Then $\hat{q}_n(\alpha)$ is 1 or 3, with $\hat{q}_n(\alpha) = 3$ iff $\alpha \in H(a_n)$. In particular, for every n we have $\hat{q}_{n+1}(\alpha)$ divides $\hat{q}_n(\alpha)$, for every $\alpha \in X$. Then q_{n+1} divides q_n by [7, 1.7]. Thus we have a tower of principal ideals :

$$(q_1) \subseteq (q_2) \subseteq \cdots \subseteq (q_n) \subseteq \cdots .$$

The ACC implies there exists a N such that $(q_N) = (q_m)$ for all $m > N$. Then $\hat{q}_N(\alpha)$ divides $\hat{q}_m(\alpha)$ for all $\alpha \in X$ and so $H(a_N) = H(a_m)$, for all $m > N$. □

We need some technical terms for the next result.

Definitions. A *fan tower* is a strictly decreasing tower of fans $F_1 > F_2 > \cdots > F_n > \cdots$, each of finite index plus a fixed choice of complements C_n where $G = C_n \times F_n$. We set $F_\infty = \cap F_n$. A *separating set of fan towers* is a finite set of fan towers $s_1, \ldots, s_\ell$, with $s_i = \{F_{in}\}$ such that

(1) Given any $q \in R$ there exists m, possibly depending on q, such that all entries of q are in $C_{im}F_{i\infty}$, for each i between 1 and ℓ.

(2) Given $K \subset \mathbb{Z}$ and forms $q_1, q_2 \in R$, there exists N, depending on q_1 and q_2 but not K, such that if for some $n > N$

$$\hat{q}_1^{-1}(K) \cap (X/F_{in}) = \hat{q}_2^{-1}(K) \cap (X/F_{in})$$

for all i then $\hat{q}_1^{-1}(K) = \hat{q}_2^{-1}(K)$.

Example. For a simple example, let $R = \mathbb{Z}[E]$ with E countably infinite. Let $F_i = sp\{t_{i+1}, t_{i+2}, \dots\}$ and $C_i = sp\{-1, t_1, \dots, t_i\}$. Then each F_i is a fan of finite index, each C_i is a complement and the F_i are strictly decreasing. Hence $\{F_i\}$ is a fan tower. Note that here $F_\infty = 1$. This fan tower is a separating (singleton) set of fan towers. A given form q has entries involving only a finite number of t_i's and so its entries lie in some C_m; this is the first condition. If we are given two forms q_1 and q_2 then again all of their entries lie in some C_N. So the signatures of the q_i depend only on the signs of $t_1, \dots t_N$ in that ordering. Hence if $\hat{q}_1$ and $\hat{q}_2$ agree on X/F_N then they agree at every ordering. This is the second condition.

Roughly, our fan towers will look like this example. When there is a product we will need one tower in each coordinate, hence a separating set.

Lemma 2.2. *If R has finite chain length then R has a separating set of fan towers.*

Proof. We prove this by induction on the chain length. When $\mathrm{cl}(R) = 1$ then $R = \mathbb{Z}$ and the result is clear. We first consider the case $R = S_1 \sqcap S_2$. Write G_1 and X_1 for G_{S_1} and X_{S_1} and similarly for G_2 and X_2. Let $\{s_1^1, \dots, s_{\ell_1}^1\}$ be a separating set of fan towers for S_1. Here $s_k^1 = \{F_{ki}^1\}$ with complements C_{ki}^1. Set $F_{ki} = F_{ki}^1 \times G_2$, which is a fan in $G = G_1 \times G_2$ with complement $C_{ki} = C_{ki}^1 \times 1$. Then for $1 \le k \le \ell_1$, $r_k = \{F_{ki}\}$ is a fan tower. Note that $F_{k\infty} = F_{k\infty}^1 \times G_2$.

Similarly, let $\{s_1^2, \dots, s_{\ell_2}^2\}$ be a separating set of fan towers for S_2, with $s_k^2 = \{F_{ki}^2\}$ and complements C_{ki}^2. Set $F_{\ell_1+k\,i} = G_1 \times F_{ki}^2$ and $C_{\ell_1+k\,i} = 1 \times C_{ki}^2$. Then for $1 \le k \le \ell_2$, $r_{\ell_1+k} = \{F_{\ell_1+k\,i}\}$ is a fan tower. We check that $r_1, \dots, r_{\ell_1}, r_{\ell_1+1}, \dots, r_{\ell_1+\ell_2}$ is a separating set of fan towers for R.

We check the first condition. We are given a form $q = \langle (a_1, b_1), \dots, (a_n, b_n)\rangle \in R$. By induction, there exists a m_1 such that $a_1, \dots, a_n \in C_{km_1}^1 F_{k\infty}^1$ for all k. So

$$(a_1, b_1), \dots, (a_n, b_n) \in C_{km_1} F_{k\infty} = C_{km_1}^1 F_{k\infty}^1 G_2,$$

for all k with $1 \le k \le \ell_1$. Similarly, there exists a m_2 such that $b_1, \dots, b_n \in C_{km_2}^2 F_{k\infty}^2$, for all $1 \le k \le \ell_2$. Hence $(a_1, b_1), \dots, (a_n, b_n) \in C_{km_2} F_{k\infty}$ for all k with $\ell_1 < k \le \ell_1 + \ell_2$. So take m to be the maximum of m_1 and m_2.

We next check the second condition. We are given $K \subset \mathbb{Z}$ and forms $q_1 = (u_1, v_1)$ and $q_2 = (u_2, v_2)$. Note that $\hat{q}_1^{-1}(K) = \hat{u}_1^{-1}(K) \cup \hat{v}_1^{-1}(K) \subset X_1 \cup X_2$, a disjoint union. By induction there exists a N_1 satisfying the second condition for K, u_1 and u_2 and a N_2 satisfying the second condition for K, v_1 and v_2. Let N be the maximum of N_1 and N_2. Suppose for some $n > N$ we have

$$\hat{q}_1^{-1}(K) \cap (X/F_{kn}) = \hat{q}_2^{-1}(K) \cap (X/F_{kn}),$$

for all $1 \leq k \leq \ell_1 + \ell_2$. For $1 \leq k \leq \ell_1$ we have:

$$\begin{aligned} \hat{q}_1^{-1}(K) \cap (X/F_{kn}) &= \hat{q}_1^{-1}(K) \cap (X_1/F^1_{kn}) \\ &= \hat{u}_1^{-1}(K) \cap (X_1/F^1_{kn}). \end{aligned}$$

We thus obtain

$$\hat{u}_1^{-1}(K) \cap (X_1/F^1_{kn}) = \hat{u}_2^{-1}(K) \cap (X_1/F^1_{kn}),$$

for all $1 \leq k \leq \ell_1$. By the second condition on S_1 we have $\hat{u}_1^{-1}(K) = \hat{u}_2^{-1}(K)$. Similarly, $\hat{v}_1^{-1}(K) = \hat{v}_2^{-1}(K)$ and so $\hat{q}_1^{-1}(K) = \hat{q}_2^{-1}(K)$.

Now suppose $R = S[E]$. Set $T_i = sp\{t_{i+1}, t_{i+2}, \dots\}$. Let $\{s_1, \dots, s_\ell\}$ be a separating set of fan towers for S where $s_k = \{F'_{ki}\}$ and the complements are C'_{ki}. Then $F_{ki} = F'_{ki} T_i$ is a fan of finite index in R with complement $C_{ki} = C'_{ki} sp\{t_1, \dots, t_i\}$. Then $r_k = \{F_{ki}\}$ is a fan tower. Note that $F_{k\infty} = F'_{k\infty}$. We show that $\{r_1, \dots, r_\ell\}$ is a separating set of fan towers for R.

For the first condition we are given a form $q \in R = S[E]$. There exists a p such that $q \in S[E_p]$. Write $q = \sum a_i x_i$ where each $a_i \in S$ and the x_i's are some list of the elements of E_p. By induction, for each i there exists a $m(i)$ such that every entry of a_i is in $C'_{km(i)} F'_{k\infty}$ for all k, $1 \leq k \leq \ell$. Let m be the maximum of the $m(i)$ and p. Then every entry of every a_i lies in $C'_{km} F'_{k\infty} \subset C_{km} F_{k\infty}$ and each x_i lies in $sp\{t_1, \dots, t_p\} \subset C_{km}$. So every entry of q lies in $C_{km} F_{k\infty}$, for all k.

For the second condition we are given $K \subset \mathbb{Z}$ and two forms $q_1, q_2 \in R$. Again there exists a p such that $q_1, q_2 \in S[E_p]$. Write $q_1 = \sum a_i x_i$ and $q_2 = \sum b_i x_i$ with $a_i, b_i \in S$ and the x_i as before. Let $\epsilon \in \{\pm 1\}^p$ be a choice of sign for $t_1, \dots, t_p$. Let $\epsilon(x_i)$ be the resulting sign of x_i. Set:

$$q_1^\epsilon = \sum a_i \epsilon(x_i) \qquad q_2^\epsilon = \sum b_i \epsilon(x_i),$$

both forms in S. For each ϵ there exists a N_ϵ so that condition 2 holds for q_1^ϵ and q_2^ϵ. Let N be the maximum of the N_ϵ and p.

If $\alpha \in X_S$ we let α^ϵ be the extension of α to $S[E_p]$ with $t_i > 0$ iff $\epsilon(t_i) = 1$. Then we claim that:

$$\hat{q}_1^{-1}(K) \cap X_{S[E_p]} = \bigcup_\epsilon [(\hat{q}_1^\epsilon)^{-1}(K)]^\epsilon.$$

Namely if $\alpha^\epsilon \in X_{S[E_p]}$ and $\hat{q}_1(\alpha^\epsilon) \in K$ then

$$\hat{q}_1(\alpha^\epsilon) = \sum \hat{a}_i(\alpha)\epsilon(x_i) = \hat{q}_1^\epsilon(\alpha).$$

Hence $\alpha^\epsilon \in (\hat{q}_1^\epsilon)^{-1}(K)^\epsilon$. The reverse inclusion is similar.

Now let $\alpha^{\epsilon e}$ denote any extension of α^{ϵ} to $R = S[E]$. Then by the claim we have:

$$\hat{q}_1^{-1}(K) = \bigcup_e \left(\bigcup_\epsilon [(\hat{q}_1^\epsilon)^{-1}(K)]^\epsilon \right)^e \tag{2.3}$$

So $\hat{q}_1^{-1}(K) \cap (X/F_{kn}) = \hat{q}_2^{-1}(K) \cap (X/F_{kn})$ implies that

$$(\hat{q}_1^\epsilon)^{-1}(K) \cap (X_s/F'_{kn}) = (\hat{q}_2^\epsilon)^{-1}(K) \cap (X_s/F'_{kn}),$$

for all sign choices ϵ. Hence by condition 2 applied to S we obtain $(\hat{q}_1^\epsilon)^{-1}(K) = (\hat{q}_2^\epsilon)^{-1}(K)$ for all ϵ. Then (2.3) gives $\hat{q}_1^{-1}(K) = \hat{q}_2^{-1}(K)$. □

Lemma 2.4. *Suppose R has a separating set of fan towers $\{s_1, \dots, s_\ell\}$. Let $q \in R$ and $K \subset \mathbb{Z}$. Let m be the index such that every entry of q lies in $C_{km}F_{k\infty}$, for all $1 \le k \le \ell$. Let $n > m$. Then for each k we have:*

$$|\hat{q}^{-1}(K) \cap (X/F_{kn})| = \frac{|X/F_{kn}|}{|X/F_{km}|} |\hat{q}^{-1}(K) \cap (X/F_{km})|.$$

Proof. Pick a k with $1 \le k \le \ell$. $F_{kn} \subset F_{km}$ are both fans of finite index so we can write $F_{km} = H \times F_{kn}$ with H spanned by $h_1, \dots, h_p$, where $2^p = |X/F_{kn}|/|X/F_{km}|$. Every $\alpha \in X/F_{km}$ has 2^p extensions to X/F_{kn}, one for each choice of signs (± 1) for the h_i. Specifically, if ϵ is a sign choice for the h_i and $h \in H$, let $\epsilon(h)$ be the resulting sign of h. Since $G = C_{km}HF_{kn}$, the extension of $\alpha \in X/F_{km}$ to X/F_{kn} via ϵ is: $\alpha^\epsilon(chf) = \alpha(c)\epsilon(h)$, where $c \in C_{km}$, $h \in H$ and $f \in F_{kn}$. We thus have

$$X/F_{kn} = \bigcup_\epsilon (X/F_{km})^\epsilon.$$

Write $q = \langle a_1, a_2, \dots \rangle$. By assumption, each a_i is in $C_{km}F_{k\infty} \subset C_{km}F_{kn}$. Hence $\alpha^\epsilon(a_i) = \alpha(a_i)$. Thus :

$$\hat{q}^{-1}(K) \cap (X/F_{kn}) = \bigcup_\epsilon \left(\hat{q}^{-1}(K) \cap (X/F_{km}) \right)^\epsilon.$$

So $|\hat{q}^{-1}(K) \cap (X/F_{kn}) = 2^p |\hat{q}^{-1}(K) \cap (X/F_{km})|$, and the result follows. □

Lemma 2.5. *Let $q \in R$ be a form of dimension n. Let F be a fan of finite index and let $K \subset \mathbb{Z}$. Then :*

$$|\hat{q}^{-1}(K) \cap (X/F)| = \frac{k}{2^n} |X/F|,$$

for some integer k, $0 \leq k \leq 2^n$.

Proof. Write $q = \langle a_1, \ldots, a_n \rangle$. Then $\hat{q}^{-1}(K)$ is a disjoint union of $H(\epsilon_1 a_1, \ldots, \epsilon_n a_n)$ for various choices of $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{\pm 1\}^n$. Set $\rho_\epsilon = \langle\langle \epsilon_1 a_1, \ldots, \epsilon_n a_n \rangle\rangle$. Then by the easy half of the representation theorem

$$\sum_{\alpha \in X/F} \hat{\rho}_\epsilon(\alpha) \equiv 0 \pmod{|X/F|}$$
$$2^n |H(\epsilon_1 a_1, \ldots, \epsilon_n a_n) \cap (X/F)| = k_\epsilon |X/F|,$$

for some non-negative integer k_ϵ. Then :

$$|\hat{q}^{-1}(K) \cap (X/F)| = \sum_\epsilon \frac{k_\epsilon}{2^n} |X/F| = \frac{k}{2^n} |X/F|,$$

for some non-negative integer k. □

The following is essentially from [9]. For a form $q = \langle a_1, \ldots, a_n \rangle$ the *discriminant* is $\operatorname{dis} q = (-1)^{n(n-1)/2} a_1 \cdots a_n$. This is sometimes called the signed discriminant.

Lemma 2.6. *Let q be an odd dimensional form.*

1. *$dis\, q >_\alpha 0$ iff $\hat{q}(\alpha) \equiv 1 \pmod 4$.*
2. *$sgn_\alpha dis(q) q \equiv 1 \pmod 4$ for all $\alpha \in X$.*
3. *If $0 \neq a = bc$ and $\hat{a}(\alpha) = \pm \hat{b}(\alpha)$ for all $\alpha \in X$ with $\hat{a}(\alpha) \neq 0$ then there exists $d \in G$ such that $\langle d \rangle a = b$.*

Proof. (1) Suppose $n = \dim q$. Let $s = \hat{q}(\alpha)$. If r is the number of α-negative entries in q then

$$\operatorname{sgn}_\alpha \operatorname{dis} q = (-1)^{n(n-1)/2}(-1)^r = (-1)^{\frac{n(n-1)}{2} + \frac{n-s}{2}} = (-1)^{(n^2-s)/2}.$$

This is positive iff $n^2 - s \equiv 0 \pmod 4$. As n is odd we get that the discriminant is positive iff $\hat{q}(\alpha) = s \equiv n^2 \equiv 1 \pmod 4$.
(2) is easy to check. For (3), let $A = \{\alpha \in X : \hat{a}(\alpha) \neq 0\}$. Then $\hat{c}(\alpha) = \pm 1$ for all $\alpha \in A$. In particular c is odd dimensional and $\hat{a}(\alpha) = 0$ iff $\hat{b}(\alpha) = 0$.Let $d = \operatorname{dis} c$. Then $\langle d \rangle c$ has signature 1 for all $\alpha \in A$ by (2). Hence $\langle d \rangle bc$ and b have the same signature at each $\alpha \in B$, and also at each $\alpha \notin A$ (as both have signature 0 there). Thus $\langle d \rangle a = \langle d \rangle bc = b$. □

Theorem 2.7. *Let R be a reduced Witt ring. Then ACC holds for principal ideals iff the chain length of R is finite.*

Proof. (2.1) gives ($\longrightarrow$). For the converse, let $(q) \subset (q_1) \subset (q_2) \subset \cdots$ be an ascending chain of principal ideals in R. Note that as each q_i divides q we have $M(q_i) \leq M(q)$. Let $M = M(q)$. Then (1.3) gives $\dim q_i \leq \frac{1}{2}\operatorname{cl}(R) M^2$ for all i (note q is not one-dimensional else all $(q_i) = R$).

We begin with some simple reductions. If all q_i are 0 then the result is clear. If some q_i is not zero then all the later q_i's are not zero. We may start our tower there, that is, we may assume $q \neq 0$. For a non-zero form φ define $\deg \varphi$ to be the largest d such that 2^d divides $\hat{\varphi}(\alpha)$ for all $\alpha \in X$. Since q_{i+1} divides q_i we have $\deg q_{i+1} \leq \deg q_i$. Let d_0 be the minimum of the degrees of the q_i. We may start our tower at a q_j of minimal degree, that is, we may assume that $\deg q = \deg q_i$ for all i. Now we may write $q = q_i \varphi_i$ for some form φ_i. We check that φ_i is odd dimensional. If instead φ_i is even dimensional then 2 divides $\hat{\varphi}_i(\alpha)$ for all α and so 2^{d+1} divides $\hat{q}(\alpha)$ for all α, contradicting our reduction to a tower of uniform degree. Hence φ_i is odd dimensional. In particular, $\hat{q}(\alpha) = 0$ iff $\hat{q}_i(\alpha) = 0$.

Let D be the set of integers $d > 1$ that divide some non-zero $\hat{q}(\alpha)$, $\alpha \in X$. Write $D = \{d_1, \dots, d_z\}$ with $d_1 < d_2 < \cdots < d_z$. Set $A(i, d_j) = \hat{q}_i^{-1}(\pm d_j)$. Let d_k be the largest element of D (if any) for which $\{A(i, d_k) : i \geq 1\}$ is not finite. Our goal is to show that there is in fact no such d_k. Our assumption on d_k means that for each $j > k$ we have a t_j such that $A(t, d_j) = A(t_j, d_j)$ for all $t \geq t_j$. Let T be the maximum of the t_j, $j > k$. Then by starting our tower of ideals with q_T, we may assume $A(i, d_j) = A(1, d_j)$ for all $j > k$ and all $i \geq 1$.

We first check that $A(i+1, d_k) \subset A(i, d_k)$ for any i. Namely, $q_i = q_{i+1}\varphi$ for some form φ. So if $\alpha \in A(i+1, d_k)$ then $\pm d_k$ divides $\hat{q}(\alpha)$. Also $|\hat{q}_i(\alpha)|$ is not of the d_j with $j > k$ else $\alpha \in A(i, d_j) = A(i+1, d_j)$, which is impossible as $\alpha \in A(i+1, d_k)$. Thus $|\hat{q}_i(\alpha)| \leq d_k$ and is divisible by d_k. Hence $\hat{q}_i(\alpha) = \pm d_k$ and $\alpha \in A(i, d_k)$ as desired.

Let $s = \{F_m\}$ be one fan tower in a separating set of fan towers for R (which exists by (2.2)). The first condition for a separating set, plus a simple induction argument, shows that for each i there exists a least $m(i)$ with every entry of $q_1, \dots, q_i$ in $C_{m(i)}F_\infty$. Note that $m(i+1) \geq m(i)$. Let $p(i)$ be the number of distinct values of

$$\frac{|A(j, d_k) \cap (X/F_{m(i)})|}{|X/F_{m(i)}|} \equiv \gamma(i, j),$$

over j with $1 \leq j \leq i$. Now, by (2.4)

$$\begin{aligned}\gamma(i+1, j) &= \frac{|A(j, d_k) \cap (X/F_{m(i+1)})|}{|X/F_{m(i+1)}|} \\ &= \frac{|X/F_{m(i+1)}|}{|X/F_{m(i)}|}\frac{|A(j, d_k) \cap (X/F_{m(i)})|}{|X/F_{m(i+1)}|} \\ &= \gamma(i, j).\end{aligned}$$

Hence $p(i+1) \geq p(i)$, with only $\gamma(i+1, i+1)$ possibly being a new value.

Since every $\dim q_i \leq \frac{1}{2}\mathrm{cl}(R)M^2$, (2.5) implies each $p(i) \leq 2^{\mathrm{cl}(R)M^2/2} + 1$. Hence there is a t_0 such that $p(t) = p(t_0)$ for all $t \geq t_0$. Let $p = p(t_0)$ and $m = m(t_0)$. Say $\gamma(t_0, j_1), \dots, \gamma(t_0, j_p)$ are the distinct γ-values over $1 \leq j \leq t_0$. Let $t > t_0$

and set $n = m(t)$. Then $\gamma(t,t) = \gamma(t_0, j_s)$ for some j_s. That is,

$$\frac{|A(t,d_k) \cap (X/F_n)|}{|X/F_n|} = \frac{|A(j_s,d_k) \cap (X/F_m)|}{|X/F_m|} = \frac{|A(j_s,d_k) \cap (X/F_n)|}{|X/F_n|},$$

using (2.4) again. Further, $A(t,d_k) \subset A(t_0,d_k) \subset A(j_s,d_k)$ so that we have

$$|A(t,d_k) \cap (X/F_n)| = |A(t_0,d_k) \cap (X/F_n)|,$$

and this holds for all $t \geq t_0$.

We can repeat this argument for each fan tower in the separating set. Let $\{s_1, \dots, s_\ell\}$ be the separating set and let $s_i = \{F_{in}\}$. Hence there exist an N and a T such that $|A(t,d_k) \cap (X/F_{in})| = |A(T,d_k) \cap (X/F_{in})|$ for all $1 \leq i \leq \ell$ and all $t \geq T$. By the second property of a separating set we have $A(t,d_k) = A(T,d_k)$ for all $t \geq T$. This contradicts our choice of d_k.

Hence we have a T such that $A(t,d_j) = A(T,d_j)$ for all $t \geq T$ and all $d_j \in D$. Thus $\hat{q}_t(\alpha) = \pm\hat{q}_T(\alpha)$ for all α in the union of the $A(T,d_j)$, that is, for all α with $\hat{q}(\alpha) \neq 0$. By our early reduction, $\hat{q}(\alpha) \neq 0$ iff $\hat{q}_T(\alpha) \neq 0$. Thus $\hat{q}_t(\alpha) = \pm\hat{q}_T(\alpha)$ for all α with $\hat{q}_T(\alpha) \neq 0$ and also q_t divides q_T. By (2.6) we obtain $(q_t) = (q_T)$, for all $t \geq T$. □

Corollary 2.8. *Let R be a real (but necessarily reduced) Witt ring. If R has finite chain length then ACC holds for principal ideals generated by odd dimensional forms.*

Proof. Every ideal containing an odd dimensional form contains the torsion ideal R_t by [7, 1.5]. Hence passing to the reduced Witt ring maintains a tower of principal ideals generated by odd dimensional forms. This reduced tower stabilizes by (2.7). Hence the original tower stabilizes. □

Corollary 2.9. *Let (G,X) be a space of orderings. Let $\mathcal{S}$ denote the collection of subsets of G of order n. If X has finite chain length then any tower*

$$H(S_1) \subset H(S_2) \subset \cdots H(S_k) \subset \cdots .$$

with each $S \in \mathcal{S}$, stabilizes.

Proof. Suppose $S_i = \{a_{i1}, \dots, a_{in}\}$. Set $q_i = \langle\langle a_{i1}, \dots, a_{in}\rangle\rangle + 1$. Then $\hat{q}_i(X) = \{1, 2^n+1\}$ and $\hat{q}_i^{-1}(2^n+1) = H(S_i)$. Thus $\hat{q}_{i+1}(\alpha)$ divides $\hat{q}_i(\alpha)$ for all $\alpha \in X$. So q_{i+1} divides q_i by [7, 1.7]. We thus have a tower of principal ideals $(q_1) \subset (q_2) \subset \cdots$. This stabilizes by (2.7) and so the tower of $H(S_i)$'s also stabilizes. □

3. Factorization.

Anderson and Valdes-Leon [1] have several notions of an associate in a commutative ring R. We need three of these. Two elements a and b are *associates* if their principal ideals are equal, $(a) = (b)$. They are *strong associates* if $a = bu$, for some unit $u \in R$. Lastly, a and b are *very strong associates* if $(a) = (b)$ and either $a = b = 0$ or $a \neq 0$ and $a = br$ implies r is a unit.

An non-unit a is *irreducible* if $a = bc$ implies either b or c is an associate of a. Similarly, a is *strongly irreducible* (*very strongly irreducible*) if $a = bc$ implies either b or c is a strong associate (respectively, very strong associate) of a. Lastly, R is *atomic* if every non-zero non-unit of R can be written as a finite product of irreducible elements. Define *strongly atomic* and *very strongly atomic* similarly.

Proposition 3.1. *Let R be a reduced Witt ring and let $a, b \in R$. Then a, b are associates iff a, b are strong associates. In particular, R is atomic iff R is strongly atomic.*

Proof. Strong associates are always associates so we check the converse. Suppose $(a) = (b)$. Write $a = bx$ and $b = ay$. Then $a = axy$ and $a(1 - xy) = 0$. Let $Z = \{\alpha \in X : \hat{a}(\alpha) = 0\}$. Then for all $\alpha \notin Z$ we have $\hat{x}(\alpha) = \pm 1$. From $a = bx$ and (2.6) we get $\langle d\rangle a = b$ for some $d \in G$. Clearly $\langle d\rangle$ is a unit. $\square$

Strong associates need not be very strong associates in a reduced Witt ring. If $\pm 1 \neq g \in G$ then $\langle 1, g\rangle$ is not even a very strong associate of itself. Namely, $\langle 1, g\rangle = \langle 1, g\rangle\langle 1, 1, -g\rangle$ and $\langle 1, g\rangle \neq 0$ and $\langle 1, 1, -g\rangle$ is not a unit. So, except for $R = \mathbb{Z}$, R will not be very strongly atomic.

Corollary 3.2. *Let R be a real Witt ring (not necessarily reduced) and suppose R has finite chain length.*

1. *Every odd dimensional form can be written as a finite product of irreducible forms.*
2. *If R is reduced then R is atomic.*

Proof. These are standard consequences of (2.8) and (2.7), see [1, 3.2]. $\square$

We are unable to prove the converse to (3.2)(2) for all reduced Witt rings R. However, we can prove the converse for a wide class of rings. For this we need Marshall's notion of a sheaf product [11]. Start with a non-empty Boolean space I, a collection of reduced Witt rings R_C, one for each clopen $C \subset I$ and a collection of ring homomorphisms $\mathrm{res}_{C.D} : R_C \to R_D$, defined whenever $D \subset C$ are clopen in I. We assume the usual sheaf properties, namely,

1. $R_\emptyset = \mathbb{Z}/2\mathbb{Z}$ and $R_C \neq \mathbb{Z}/2\mathbb{Z}$ if $C \neq \emptyset$.
2. $\mathrm{res}_{C,C}$ is the identity map on C.
3. If $E \subset D \subset C$ then $\mathrm{res}_{C,E} = \mathrm{res}_{D,E}\mathrm{res}_{C,D}$.
4. If $C = \cup_j C_j$ and if $r_j \in R_j$ are given such that

$$\mathrm{res}_{C_j,C_j\cap C_k}(r_j) = \mathrm{res}_{C_k,C_j\cap C_k}(r_k),$$

for all j, k, then there exists a unique $r \in R_C$ such that $\mathrm{res}_{C,C_j}(r) = r_j$, for all j.

For fixed $i \in I$ we form the *stalk*

$$R_i = \varinjlim_{i \in C} R_C.$$

Each R_i is a reduced Witt ring. We call the reduced Witt ring R_I the *sheaf product* of the R_i's and write $R_I = \prod_{i \in I} R_i$. When I is finite and discrete this is the usual product of Witt rings.

We next define a sequence of classes of reduced Witt rings (which is slightly different from the sequence of Marshall [11, p. 219]). Let $\mathcal{C}_1$ denote the class of finitely generated reduced Witt rings. Inductively define $\mathcal{C}_n$ to be sheaf products of $R_i[E^i]$, where E^i is a group of exponent two (not necessarily finite) and $R_i \in \mathcal{C}_m$ for some $m < n$. Lastly, let $\mathcal{C}_\omega$ be the union of all $\mathcal{C}_n$. This is a large class. Already $\mathcal{C}_2$ contains all SAP reduced Witt rings and $\mathcal{C}_\omega$ contains all reduced Witt rings where X has only a finite number of accumulation points [11, 8.17].

We will prove that $R \in \mathcal{C}_\omega$ atomic implies R has finite chain length. We begin with a lemma.

Lemma 3.3. *Let $S = R[E]$ and let $T \subset G_S$ be a fan of finite index. Set $T_0 = T \cap G_R$.*

1. *T_0 is a fan in G_R.*
2. *Suppose $X_R/T_0 = \{P, Q\}$. Then X_S/T consists of extensions of P, Q to S. If $x \in G_S \setminus G_R$ then either none, exactly half or all of the extensions of P that lie in X_S/T make x positive.*

Proof. (1) Write $T = T_0 H$ for some subgroup H of G_S with $H \cap G_R = 1$. Extend H to subgroup L of G_S such that $G_S = G_R \times L$. Suppose $P \subset G_R$ is a subgroup of index 2, containing T_0 but not -1. Then PL is a subgroup of index at most 2 containing T. If $-1 \in PL$ then for some $p \in P$ and $y \in L$ we have $-p = y \in P \cap L = 1$. But then $-1 = p \in P$, a contradiction. Thus PL is an ordering in G_S. It is easy to check that P is then an ordering in G_R. This shows T_0 is a fan.

(2) The first statement is clear. Suppose $P_1, \ldots P_m, Q_1, \ldots, Q_m$ are the extensions of P, Q that lie in X_S/T. Pick $a \in G_R$ with $\hat{a}(P) = 1$ and $\hat{a}(Q) = -1$. Let k be the number of P_i for which x is positive. From the easy half of the Representation Theorem [11, 7.13]

$$\sum_{\alpha \in X_S/T} \mathrm{sgn}_\alpha \langle\langle a, x \rangle\rangle \equiv 0 \pmod{2m}$$

$$4k \equiv 0 \pmod{2m}.$$

So m divides $2k$ and clearly $k \leq m$. Hence $k = 0, \frac{1}{2}m$ or m. □

Our proof that $R \in \mathcal{C}_\omega$ atomic implies finite chain length is not the usual induction argument since we are unable to show $R[E]$ atomic implies R atomic. Instead we explicitly construct a form which does not factor into a finite product of irreducibles. Unfortunately, the construction requires considerable notation. We introduce this notation by first looking at a special case. Let $*$ denote a group ring extension. A ring in $\mathcal{C}_n$ looks like

$$\begin{aligned} R &= \prod_{\alpha \in A_1} W(\alpha)^* \\ &= \prod_{\alpha \in A_1} \Big(\prod_{\beta \in A_2(\alpha)} W(\alpha,\beta)^* \Big)^* \\ &= \prod_{\alpha \in A_1} \Big(\prod_{\beta \in A_2(\alpha)} \Big(\prod_{\gamma \in A_3(\alpha,\beta)} W(\alpha,\beta,\gamma)^* \Big)^* \Big)^*, \end{aligned}$$

where each $A_1, A_2(\alpha)$ and $A_3(\alpha,\beta)$ is a Boolean space and each $W(\alpha,\beta,\gamma)$ is in $\mathcal{C}_m$, for some $m \leq n-3$.
Suppose we want to single out the product over $A_3(\alpha_0,\beta_0)$, for some particular α_0 and β_0. We set :

$$\begin{aligned} R_1 &= \prod_{\gamma \in A_3(\alpha_0,\beta_0)} W(\alpha_0,\beta_0,\gamma)^* \\ R_2 &= \prod_{\substack{\beta \in A_2(\alpha_0) \\ \beta \neq \beta_0}} \Big(\prod_{\gamma \in A_3(\alpha_0,\beta)} W(\alpha_0,\beta,\gamma)^* \Big)^* \\ R_3 &= \prod_{\substack{\alpha \in A_1 \\ \alpha \neq \alpha_0}} \Big(\prod_{\beta \in A_2(\alpha)} \Big(\prod_{\gamma \in A_3(\alpha,\beta)} W(\alpha,\beta,\gamma)^* \Big)^* \Big)^*. \end{aligned}$$

Then $R = ((R_1^* \sqcap R_2^*)^* \sqcap R_3^*)^*$.
We will want to single out the first infinite sheaf product. We have:

$$R = ((\ldots((R_1^* \sqcap R_2^*)^* \sqcap R_3^*)^* \sqcap \ldots)^* \sqcap R_s^*)^*,$$

with R_1 an infinite sheaf product, say

$$R_1 = \prod_{\delta \in A} W(\delta)^*,$$

and each $W(\delta)$ in some $\mathcal{C}_m$, $m \leq n-s$. We will need explicit extension groups. We use the notation

$$R = (\ldots((R_1[E^1] \sqcap R_2[F^1])[E^2] \sqcap R_3[F^2])[E^3] \sqcap \ldots \sqcap R_s[F^{s-1}])[E^s].$$

We further take $\{t_j^i\}$ as generators of E^i.

Lastly, we need notation to express the orderings on R. Let X_i denote X_{R_i}. Let $X_1(\epsilon_1)$ denote the extensions of X_1 to $R_1[E^1]$. Here ϵ_1 is an arbitrary choice of signs. The extension is determined by the values $\epsilon_1(t^1_j) \in \{\pm 1\}$. To save on indices we will write $\epsilon_1(j)$ for $\epsilon_1(t^1_j)$. Next, $X_2(\eta_1)$ denotes the extensions from R_2 to $R_2[F^1]$. $X_1(\epsilon_1, \epsilon_2)$ denotes the extensions from R_1 to $(R_1[E^1] \sqcap R_1[F^1])[E^2]$, with ϵ_2 a sign choice for E^2. Continue with this pattern. We obtain for X_R

$$\bigcup_{\epsilon,\eta}[X_1(\epsilon_1,\dots,\epsilon_s) \cup X_2(\eta_1,\epsilon_2,\dots,\epsilon_s) \cup X_3(\eta_2,\epsilon_3,\dots,\epsilon_s) \cup \dots \cup X_s(\eta_{s-1},\epsilon_s)].$$

Theorem 3.4. *Suppose $R \in \mathcal{C}_\omega$. The following are equivalent:*

(1) *R has finite chain length.*
(2) *R has ACC on principal ideals.*
(3) *R is atomic.*

Proof. We need only show R atomic implies R has finite chain length, by (2.7) and (3.2). Suppose $R \in \mathcal{C}_n$ and let s be the first level (if any) with an infinite sheaf product. We follow the above notation. Fix some $\delta_0 \in A$ and define $a \in G_{R_1}$ with -1 in the δ_0 coordinate and 1 in the other coordinates. Set

$$b = ((\dots(a,-1),-1),\dots),-1) \in G_R,$$

and set $q = \langle b, t^1_1, bt^1_1\rangle$.
Let X_δ be the orderings on $W(\delta)^*$ so that $X_1 = \cup X_\delta$. Set $C = \hat{q}^{-1}(3)$. Then:

$$C = \bigcup_{\substack{\epsilon,\eta \\ \epsilon_1(1)=1}} \left[\left(\bigcup_{\delta\neq\delta_0} X_\delta\right)(\epsilon_1,\dots,\epsilon_s) \cup X_2(\eta_1,\epsilon_2,\dots,\epsilon_s) \cup \dots \cup X_s(\eta_{s-1},\epsilon_s))\right].$$

We are assuming R is atomic, so let $q = \varphi_1 \cdots \varphi_r$ with each φ_i irreducible. We may assume $\hat{\varphi}_i(X) = \{3,-1\}$ by (2.6). Set $D_i = \hat{\varphi}_i^{-1}(3)$. Note $D_i \subset C$. We will show that in fact one of the φ_i factors and hence that no sheaf product in R is infinite.
Our first goal is to show that each D_i consists of all extensions, with t^1_1 positive, of some subset of X_1. Pick $P \in X_{\delta_0}$ and $Q \in X_\delta$ with $\delta \neq \delta_0$. Fix some k and j. Let

$$e^k = sp\{t^k_1,\dots,t^k_{j-1},t^k_{j+1},\dots\}$$
$$e^1 = sp\{t^1_2,t^1_3,\dots\}.$$

Let T be the fan

$$(\dots(((P\cap Q)[e^1] \sqcap G_{R_2}[F^1])[E^2]\dots)[e^k] \sqcap \dots \sqcap G_{R_s}[F^{s-1}])[E^s].$$

Then X/T has 8 orderings, namely the extensions of P and Q with all t^i_ℓ positive except possibly t^1_1 and t^j_k. Write these orderings as $P(\pm 1, \pm 1)$ and $Q(\pm 1, \pm 1)$, where the first coordinate gives the sign of t^1_1 and the second gives the sign of t^k_j.

$C \cap (X/T) = \{Q(1, \pm 1)\}$ so that $|C \cap (X/T)| = 2$. To ease notation slightly, write D for one of the D_i. Let $w = |D \cap (X/T)|$. Then by the easy part of the Representation Theorem we have:

$$\sum_{\gamma \in X/T} \hat{\varphi}(\gamma) \equiv 0 \pmod{|X/T|}$$
$$3w - (8 - w) \equiv 0 \pmod 8$$
$$w \equiv 0 \pmod 2.$$

As $D \cap (X/T) \subset C \cap (X/T)$ we have $D \cap (X/T)$ is either empty or all of $C \cap (X/T)$.

Suppose for some k and j we are in the second case, $D \cap (X/T) = C \cap (X/T)$. Choose another pair g, h. Pick the fan T' generated over $P \cap Q$ by E^i for $i \neq 1, k, g$, the same e^1 as before and

$$e^{k'} = sp\{t^k_1, \dots, t^k_{j-1}, -t^k_j, t^k_{j+1} \dots\}$$
$$e^{g'} = sp\{t^g_1, \dots, t^g_{h-1}, t^g_{h+1}, \dots\}.$$

Then X/T' has 8 orderings, namely the extensions of P and Q with all t^i_ℓ positive except t^k_j negative and t^1_1, t^g_h arbitrary. Write these as $P(\pm 1, -1, \pm 1)$ and $Q(\pm 1, -1, \pm 1)$ with the first coordinate the sign of t^1_1, the second coordinate indicating that t^k_j is negative and the third coordinate the sign of t^g_h.

Again $C \cap (X/T')$ consists of two orderings, $Q(1, -1, \pm 1)$. And as before we get that $D \cap (X/T')$ is either empty or all of $C \cap (X/T')$. But $Q(1, -1, 1)$ is the same ordering that was denoted by $Q(1, -1)$ before (that is, with t^1_1 positive, t^k_j negative and all other t's positive). Hence we have $D \cap (X/T') = C \cap (X/T')$.

We continue to assume $D \cap (X/T) = C \cap (X/T)$. If we repeat this argument (first with a fan having t^k_j and t^g_h negative) we get that any extension Q with t^1_1 positive and only a finite number of t^i_ℓ negative is in D. Now $D = \hat{\varphi}^{-1}(3)$ and the entries of φ involve only a finite number of t^i_ℓ. Hence we have that any extension of Q with t^1_1 positive is in D.

The assumption that $D \cap (X/T) \neq \emptyset$ means we are assuming some extension of Q with t^1_1 positive is in D. From this we conclude that all such extensions are in D.

Let X^*_1 denote the orderings on $R_1[E^1]$, namely the extensions ϵ_1 of X_1. Write $D|X^*_1$ for the orderings in D restricted to $R_1[E^1]$. We have shown that $D|X^*_1$ consists of all extensions, with t^1_1 positive, of some subset (call it $D|X_1$) of X_1. Each factor φ_i of q has its set D_i. We have $C = \cup D_i$ and

$$\cup(D_i|X_1) = C|X_1 = \bigcup_{\substack{\delta \in A \\ \delta \neq \delta_0}} X_\delta.$$

A is infinite so some $D_i|X_1$ meets at least two X_δ's. For simplicity, call this D_i simply D and the corresponding form φ. Suppose $D|X_1$ meets X_{δ_1} and X_{δ_2}, $\delta_1 \neq \delta_2$. Set

$$D_0 = \bigcup_{\epsilon(1)=1} [(D|X_1) \cap X_{\delta_1}](\epsilon_1) \subset X_1^*.$$

In words, D_0 consists of the extensions for X_{δ_1} that lie in $D|X_1^*$. We will use D_0 to construct a factor of φ.

Let $f : X_1^* \to \mathbb{Z}$ by $f(P) = 3$ if $P \in D_0$ and $f(P) = -1$ if $P \notin D_0$. We want to use the Representation Theorem [11,7.13] to show f is represented by a form in $R_1[E^1]$. Let $T \subset G_{R_1}E^1$ be a fan of finite index. Then $T_1 = T \cap G_{R_1}$ is a fan in G_{R_1} by (3.3)

Case 1 : $(X_1/T_1) \subset X_\delta$ for some $\delta \in A$.

Here $X_1^*/T = (X_\delta/T_1)(\epsilon)$, over some set of extensions ϵ to E^1. If $\delta \neq \delta_1$ then $f(P) = -1$ for all $P \in (X_1^*/T)$ since D_0 only has extensions from X_{δ_1}. Thus

$$\sum_{P \in X_1^*/T} f(P) = -|X_1^*/T| \equiv 0 \pmod{|X_1^*/T|}.$$

If $\delta = \delta_1$ then $P \in D_0$ iff $P \in D|X_1^*$ iff some (equivalently, every) extension, with t_1^1 positive, of P to X_R lies in D iff $\hat{\varphi}(P) = 3$. So $f(P) = \hat{\varphi}(P)$ for all $P \in X_1^*/T$. We obtain

$$\sum_{P \in X_1^*/T} f(P) = \sum_{P \in X_1^*/T} \hat{\varphi}(P) \equiv 0 \pmod{|X_1^*/T|}.$$

Case 2 : $(X_1/T_1) \not\subset X_\delta$ for some $\delta \in A$.

Here we must have $|X_1/T_1| = 2$ by [11, 8.12] Write $X_1/T_1 = \{P_\alpha, P_\beta\}$ where α, β are distinct elements of A and $P_\alpha \in X_\alpha$ and $P_\beta \in X_\beta$. Then X_1^*/T consists of some set of extensions, to E^1, applied to P_α and P_β.

Again, if neither α nor β are δ_1 then all $f(P) = -1$ and we are done. So say $\alpha = \delta_1$ (and so $\beta \neq \delta_1$). If $P_\alpha \notin D|X_1$ then no extension is in D_0 and all $f(P) = -1$ again. So suppose $P_\alpha \in (D|X_1) \cap X_{\delta_1}$. Since $P_\beta \notin X_{\delta_1}$ no extension of P_β in X_1^*/T is in D_0. This is half of X_1^*/T. The other half consists of extensions of P_α and by (3.3) either none, exactly half or all of these extensions make t_1^1 positive, and hence lie in D_0. Thus $|D_0 \cap (X_1^*)| = d|X_1^*/T|$, where d is either (i) 0, or (ii) $\frac{1}{4}$ or (iii) $\frac{1}{2}$. In case (i) we have

$$\sum_{P \in X_1^*} f(P) = -|X_1^*/T| \equiv 0 \pmod{|X_1^*/T|}.$$

In case (ii) we have

$$\sum_{P \in X_1^*} f(P) = \tfrac{1}{4}|X_1^*/T| \cdot 3 + \tfrac{3}{4}|X_1^*/T| \cdot (-1) \equiv 0 \pmod{|X_1^*/T|}.$$

In case (iii) we have

$$\sum_{P\in X_1^*} f(P) = \tfrac{1}{2}|X_1^*/T|\cdot 3 + \tfrac{1}{2}|X_1^*/T|\cdot(-1) \equiv 0 \pmod{|X_1^*/T|}.$$

Thus in all cases we have $\sum f(P) \equiv 0 \pmod{|X_1^*/T|}$. By the non-trivial half of the Representation Theorem we have $f = \hat{\psi}$ for some form $\psi \in R_1[E^1]$. By construction $\hat{\psi}(X_1^*) = \{3, -1\}$ and $\hat{\psi}^{-1}(3) = D_0 < D$. Hence by [7, 1.7] ψ is a proper divisor of φ. Hence φ is not irreducible, a contradiction.
We thus have if $R \in \mathcal{C}_n$ is atomic then all sheaf products are finite. Hence $\mathrm{cl}(R) < \infty$, using [12, 4.2.1]. $\square$

Corollary 3.5. *Let $R \in \mathcal{C}_\omega$. If $R[E]$ is atomic then so is R.*

Proof. $R[E]$ atomic implies $R[E]$ has finite chain length by (3.4). Then , as $\mathrm{cl}(R[E]) = \mathrm{cl}(R)$, R has finite chain length and so is atomic by (3.2). $\square$

It is unknown if the reduced Witt rings of finite stability index lie in $\mathcal{C}_\omega$ so the following may improve (3.4), although (3.4) includes many atomic Witt rings with X infinite.

Proposition 3.6. *Suppose R has finite stability index. The following are equivalent:*

(1) *R has finite chain length.*
(2) *R has ACC on principal ideals.*
(3) *R is atomic.*
(4) *X is finite.*

Proof. (1) and (4) are equivalent by [10] (first shown, in the field case in [4]). As in the proof of (3.4) we need only show (3) implies (1). Suppose the stability index of R is n. We can find a prime p congruent to 1 mod 2^n by Dirichlet's Theorem. R is atomic so $p = \varphi_1 \cdots \varphi_t$ for some irreducible elements φ_i. Note that for each i we have $|\hat{\varphi}_i(X)| = \{p, 1\}$. Let $A_i = \hat{\varphi}_i^{-1}(\pm p)$. The A_i's form a clopen cover of X.
We wish to show R has finite chain length. So suppose we have a tower

$$H(a_1) > H(a_2) > H(a_3) > \cdots.$$

First suppose there is an s, $1 \le s \le t$ and a k such that $A_s \cap H(a_k)$ is a non-empty, proper subset of A_s. Define $f : X \to \mathbb{Z}$ by

$$f(\alpha) = \begin{cases} p, & \text{if } \alpha \in A_s \cap H(a_k) \\ 1, & \text{if } \alpha \notin A_s \cap H(a_k). \end{cases}$$

Let T be a fan, $|X/T| = 2^m$, where $m \le n$ by definition of the stability index. Set $w = |A_s \cap H(a_k) \cap (X/T)|$. Then

$$\sum_{\alpha\in X/T} f(\alpha) = wp + (2^m - w) = w(p-1) \equiv 0 \pmod{2^m},$$

since $p-1$ is a multiple of 2^n. By the Representation Theorem, $f = \hat{\psi}$ for some form ψ. Then $\hat{\psi}(\alpha)$ divides $\hat{\varphi}_s(\alpha)$ for all α and for $\alpha \in A_s \setminus H(a_k)$, $\hat{\psi}(\alpha) \neq \pm\hat{\varphi}_s(\alpha)$. So, using [7, 1.7], we have ψ is proper divisor of φ_s, which is impossible.
Thus there does not exists a pair s, k such that $H(a_k) \cap A_s$ is a non-empty, proper subset of A_s. That is, for all i, j we have $H(a_i) \cap A_j \neq \emptyset$ implies $A_j \subset H(a_i)$. The A_j's cover X so each $H(a_i)$ is a union of A_j's. Let $n(i)$ be the number of A_j's required to cover $H(a_i)$. Then $1 \leq n(i+1) < n(i) \leq t$ for all i. Thus the tower is finite and we are done. □

4. Irreducible elements.

We look at some examples to illustrate factorization in reduced Witt rings.

Proposition 4.1. *If $1 \neq a \in G$ then $\langle 1, -a\rangle$ is irreducible in R.*

Proof. Suppose $\langle 1, -a\rangle = q\varphi$ in R. We may assume q is even dimensional and φ is odd dimensional. If $a <_\alpha 0$ then $2 = \hat{q}(\alpha)\hat{\varphi}(\alpha)$. Thus $\hat{q}(\alpha) = \pm 2 = \pm\mathrm{sgn}_\alpha\langle 1, -a\rangle$, for all α with $\mathrm{sgn}_\alpha\langle 1, -a\rangle \neq 0$. By (2.6) there exists a $d \in G$ such that $\langle d\rangle\langle 1, -a\rangle = q$ and so q is an associate of $\langle 1, -a\rangle$. □

Example. If $R \neq \mathbb{Z}$ then factorization into irreducible elements is not unique. Namely, if $a \neq \pm 1$ then $\langle 1, -a\rangle\langle 1, -a\rangle = \langle 1, 1\rangle\langle 1, -a\rangle$ gives two different factorizations of the Pfister form. This is quite different from the case of factoring odd dimensional forms. When X is finite there is unique factorization of odd dimensional forms if the ideal class group of R is trivial or, equivalently, the stability index is at most 2, by [6, 2.7] and [7, 1.17].

We next find the irreducible elements in $\mathbb{Z}[E_1]$. Note that any form q in this ring is associate to some $n + mt$ with $n \geq |m|$.

Proposition 4.2. *Let $q = n + mt \in \mathbb{Z}[E_1]$ with $n \geq |m|$. Then q is irreducible iff (n, m) or $(n, -m)$ equals one of the following:*

(1) $(1, 1)$
(2) $(2^k + 1, 2^k - 1)$, *for some $k \geq 0$*
(3) $(\frac{1}{2}(p+1), \frac{1}{2}(p-1))$, *for some odd prime p.*

Proof. Let q be irreducible. First suppose q is even dimensional. If both n and m are even then 2 is a factor of q. So we have n and m odd. If $n = \pm m$ then n is a factor of q and we must have $n = 1$. Thus $(n, m) = (1, \pm 1)$. We may thus suppose $n + m$ and $n - m$ are non-zero. Write $n + m = 2^g h$ and $n - m = 2^k \ell$ with h and ℓ odd and $g, k \geq 1$. Set

$$\varphi_1 = \tfrac{1}{2}(2^g + 2^k) + \tfrac{1}{2}(2^g - 2^k)t$$
$$\varphi_2 = \tfrac{1}{2}(h + \ell) + \tfrac{1}{2}(h - \ell)t.$$

Then $q = \varphi_1\varphi_2$ and φ_2 is odd dimensional and so not an associate of q. Thus φ_1 is an associate of q. If α is the ordering with t positive then $n + m = \hat{q}(\alpha) =$

$\pm\hat{\varphi}_1(\alpha) = \pm 2^g$. Since $n \geq -m$ we obtain $n+m = 2^g$ and $h = 1$. Similarly, taking signatures at the ordering β with t negative gives $\ell = 1$. If both g and k are at least 2 then n and m are even which is not possible. Suppose $n+m = 2^g$ and $n - m = 2$. Then we get case (2). The reverse , $n+m = 2$ and $n-m = 2^k$ gives case (2) for the pair $(n, -m)$.
Now suppose q is odd dimensional. If $n+m$ is composite, say $n+m = ab$ with $a, b > 1$, then set

$$\begin{aligned}\varphi_1 &= \tfrac{1}{2}(a+1) + \tfrac{1}{2}(a-1)t\\ \varphi_2 &= \tfrac{1}{2}(b+n-m) + \tfrac{1}{2}(b-n+m)t.\end{aligned}$$

Then $q = \varphi_1\varphi_2$. Neither φ_1 nor φ_2 is an associate of q as $\hat{q}(\alpha) = ab$ while $\hat{\varphi}_1(\alpha) = a$ and $\hat{\varphi}_2(\alpha) = b$. Hence $n+m$ is not composite. Similarly, $n-m$ is not composite. If both $n+m$ and $n-m$ are prime then set

$$\begin{aligned}\varphi_1 &= \tfrac{1}{2}(n+m+1) + \tfrac{1}{2}(n+m-1)t\\ \varphi_2 &= \tfrac{1}{2}(n-m+1) + \tfrac{1}{2}(1-n+m)t.\end{aligned}$$

We have $q = \varphi_1\varphi_2$. Neither φ_1 nor φ_2 is an associate of q as $\hat{q}(\alpha) = n+m$ while $\hat{\varphi}_2(\alpha) = 1$ and $\hat{q}(\beta) = n-m$ while $\hat{\varphi}_1(\beta) = 1$. Thus we must have $n+m = p$, p an odd prime, and $n-m = 1$ (or the reverse). This gives case (3).
It is straightforward to check the forms in cases (1) - (3) are irreducible. □

Example. Already for $\mathbb{Z}[E_1]$, and in fact for any $R \neq \mathbb{Z}$, the number of irreducible factors in factorization of a given element can be arbitrarily large. For instance, $\langle 1, 1, t\rangle$ is irreducible (take $p = 3$ in (4.2)(3)) and $\langle 1, -t\rangle\langle 1, 1, t\rangle = \langle 1, -t\rangle$. Hence

$$\langle\langle 1, -t\rangle\rangle = \langle 1, 1\rangle\langle 1, 1, t\rangle^n\langle 1, -t\rangle$$

is a factorization into irreducible elements for any n. Again the situation is quite different if we consider only factorizations of odd dimensional forms. When X is finite, the number of irreducible factors in a factorization is uniquely determined iff the stability index is at most 3 and R has no factor of the type $(\mathbb{Z}^s)[E_2]$, with $s \geq 3$, see [7].

Notice that the even prime of $\mathbb{Z}$ remains irreducible in $\mathbb{Z}[E_1]$ while the odd primes of $\mathbb{Z}$ all factor in $\mathbb{Z}[E_1]$. This holds more generally.

Proposition 4.3. *Let $q \in R$ be irreducible.*

(1) *If q is even dimensional then q remains irreducible in $R[E_1]$.*
(2) *If q is odd dimensional then q remains irreducible in $R[E_1]$ iff q is not associate to $1 + 2q_0$, for some $q_0 \in R$.*

Proof. First say $q = 1 + 2q_0$, for some $q_0 \in R$. Since q is not a unit, there exists an $\alpha \in X_R$ with $\hat{q}(\alpha) \neq \pm 1$. Let α^+ and α^- denote the extensions of α to $R[E_1]$ with, respectively, t positive and t negative. Now

$$q = (1 + q_0\langle 1, t\rangle)(1 + q_0\langle 1, -t\rangle).$$

Neither factor is an associate of q as the first has signature 1 at α^- and the second has signature 1 at α^+. Thus q is not irreducible in $R[E_1]$.
Now suppose we have an irreducible q that factors in $R[E_1]$. We want to show q is odd dimensional and associate to some $1+2q_0$. Write $q = (a + b\langle 1, t\rangle)(c + d\langle 1, -t\rangle)$, with $a, b, c, d \in R$ and neither factor an associate of q. The coefficient of t, namely $bc - ad$, must be zero and so $q = ac + ad + bc$. Then

$$q = ac + 2bc = c(a + 2b) \tag{4.4}$$

$$= ac + 2ad = a(c + 2d). \tag{4.5}$$

As q is irreducible in R, (4.4) shows that either c or $a + 2b$ is an associate of q. We may assume c is the associate of q. Namely, if $a + 2b$ is the associate then rewrite q as

$$\begin{aligned} q &= ((c + 2d) + (-d)\langle 1, t\rangle)((a + 2b) + (-b)\langle 1, -t\rangle) \\ &\equiv (a' + b'\langle 1, t\rangle)(c' + d'\langle 1, -t\rangle). \end{aligned}$$

Then $c' = a + 2b$ is associate to q.
Write $uq = c$ for some unit $c \in R$. Equation (4.5) shows that either a or $c + 2d$ is an associate of q. Assume by way of contradiction that $vq = c + 2d$ for some unit $v \in R$. Note $(v - u)q = 2d$; set $\chi = v - u$. Let $Z = \{\alpha \in X_R : \hat{q}(\alpha) \neq 0\}$. From (4.4), $q = qu(a + 2b)$ so that $\hat{u} = \hat{a} + 2\hat{b}$ on Z. Similarly, from (4.5) $q = qva$ so that $\hat{v} = \hat{a}$ on Z. Thus, on Z, $\hat{\chi} = \hat{v} - \hat{u} = -2\hat{b}$. Now u and v are units and so have signatures ± 1 at all orderings. Thus $\hat{\chi}(X_R) \subset \{2, 0, -2\}$. If b is even dimensional then we must have $\hat{b} = 0$ on Z. Then $\hat{\chi} = 0$ on Z and $0 = q\chi = 2d$. But then $d = 0$ and the second factor of q, $c + d\langle 1, -t\rangle = c = uq$ is an associate of q, a contradiction. Hence b is odd dimensional. In particular, $\hat{b}$ is never zero. So $\hat{v} - \hat{u}$ is not zero on Z. We must have $\hat{v} = -\hat{u}$ (as $\hat{u}$ and $\hat{v}$ are always ± 1). So $\hat{\chi} = 2\hat{v}$ on Z. Then $2vq = q\chi = 2d$ and $vq = d$. But then the second factor of q is $c + d\langle 1, -t\rangle = uq + vq\langle 1, -t\rangle = q(u + v - vt) = -vtq$, an associate of q. This is impossible.
Hence we must have that q is an associate of a as well as c. Write $uq = c$ and $vq = a$ for units $u, v \in R$. Equation (4.4) gives $q = uq(a + 2b)$. If q is even dimensional then $a + 2b$ is odd dimensional and so a is odd dimensional. But a is an associate of the even dimensional q so a must be even dimensional, a contradiction.
We have then that q is odd dimensional. Then $q = uq(a+2b)$ implies $u(a+2b) = 1$. So $uvq = ua = 1 - 2ub$, as desired. $\square$

It can be shown that $a + bt \in R[E_1]$ is irreducible if $a + b$ is irreducible in R and $a - b$ is a unit. Thus in the factorization of (4.3) $1 + 2q_0 = (1 + q_0 + q_0 t)(1 + q_0 - q_0 t)$, both factors are irreducible. However, not every irreducible $a + bt \in R[E_1]$ satisfies $a + b$ irreducible and $a - b$ a unit. For instance, one may easily check that $q = \langle 1\rangle + \langle\langle t_1, t_2, t_3\rangle\rangle \in \mathbb{Z}[E_3]$ is irreducible. As a form in $R[E_1]$, where $R = \mathbb{Z}[E_2]$, we have $q = a + bt_3$ with $a = \langle 1 \rangle + \langle\langle t_1, t_2 \rangle\rangle$ and $b = \langle\langle t_1, t_2 \rangle\rangle$. Then $a - b$ is a unit but $a + b = 1 + 2\langle\langle t_1, t_2 \rangle\rangle = (1 - \langle\langle t_1, t_2 \rangle\rangle)^2$. In fact, we have been unable to determine the irreducible elements of $R[E_1]$ in terms of the irreducibles of R. For products, we can determine only the irreducible odd dimensional forms.

Proposition 4.6. *If $R = R_1 \sqcap R_2$ and $(a, b) \in R$ is odd dimensional then (a, b) is irreducible iff a is irreducible in R and b is a unit or the reverse, a is a unit and b is irreducible.*

Proof. We have $(a, b) = (a, 1)(1, b)$. So (a, b) irreducible implies either a or b is a unit. Say b is a unit. If $a = xy$ then $(a, b) = (x, b)(y, 1)$, so a must be irreducible in R. □

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