

QUADRATIC QUATERNION FORMS, INVOLUTIONS AND TRIALITY

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ABSTRACT. Quadratic quaternion forms, introduced by Seip-Hornix (1965), are special cases of generalized quadratic forms over algebras with involutions. We apply the formalism of these generalized quadratic forms to give a characteristic free version of different results related to hermitian forms over quaternions:

- 1) An exact sequence of Lewis
- 2) Involutions of central simple algebras of exponent 2.
- 3) Triality for 4-dimensional quadratic quaternion forms.

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1. INTRODUCTION

Let F be a field of characteristic not 2 and let D be a quaternion division algebra over F . It is known that a skew-hermitian form over D determines a symmetric bilinear form over any separable quadratic subfield of D and that the unitary group of the skew-hermitian form is the subgroup of the orthogonal group of the symmetric bilinear form consisting of elements which commute with a certain semilinear mapping (see for example Dieudonné [3]). Quadratic forms behave nicer than symmetric bilinear forms in characteristic 2 and Seip-Hornix developed in [9] a complete, characteristic-free theory of quadratic quaternion forms, their orthogonal groups and their classical invariants. Her theory was subsequently (and partly independently) generalized to forms over algebras (even rings) with involution (see [11], [10], [1], [8]).

Similitudes of hermitian (or skew-hermitian) forms induce involutions on the endomorphism algebra of the underlying space. To generalize the case where only similitudes of a quadratic form are considered, the notion of a quadratic pair was worked out in [6]. Relations between quadratic pairs and generalized quadratic forms were first discussed by Elomary [4].

The aim of this paper is to apply generalized quadratic forms to give a characteristic free presentation of some results on forms and involutions. After briefly recalling in Section 2 the notion of a generalized quadratic form (which, following the standard literature, we call an (ε, σ) -quadratic form) we give in Section 3 a characteristic-free version of an exact sequence of Lewis (see [7], [8, p. 389] and the appendix to [2]), which connects Witt groups of quadratic and quaternion algebras. The quadratic quaternion forms of Seip-Hornix are the main ingredient. Section 4 describes a canonical bijective correspondence between quadratic pairs and (ε, σ) -quadratic forms and Section 5 discusses the Clifford algebra. In particular we compare the definitions given in [10] and in [6]. In Section 6 we develop triality for 4-dimensional quadratic quaternion forms whose associated forms (over a separable quadratic subfield) are 3-Pfister forms. Any such quadratic quaternion form θ is an element in a triple $(\theta_1, \theta_2, \theta_3)$ of forms over 3 quaternion algebras D_1, D_2 and D_3 such that $[D_1][D_2][D_3] = 1$ in the Brauer group of F . Triality acts as permutations on such triples.

2. GENERALIZED QUADRATIC FORMS

Let D be a division algebra over a field F with an involution $\sigma : x \mapsto \bar{x}$. Let V be a finite dimensional right vector space over D . An F -bilinear form

$$k : V \times V \rightarrow D$$

is *sesquilinear* if $k(xa, yb) = \bar{a}k(x, y)b$ for all $x, y \in V, a, b \in D$. The additive group of such maps will be denoted by $\text{Sesq}_\sigma(V, D)$. For any $k \in \text{Sesq}_\sigma(V, D)$ we write

$$k^*(x, y) = \overline{k(y, x)}.$$

Let $\varepsilon \in F^\times$ be such that $\varepsilon\bar{\varepsilon} = 1$. A sesquilinear form k such that $k = \varepsilon k^*$ is called ε -*hermitian* and the set of such forms on V will be denoted by $\text{Herm}_\sigma^\varepsilon(V, D)$. Elements of

$$\text{Alt}_\sigma^\varepsilon(V, D) = \{g = f - \varepsilon f^* \mid f \in \text{Sesq}_\sigma(V, D)\}.$$

are ε -*alternating forms*. We obviously have $\text{Alt}_\sigma^{-\varepsilon}(V, D) \subset \text{Herm}_\sigma^\varepsilon(V, D)$. We set

$$\text{Q}_\sigma^\varepsilon(V, D) = \text{Sesq}_\sigma(V, D) / \text{Alt}_\sigma^\varepsilon(V, D)$$

and refer to elements of $\text{Q}_\sigma^\varepsilon(V, D)$ as (ε, σ) -*quadratic forms*. We recall that (ε, σ) -quadratic forms were introduced by Tits [10], see also Wall [11], Bak [1] or Scharlau [8, Chapter 7]. For any algebra A with involution τ , let $\text{Sym}^\varepsilon(A, \tau) = \{a \in A \mid a = \varepsilon\tau(a)\}$ and $\text{Alt}^\varepsilon(A, \tau) = \{a \in A \mid a = c - \varepsilon\tau(c), c \in A\}$. To any class $\theta = [k] \in \text{Q}_\sigma^\varepsilon(V, D)$, represented by $k \in \text{Sesq}_\sigma(V, D)$, we associate a quadratic map

$$q_\theta : V \rightarrow D / \text{Alt}^\varepsilon(D, \sigma), \quad q_\theta(x) = [k(x, x)]$$

where $[d]$ denotes the class of d in $D / \text{Alt}^\varepsilon(D, \sigma)$. The ε -hermitian form

$$b_\theta(x, y) = k(x, y) + \varepsilon k^*(x, y) = k(x, y) + \varepsilon \overline{k(y, x)}$$

depends only on the class θ of k in $Q_\sigma^\varepsilon(V, D)$. We say that b_θ is the *polarization* of q_θ .

PROPOSITION 2.1. *The pair (q_θ, b_θ) satisfies the following formal properties:*

$$(1) \quad \begin{aligned} q_\theta(x + y) &= q_\theta(x) + q_\theta(y) + [b_\theta(x, y)] \\ q_\theta(xd) &= \overline{dq_\theta(x)d} \\ b_\theta(x, x) &= q_\theta(x) + \varepsilon \overline{q_\theta(x)} \end{aligned}$$

for all $x, y \in V, d \in D$. Conversely, given any pair $(q, b), q : V \rightarrow D/\text{Alt}^\varepsilon(D, \sigma), b \in \text{Herm}_\sigma^\varepsilon(V, D)$ satisfying (1), there exist a unique $\theta \in Q_\sigma^\varepsilon(V, D)$ such that $q = q_\theta, b = b_\theta$.

Proof. The formal properties are straightforward to verify. For the converse see [11, Theorem 1]. □

EXAMPLE 2.2. Let $D = F, \sigma = Id_F$ and $\varepsilon = 1$. Then sesquilinear forms are F -bilinear forms, $\text{Alt}^\varepsilon(D, \sigma) = 0$ and a (σ, ε) -quadratic form is a (classical) quadratic form. We denote the set of bilinear forms on V by $\text{Bil}(V, F)$. Accordingly we speak of ε -symmetric bilinear forms instead of ε -hermitian forms.

EXAMPLE 2.3. Let D be a division algebra with involution σ and let V be a finite dimensional (right) vector space over D . We use a basis of V to identify V with D^n and $\text{End}_D(V)$ with the algebra $M_n(D)$ of $(n \times n)$ -matrices with entries in D . For any $(n \times m)$ -matrix $x = (x_{ij})$, let $x^* = \overline{x}^t$, where t is transpose and $\overline{x} = (\overline{x}_{ij})$. In particular the map $a \mapsto a^*$ is an involution of $A = M_n(D)$. If we write elements of D^n as column vectors $x = (x_1, \dots, x_n)^t$ any sesquilinear form k over D^n can be expressed as $k(x, y) = x^*ay$, with $a \in M_n(D)$, and $k^*(x, y) = x^*a^*y$. We write $\text{Alt}_n(D) = \{a = b - \varepsilon b^*\} \subset M_n(D)$, so that $Q_\sigma^\varepsilon(V, D) = M_n(D)/\text{Alt}_n(D)$.

EXAMPLE 2.4. Let D be a quaternion division algebra, i.e. D is a central division algebra of dimension 4 over F . Let K be a maximal subfield of D which is a quadratic Galois extension of F and let $\sigma : x \mapsto \overline{x}$ be the nontrivial automorphism of K . Let $j \in K \setminus F$ be an element of trace 1, so that $K = F(j)$ with $j^2 = j + \lambda, \lambda \in F$. Let $\ell \in D$ be such that $\ell x \ell^{-1} = \overline{x}$ for $x \in K, \ell^2 = \mu \in F^\times$. The elements $\{1, j, \ell, \ell j\}$ form a basis of D and $D = K \oplus \ell K$ is also denoted $[K, \mu]$. The F -linear map $\sigma : D \rightarrow D, \sigma(d) = \text{Trd}_D(d) - d = \overline{d}$ is an involution of D (the ‘‘conjugation’’) which extends the automorphism σ of K . The element $N(d) = d\sigma(d) = \sigma(d)d$ is the reduced norm of d . We have $\text{Alt}_\sigma^{-1}(D) = F$ and $(\sigma, -1)$ -quadratic forms correspond to the quadratic quaternion forms introduced by Seip-Hornix in [9]. Accordingly we call $(\sigma, -1)$ -quadratic forms *quadratic quaternion forms*.

The restriction of the involution τ to the center Z of A is either the identity (*involutions of the first kind*) or an automorphism of order 2 (*involutions of the second kind*). If the characteristic of F is different from 2 or if the involution is of second kind there exists an element $j \in Z$ such that $j + \sigma(j) = 1$. Under

such conditions the theory of (σ, ε) -quadratic forms reduces to the theory of ε -hermitian forms:

PROPOSITION 2.5. *If the center of D contains an element j such that $j + \sigma(j) = 1$, then $\text{Herm}_\sigma^{-\varepsilon}(V, D) = \text{Alt}_\sigma^\varepsilon(V, D)$ and a (σ, ε) -quadratic form is uniquely determined by its polar form b_θ .*

Proof. If $k = -\varepsilon k^* \in \text{Herm}_\sigma^{-\varepsilon}(V, D)$, then $k = 1k = jk + \bar{j}k = jk - \bar{j}\varepsilon k^* \in \text{Alt}_\sigma^\varepsilon(V, D)$. The last claim follows from the fact that polarization induces an isomorphism $\text{Sesq}_\sigma(V, D) / \text{Herm}_\sigma^{-\varepsilon}(V, D) \xrightarrow{\sim} \text{Q}_\sigma^\varepsilon(V, D)$. \square

For any left (right) D -space V we denote by ${}^\sigma V$ the space V viewed as right (left) D -space through the involution σ . If ${}^\sigma x$ is the element x viewed as an element of ${}^\sigma V$, we have ${}^\sigma xd = \sigma(\sigma(d)x)$. Let V^* be the dual ${}^\sigma \text{Hom}_D(V, D)$ as a right D -module, i.e., $({}^\sigma f d)(x) = \sigma(\bar{d}f)(x)$, $x \in V$, $d \in D$. Any sesquilinear form $k \in \text{Sesq}_\sigma(V, D)$ induces a D -module homomorphism $\widehat{k} : V \rightarrow V^*$, $x \mapsto k(x, -)$. Conversely any homomorphism $g : V \rightarrow V^*$ induces a sesquilinear form $k \in \text{Sesq}_\sigma(V, D)$, $k(x, y) = g(x)(y)$ and the additive groups $\text{Sesq}_\sigma(V, D)$ and $\text{Hom}_D(V, V^*)$ can be identified through the map $h \mapsto \widehat{k}$. For any $f : V \rightarrow V'$, let $f^* : V'^* \rightarrow V^*$ be the transpose, viewed as a homomorphism of right vector spaces. We identify V with V^{**} through the map $v \mapsto v^{**}$, $v^{**}(f) = \overline{f(v)}$. Then, for any $f \in \text{Hom}_D(V, V^*)$, f^* is again in $\text{Hom}_D(V, V^*)$ and $\widehat{k^*} = \widehat{\widehat{k}}$. A (σ, ε) -quadratic form q_θ is called *nonsingular* if its polar form b_θ induces an isomorphism \widehat{b}_θ . A pair (V, q_θ) with q_θ nonsingular is called a (σ, ε) -quadratic space. For any vector space W , the *hyperbolic space* $V = W \oplus W^*$ equipped with the quadratic form q_θ , $\theta = [k]$ with

$$k((p, q), (p', q')) = q(p'),$$

is nonsingular. There is an obvious notion of orthogonal sum $V \perp V'$ and a quadratic space decomposes whenever its polarization does. Most of the classical theory of quadratic spaces extends to (σ, ε) -quadratic spaces. For example Witt cancellation holds and any (σ, ε) -quadratic space decomposes uniquely (up to isomorphism) as the orthogonal sum of its anisotropic part with a hyperbolic space. Moreover, if we exclude the case $\sigma = 1$ and $\varepsilon = -1$, any (σ, ε) -quadratic space has an orthogonal basis. A *similitude* of (σ, ε) -quadratic spaces $t : (V, q) \xrightarrow{\sim} (V', q')$ is a D -linear isomorphism $V \xrightarrow{\sim} V'$ such that $q'(tx) = \mu(t)q(x)$ for some $\mu(t) \in F^\times$. The element $\mu(t)$ is called the *multiplier* of the similitude. Similitudes with multipliers equal to 1 are *isometries*. As in the classical case there is a notion of Witt equivalence and corresponding Witt groups are denoted by $W^\varepsilon(D, \sigma)$.

3. AN EXACT SEQUENCE OF LEWIS

Let D be a quaternion division algebra. We fix a representation $D = [K, \mu] = K \oplus \ell K$, with $\ell^2 = \mu$, as in (2.4). Let V be a vector space over D . Any sesquilinear form $k : V \times V \rightarrow D$ can be decomposed as

$$k(x, y) = P(x, y) + \ell R(x, y)$$

with $P : V \times V \rightarrow K$ and $R : V \times V \rightarrow K$. The following properties of P and R are straightforward.

LEMMA 3.1. 1) $P \in \text{Sesq}_\sigma(V, K)$, $R \in \text{Sesq}_1(V, K) = \text{Bil}(V, K)$.
 2) $k^* = P^* - \ell R^t$, where $P^*(x, y) = \overline{P(y, x)}$ and $R^t(x, y) = R(y, x)$.

The sesquilinearity of k implies the following identities:

$$(2) \quad \begin{aligned} R(x\ell, y) &= -P(x, y), & R(x, y\ell) &= \overline{P(x, y)} \\ P(x\ell, y) &= -\mu R(x, y), & P(x, y\ell) &= \mu \overline{R(x, y)} \\ P(x\ell, y\ell) &= -\mu \overline{P(x, y)}, & R(x\ell, y\ell) &= -\mu \overline{R(x, y)} \end{aligned}$$

Let V^0 be V considered as a (right) vector space over K (by restriction of scalars) and let $T : V^0 \rightarrow V^0, x \mapsto x\ell$. The map T is a K -semilinear automorphism of V^0 such that $T^2 = \mu$. Conversely, given a vector space U over K , together with a semilinear automorphism T such that $T^2 = \mu \in F^\times$, we define the structure of a right D -module on U , $D = [K, \mu)$, by putting $x\ell = T(x)$.

LEMMA 3.2. Let V be a vector space over D . 1) Let $f_1 : V^0 \times V^0 \rightarrow K$ be a sesquilinear form over K . The form

$$f(x, y) = f_1(x, y) - \ell\mu^{-1}f_1(Tx, y)$$

is sesquilinear over D if and only if $f_1(Tx, Ty) = -\mu \overline{f_1(x, y)}$.

2) Let $f_2 : V^0 \times V^0 \rightarrow K$ be a bilinear form over K . The form

$$f(x, y) = -f_2(Tx, y) + \ell f_2(x, y)$$

is sesquilinear over D if and only if $f_2(Tx, Ty) = -\mu \overline{f_2(x, y)}$.

Proof. The two claims follow from the identities (2). □

Let f be a bilinear form on a space U over K and let $\lambda \in K^\times$. A semilinear automorphism t of U such that $f(tx, ty) = \lambda \overline{f(x, y)}$ for all $x \in U$ is a *semilinear similitude* of (U, f) , with *multiplier* λ . In particular $Tx = x\ell$ is a semilinear similitude of R on V^0 , such that $T^2 = \mu$ and with multiplier $-\mu$. The following nice observation of Seip-Hornix [9, p. 328] will be used later:

PROPOSITION 3.3. Let R be a K -bilinear form over U and let T be a semilinear similitude of U with multiplier $\lambda \in K^\times$ and such that $T^2 = \mu$. Then:

- 1) $\mu \in F$,
- 2) For any $\xi \in K$ and $x \in U$, let $\rho_\xi(x) = x\xi$. There exists $\nu \in K^\times$ such that $T' = \rho_\nu \circ T$ satisfies $T'^2 = \mu'$ and $R(T'x, T'y) = -\mu' \overline{R(x, y)}$.

Proof. The first claim follows from $\mu = \lambda\bar{\lambda}$. For the second we may assume that $\lambda \neq \mu$ (if $\lambda = \mu$ replace T by $T \circ \rho_k$ for an appropriate k). For $\nu = (1 - \mu\lambda^{-1})$ we have $\mu' = 2\mu - \lambda - \bar{\lambda}$. □

Assume that $k \in \text{Sesq}_\sigma(V, D)$ defines a (σ, ε) -quadratic space $[k]$ on V over D . It follows from (3.1) that P defines a (σ, ε) -quadratic space $[P]$ on V^0 over K and R a $(\text{Id}, -\varepsilon)$ -quadratic space $[R]$ on V^0 over K . Let $K = F(j)$ with $j^2 = j + \lambda$. Let $r(x, y) = R(x, y) - \varepsilon R(y, x)$ be the polar of R .

- PROPOSITION 3.4. 1) $q_{[P]}(x) = \bar{\varepsilon}j[r(x, Tx)]$
 2) $q_{[k]}(x) = \bar{\varepsilon}j[r(x, Tx)] + \ell q_{[R]}(x)$
 3) The map T is a semilinear similitude of $(q_{[R]}, V^0)$ with multiplier $-\mu$.

Proof. It follows from the relations (2) that

$$(3) \quad \overline{P(x, x)} + \varepsilon P(x, x) = R(x, Tx) - \varepsilon R(Tx, x) = r(x, Tx)$$

and obviously this relation determines $P(x, x)$ up to a function with values in $\text{Sym}^{-\varepsilon}(K, \sigma)$. Since $\text{Sym}^{-\varepsilon}(K, \sigma) = \text{Alt}^{+\varepsilon}(K, \sigma)$ by (2.5), $[P]$ is determined by (3). Since $r(x, Tx) = \bar{\varepsilon}j r(x, Tx)$ by (2), we have $\bar{\varepsilon}j r(x, Tx) + \varepsilon(\bar{\varepsilon}j r(x, Tx)) = r(x, Tx)$ and 1) follows. The second claim follows from 1) and 3) is again a consequence of the identities (2). \square

COROLLARY 3.5. Any pair $([R], T)$ with $[R] \in Q_1^\varepsilon(U, K)$ and T a semilinear similitude with multiplier $-\mu \in F^\times$ and such that $T^2 = \mu$, determines the structure of a (σ, ε) -quadratic space on U over $D = [K, \mu]$.

PROPOSITION 3.6. The assignments $h \mapsto P$ and $h \mapsto R$ induce homomorphisms of groups $\pi_1 : W^\varepsilon(D, -) \rightarrow W^\varepsilon(K, -)$ and $\pi_2 : W^{-\varepsilon}(D, -) \rightarrow W^\varepsilon(K, \text{Id})$.

Proof. The assignments are obviously compatible with orthogonal sums and Witt equivalence. \square

We recall that $W^\varepsilon(K, -)$ can be identified with the corresponding Witt group of ε -hermitian forms (apply (2.5)). However, it is more convenient for the following computations to view ε -hermitian forms over K as (σ, ε) -quadratic forms. Let $i \in K^\times$ be such that $\sigma(i) = -i$ (take $i = 1$ if $\text{Char } F = 2$). The map $k \mapsto ik$ induces an isomorphism $s : W^\varepsilon(K, -) \xrightarrow{\sim} W^{-\varepsilon}(K, -)$ (“scaling”). For any space U over K , let $U_D = U \otimes_K D$. We identify U_D with $U \oplus U\ell$ through the map $u \otimes (x + \ell y) \mapsto (ux, u\bar{y}\ell)$ and get a natural D -module structure on $U_D = U \oplus U\ell$. Any K -sesquilinear form k on U extends to a D -sesquilinear form k_D on U_D through the formula

$$k_D(x \otimes a, y \otimes b) = \bar{a}k(x, y)b$$

for $x, y \in U$ and $a, b \in D$.

LEMMA 3.7. The assignment $k \mapsto (ik)_D$ induces a homomorphism

$$\beta : W^\varepsilon(K, -) \rightarrow W^{-\varepsilon}(D, -)$$

Proof. Let $\tilde{k} = (ik)_D$. We have $(\tilde{k})^* = -\tilde{k}^*$. \square

THEOREM 3.8 (Lewis). *With the notations above, the sequence*

$$W^\varepsilon(D, -) \xrightarrow{\pi_1} W^\varepsilon(K, -) \xrightarrow{\beta} W^{-\varepsilon}(D, -) \xrightarrow{\pi_2} W^\varepsilon(K, Id)$$

is exact.

Proof. This is essentially the proof given in Appendix 2 of [2] with some changes due to the use of generalized quadratic forms, instead of hermitian forms. We first check that the sequence is a complex. Let $[k] \in Q_\sigma^\varepsilon(V, D)$ and let $V^0 = U$. We write elements of $U_D = U \oplus U\ell$ as pairs $(x, y\ell)$ and decompose $k_D = P + \ell R$. By definition we have $\beta\pi_1([k]) = [\beta(P)]$ and

$$\begin{aligned} \beta(P)((x_1, y_1), (x_2, y_2)) &= i(P(x_1, x_2) + P(x_1, y_2)\ell + \ell P(y_1, x_2) \\ &\quad + \ell P(y_1, y_2)\ell). \end{aligned}$$

Let $(x\ell, x\ell) \in U \oplus U\ell$. We get $\beta(P)((x\ell, x\ell), (x\ell, x\ell)) = 0$ hence $W = \{(x\ell, x\ell)\} \subset U \oplus U\ell$ is totally isotropic. It is easy to see that $W \subset W^\perp$, so that $[\beta(P)]$ is hyperbolic and $\beta \circ \pi_1 = 0$. Let $[g] \in Q_\sigma^\varepsilon(U, K)$. The subspace $W = \{(x, 0) \in U \oplus U\ell\}$ is totally isotropic for $\pi_2\beta([g])$ and $W \subset W^\perp$. Hence $\pi_2\beta([g]) = 0$. We now prove exactness at $W^\varepsilon(K, -)$. Since the claim is known if $\text{Char} \neq 2$, we may assume that $\text{Char} = 2$ and $\varepsilon = 1$. Let $[g] \in Q_\sigma^\varepsilon(U, K)$ be anisotropic such that $\beta([g]) = 0 \in W^{-\varepsilon}(D, -)$. In particular $\beta([g]) \in Q_\sigma^{-\varepsilon}(U_D, D)$ is isotropic. Hence there exist elements $x_1, x_2 \in U$ such that $[\tilde{g}]((x_1, x_2\ell), (x_1, x_2\ell)) = 0$. This implies (in $\text{Char } 2$) that

$$(4) \quad g(x_1, x_1) + \overline{\mu g(x_2, x_2)} \in F, \quad g(x_1, x_2)\ell + \ell g(x_2, x_1) = 0.$$

Let V_1 be the K -subspace of V generated by x_1 and x_2 . Since $[g]$ is anisotropic, $[g] = [g_1] \perp [g_2]$ with $g_1 = g|_{V_1}$. We make V_1 into a D -space by putting

$$(x_1a_1 + x_2a_2)\ell = \mu x_2\bar{a}_1 + x_1\bar{a}_2$$

To see that the action is well-defined, it suffices to show that $\dim_K V_1 = 2$. The elements x_1 and x_2 cannot be zero since $[g]$ is anisotropic, so assume $x_2 = x_1c$, $c \in K^\times$. Then (4) implies $g(x_1, x_1) + \mu c\bar{c}g(x_1, x_1) \in F$, which contradicts the fact that g is anisotropic. Let $g_1(x_1, x_1) + \mu g_1(x_2, x_2) = z \in F$. Let $f \in \text{Sesq}_\sigma(V_1, K)$. Replacing g_1 by $g_1 + f + f^*$ defines the same class in $Q_\sigma^\varepsilon(V_1, K)$ (recall that $\text{Char } F = 2$). Choosing f as

$$f(x_1, x_1) = jz, \quad f(x_2, x_2) = 0, \quad f(x_1, x_2) = f(x_2, x_1) = 0,$$

we may assume that

$$(5) \quad g_1(x_1, x_1) + \overline{\mu g_1(x_2, x_2)} = 0, \quad g_1(x_1, x_2)\ell + \ell g_1(x_2, x_1) = 0.$$

By (3.2) we may extend g_1 to a sesquilinear form

$$g'(x, y) = g_1(x, y) + \ell\mu^{-1}g_1(x\ell, y)$$

over D if g_1 satisfies

$$g_1(x\ell, y\ell) = -\overline{\mu g_1(x, y)}$$

This can easily be checked using (5) (and the definition of $x\ell$). Then g_1 is in the image of π_1 . Exactness at $W^\varepsilon(K, -)$ now follows by induction on the dimension

of U . We finally check exactness at $W^{-\varepsilon}(D, -)$. Let $[k]$ be anisotropic such that $\pi_2([k]) = 0$ in $W^{-\varepsilon}(K, Id)$. In particular $\pi_2([k])$ is isotropic; let $x \neq 0$ be such that $\pi_2 k(x, x) = 0$ and let W be the D -subspace of V generated by x . Since $[k]$ is anisotropic, $[k'] = [k|_W]$ is nonsingular and $[k] = [k'] \perp [k'']$. The condition $\pi_2 k(x, x) = 0$ implies $k(x, x) \in K$. Let W_1 be the K -subspace of W generated by x . Define $g : W_1 \times W_1 \rightarrow K$ by $g(xa, xb) = k(xa, xb)i^{-1}$ for $a, b \in K$. Then clearly $[g]$ defines an element of $W^\varepsilon(K, -)$ and $\beta(g) = k'$. Once again exactness follows by induction on the dimension of V . \square

4. INVOLUTIONS ON CENTRAL SIMPLE ALGEBRAS

Let D be a central division algebra over F , with involution σ and let $b : V \times V \rightarrow D$ be a nonsingular ε -hermitian form on a finite dimensional space over D . Let $A = \text{End}_D(V)$. The map $\sigma_b : A \rightarrow A$ such that $\sigma_b(\lambda) = \sigma(\lambda)$ for all $\lambda \in F$ and

$$b(\sigma_b(f)(x), y) = b(x, f(y))$$

for all $x, y \in V$, is an involution of A , called the involution *adjoint to b* . We have $\sigma_b(f) = \widehat{b}^{-1} f^* \widehat{b}$, where $\widehat{b} : V \xrightarrow{\sim} V^*$ is the adjoint of b . Conversely, any involution of A is adjoint to some nonsingular ε -hermitian form b and b is uniquely multiplicatively determined up to a σ -invariant element of F^\times .

Any automorphism ϕ of A compatible with σ_b , i.e., $\sigma_b(\phi(a)) = \phi(\sigma_b(a))$, is of the form $\phi(a) = uau^{-1}$ with $u : V \xrightarrow{\sim} V$ a similitude of b . We say that an involution τ of A is a *q -involution* if τ is adjoint to the polar b_θ of a (σ, ε) -quadratic form θ . We write $\tau = \sigma_\theta$. Two algebras with q -involutions are *isomorphic* if the isomorphism is induced by a similitude of the corresponding quadratic forms. Over fields q -involutions differ from involutions only in characteristic 2 and for symplectic involutions. In view of possible generalizations (for example rings in which $2 \neq 0$ is not invertible) we keep to the general setting of (σ, ε) -quadratic forms. Let F_0 be the subfield of F of σ -invariant elements and let T_{F/F_0} be the corresponding trace.

LEMMA 4.1. *The symmetric bilinear form on A given by $\text{Tr}(x, y) = T_{F/F_0}(\text{Trd}_A(xy))$ is nonsingular and $\text{Sym}(A, \tau)^\perp = \text{Alt}(A, \tau)$.*

Proof. If τ is of the first kind $F_0 = F$ and the claim is (2.3) of [6]. Assume that τ is of the second kind. Since the bilinear form $(x, y) \rightarrow \text{Trd}_A(xy)$ is nonsingular, Tr is also nonsingular and it is straightforward that $\text{Alt}(A, \tau) \subset \text{Sym}(A, \tau)^\perp$. Equality follows from the fact that $\dim_{F_0} \text{Alt}(A, \tau) = \dim_{F_0} \text{Sym}(A, \tau) = \dim_F A$. \square

PROPOSITION 4.2. *Let (V, θ) , $\theta = [k]$ be a (σ, ε) -quadratic space over D and let $h = \widehat{k} + \varepsilon \widehat{k}^* : V \xrightarrow{\sim} V^*$. The F_0 -linear form*

$$f_\theta : \text{Sym}(A, \sigma_\theta) \rightarrow F_0, \quad f_\theta(s) = \text{Tr}(h^{-1} \widehat{k} s), \quad s \in \text{Sym}(A, \sigma_\theta)$$

depends only on the class θ and satisfies $f_\theta(x + \sigma_\theta(x)) = \text{Tr}(x)$.

Proof. The first claim follows from (4.1) and the fact that if $k \in \text{Alt}_\sigma^\varepsilon(V, D)$ then $h^{-1}\widehat{k} \in \text{Alt}_{\sigma_\theta}^1(V, D)$. For the last claim we have:

$$\begin{aligned} f_\theta(x + \sigma_\theta(x)) &= \text{Tr}(h^{-1}\widehat{k}(x + \sigma_\theta(x))) \\ &= \text{Tr}(h^{-1}\widehat{k}x) + \text{Tr}(h^{-1}\widehat{k}h^{-1}x^*h) \\ &= \text{Tr}(h^{-1}\widehat{k}x) + \text{Tr}(\widehat{k}h^{-1}x^*) \\ &= \text{Tr}(h^{-1}\widehat{k}x) + \text{Tr}(x(h^{-1})^*\widehat{k}^*) \\ &= \text{Tr}(h^{-1}\widehat{k}x) + \text{Tr}(h^{-1}\varepsilon\widehat{k}^*x) = \text{Tr}(x). \end{aligned}$$

□

LEMMA 4.3. *Let τ be an involution of $A = \text{End}_D(V)$ and let f be a F_0 -linear form on $\text{Sym}(A, \tau)$ such that $f(x + \tau(x)) = \text{Tr}(x)$ for all $x \in A$. There exists an element $u \in A$ such that $f(s) = \text{Tr}(us)$ and $u + \tau(u) = 1$. The element u is uniquely determined up to additivity by an element of $\text{Alt}(A, \tau)$. We take $u = 1/2$ if $\text{Char } F \neq 2$.*

Proof. The proof of (5.7) of [6] can easily be adapted. □

PROPOSITION 4.4. *Let τ be an involution of $A = \text{End}_D(V)$ and let f be a F_0 -linear form on $\text{Sym}(A, \tau)$ such that $f(x + \tau(x)) = \text{Tr}(x)$ for all $x \in A$.*

- 1) *There exists a nonsingular (σ, ε) -quadratic form θ on V such that $\tau = \sigma_\theta$ and $f = f_\theta$.*
- 2) *$(\sigma_\theta, f_\theta) = (\sigma_{\theta'}, f_{\theta'})$ if and only if $\theta' = \lambda\theta$ for $\lambda \in F_0$.*
- 3) *If $\tau = \sigma_\theta$ and $f = f_\theta$ with $f_\theta(s) = \text{Tr}(us)$, the class of u in $A/\text{Alt}(A, \sigma_\theta)$ is uniquely determined by θ .*

Proof. Here the proof of (5.8) of [6] can be adapted. We prove 1) for completeness. Let $\tau(x) = h^{-1}x^*h$, $h = \varepsilon h^* : V \xrightarrow{\sim} V^*$. Let $f(s) = \text{Tr}(us)$ with $u + \tau(u) = 1$ and let $k \in \text{Sesq}_\sigma(V, D)$ be such that $\widehat{k} = hu : V \rightarrow V^*$. We set $\theta = [k]$. It is then straightforward to check that $h = k + \varepsilon k^*$. □

PROPOSITION 4.5. *Let $\phi : (\text{End}_D(V), \sigma_\theta) \xrightarrow{\sim} (\text{End}_D(V'), \sigma_{\theta'})$ be an isomorphism of algebras with involution. Let $f_\theta(s) = \text{Tr}(us)$ and $f_{\theta'}(s') = \text{Tr}(u's')$. The following conditions are equivalent:*

- 1) *ϕ is an isomorphism of algebras with q -involutions.*
- 2) *$f_{\theta'}(\phi(s)) = f_\theta(s)$ for all $s \in \text{Sym}(\text{End}_D(V), \sigma_\theta)$.*
- 3) *$[\phi(u)] = [u'] \in \text{End}_D(V')/\text{Alt}(\text{End}_D(V'), \sigma_{\theta'})$.*

Proof. The implication 1) \Rightarrow 2) is clear. We check that 2) \Rightarrow 3). Let ϕ be induced by a similitude $t : (V, b_\theta) \xrightarrow{\sim} (V', b_{\theta'})$. Since $f_{\theta'}(\phi s) = f_\theta(s)$, we have $\text{Tr}(t^{-1}u'ts) = \text{Tr}(u'tst^{-1}) = \text{Tr}(us)$ for all $s \in \text{Sym}(\text{End}_D(V), \sigma_\theta)$, hence $[\phi(u)] = [u']$. The implication 3) \Rightarrow 1) follows from the fact that u can be chosen as $h^{-1}\widehat{k}$, $h = \widehat{k} + \varepsilon\widehat{k}^*$. □

REMARK 4.6. We call the pair $(\sigma_\theta, f_\theta)$ a (σ, ε) -quadratic pair or simply a quadratic pair. It determines θ up to the multiplication by a σ -invariant scalar $\lambda \in F^\times$. In fact σ_θ determines the polar b_θ up to λ and f_θ determines u . We have $\theta = [\widehat{b_\theta u}]$.

EXAMPLE 4.7. Let $q : V \rightarrow F$ be a nonsingular quadratic form. The polar b_q induces an isomorphism $\psi : V \otimes_F V \xrightarrow{\sim} \text{End}_F(V)$ such that $\sigma_q(\psi(x \otimes y)) = \psi(y \otimes x)$. Thus $\psi(x \otimes x)$ is symmetric and $f_q(\psi(x \otimes x)) = q(x)$ (see [6, (5.11)]). More generally, if V is a right vector space over D , we denote by *V the space V viewed as a left D -space through the involution σ of D . The adjoint $\widehat{b_\theta}$ of a (σ, ε) -quadratic space (V, θ) induces an isomorphism $\psi_\theta : V \otimes_D {}^\sigma V \xrightarrow{\sim} \text{End}_D(V)$ and $\psi_\theta(xd \otimes x)$ is a symmetric element of $(\text{End}_D(V), \sigma_\theta)$ for all $x \in V$ and all ε -symmetric $d \in D$. One has $f_\theta(\psi(xd \otimes x)) = [dk(x, x)]$, where $\theta = [k]$ (see [4, Theorem 7]).

5. CLIFFORD ALGEBRAS

Let σ be an involution of the first kind on D and let θ be a nonsingular (σ, ε) -quadratic form on V . Let σ_θ be the corresponding q -involution on $A = \text{End}_D(V)$. We assume in this section that over a splitting $A \otimes_F \tilde{F} \xrightarrow{\sim} \text{End}_{\tilde{F}}(M)$ of A , $\theta_{\tilde{F}} = \theta \otimes 1_{\tilde{F}}$ is a $(Id, 1)$ -quadratic form \tilde{q} over \tilde{F} , i.e. $\theta_{\tilde{F}}$ is a (classical) quadratic form. In the terminology of [6] this means that σ_θ is orthogonal if $\text{Char} \neq 2$ and symplectic if $\text{Char} = 2$. From now on we call such forms over D *quadratic forms over D* , resp. *quadratic spaces over D* if the forms are non-singular.

Classical invariants of quadratic spaces (V, θ) are the dimension $\dim_D V$ and the discriminant $\text{disc}(\theta)$ and the Clifford invariant associated with the Clifford algebra. We refer to [6, §7] for the definition of the discriminant. We recall the definition of the Clifford algebra $\text{Cl}(V, \theta)$, following [10, 4.1]. Given (V, θ) as above, let $\theta = [k]$, $k \in \text{Sesq}_\sigma(V, D)$, $b_\theta = k + \varepsilon k^*$ and $h = \widehat{b_\theta} \in \text{Hom}_D(V, V^*)$. Let $A = \text{End}_D(V)$, $B = \text{Sesq}_\sigma(V, D)$ and $B' = V \otimes_D {}^\sigma V$. We identify A with $V \otimes_D {}^\sigma V^*$ through the canonical isomorphism $(x \otimes {}^\sigma f)(v) = xf(v)$ and B with $V^* \otimes_D {}^\sigma V^*$ through $(f \otimes {}^\sigma g)(x, y) = \overline{g(x)}f(y)$. The isomorphism h can be used to define further isomorphisms:

$$\varphi_\theta : B' = V \otimes_D {}^\sigma V \xrightarrow{\sim} A = \text{End}_D(M), \quad \varphi_\theta : x \otimes y \mapsto x \otimes h(y)$$

and the isomorphism ψ_θ already considered in (4.7):

$$\psi_\theta : A \xrightarrow{\sim} B, \quad \psi_\theta : x \otimes {}^\sigma f \mapsto h(x) \otimes {}^\sigma f.$$

We use φ_θ and ψ_θ to define maps $B' \times B \rightarrow A$, $(b', b) \mapsto b'b$ and $A \times B' \rightarrow B'$, $(a, b') \mapsto ab'$:

$$(x \otimes {}^\sigma y)(h(u) \otimes g) = xb(y, u) \otimes {}^\sigma f \text{ and } (x \otimes, {}^\sigma f)(u \otimes, {}^\sigma v) = xf(u) \otimes {}^\sigma h(v)$$

Furthermore, let $\tau_\theta = \varphi_\theta^{-1} \sigma_\theta \varphi_\theta : B' \rightarrow B'$ be the transport of the involution σ_θ on A . We have $\tau_\theta(x \otimes {}^\sigma y) = \varepsilon y \otimes {}^\sigma x$. Let $S_1 = \{s_1 \in B' \mid \tau_\theta(s_1) = s_1\}$. We

have $S_1 = (\text{Alt}^\varepsilon(V, D))^\perp$ for the pairing $B' \times B \rightarrow F$, $(b', b) \mapsto \text{Trd}_A(b'b)$. Let Sand be the bilinear map $B' \otimes B' \times B \rightarrow B'$ defined by $\text{Sand}(b'_1 \otimes b'_2, b) = b'_2 b b'_1$. The Clifford algebra $\text{Cl}(V, \theta)$ of the quadratic space (V, θ) is the quotient of the tensor algebra of the F -module B' by the ideal I generated by the sets

$$\begin{aligned} I_1 &= \{s_1 - \text{Trd}_A(s_1 k)1, s_1 \in S_1\} \\ I_2 &= \{c - \text{Sand}(c, k) \mid \text{Sand}(c, \text{Alt}^\varepsilon(V, D)) = 0\}. \end{aligned}$$

The Clifford algebra $\text{Cl}(V, \theta)$ has a canonical involution σ_0 induced by the map τ . We have $\text{Cl}(V, \theta) \otimes_F \tilde{F} = \text{Cl}(V \otimes_F \tilde{F}, \theta \otimes 1_{\tilde{F}})$ for any field extension \tilde{F} of F and $\text{Cl}(V, q)$ is the even Clifford algebra $C_0(V, q)$ of (V, q) if $D = F$ ([10, Théorème 2]). The reduction is through Morita theory for hermitian spaces (see for example [5, Chapter I, §9] for a description of Morita theory). In [6, §8] the Clifford algebra $C(A, \sigma_\theta, f_\theta)$ of the triple $(A, \sigma_\theta, f_\theta)$ is defined as the quotient of the tensor algebra $T(A)$ of the F -space A by the ideal generated by the sets

$$\begin{aligned} J_1 &= \{s - \text{Trd}_A(us), s \in \text{Sym}(A, \sigma_\theta)\} \\ J_2 &= \{c - \text{Sand}'(c, u), c \in A \text{ with } \text{Sand}'(c, \text{Alt}(A, \sigma_\theta)) = 0\} \end{aligned}$$

where $u = \widehat{b}_\theta^{-1} k$ and $\text{Sand}' : (A \otimes A, A) \rightarrow A$ is defined as $\text{Sand}'(a \otimes b, x) = axb$. The two definitions give in fact isomorphic algebras:

PROPOSITION 5.1. *The isomorphism $\varphi_\theta : V \otimes_D {}^\sigma V \xrightarrow{\sim} \text{End}_D(V)$ induces an isomorphism $\text{Cl}(V, \theta) \xrightarrow{\sim} C(A, \sigma_\theta, f_\theta)$.*

Proof. We only check that φ_θ maps I_1 to J_1 . By definition of τ and S_1 , $s = \varphi_\theta(s_1)$ is a symmetric element of A . On the other hand we have by definition of the pairing $B' \times B \rightarrow A$,

$$\begin{aligned} \text{Trd}_A(s_1 k) &= \text{Trd}_A(\varphi_\theta(s_1) \psi_\theta^{-1}(k)) \\ &= \text{Trd}_A(sh^{-1} \widehat{k}) = \text{Trd}_A(su) = \text{Trd}_A(us), \end{aligned}$$

hence the claim. □

In particular we have $C(\text{End}_F(V), \sigma_q, f_q) = C_0(V, q)$ for a quadratic space (V, q) over F . It is convenient to use both definitions of the Clifford algebra of a generalized quadratic space.

Let $D = [K, \mu] = K \oplus \ell K$ be a quaternion algebra with conjugation σ . Let V be a D -module and let V^0 be V as a right vector space over K (through restriction of scalars). Let $T : V^0 \rightarrow V^0$, $Tx = x\ell$. We have $\text{End}_D(V) \subset \text{End}_K(V^0)$ and

$$\text{End}_D(V) = \{f \in \text{End}_K(V^0) \mid fT = Tf\}.$$

Let $\theta = [k]$ be a $(\sigma, -1)$ -quadratic space and let $k(x, y) = P(x, y) + \ell R(x, y)$ as in Section 3. It follows from (3.1) that R defines a quadratic space $[R]$ on V^0 over K .

PROPOSITION 5.2. *We have $\sigma_{[R]}|_{\text{End}_D(V)} = \sigma_\theta$ and $f_\theta = f_{[R]}|_{\text{End}_D(V)}$.*

Proof. We have an embedding $D \hookrightarrow M_2(K)$, $a + \ell b \mapsto \begin{pmatrix} a & \mu\bar{b} \\ b & \bar{a} \end{pmatrix}$ and conjugation given by $x \mapsto x^* = c^{-1}x^t c$, $c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The choice of a basis of V over D identifies V with D^n , V^0 with K^{2n} , $\text{End}_D(V)$ with $M_n(D)$ and $\text{End}_K(V^0)$ with $M_{2n}(K)$, where $n = \dim_D V$. We further identify V and V^* through the choice of the dual basis. We embed any element $x = x_1 + \ell x_2 \in M_{k,l}(D)$, $x_i \in M_{k,l}(K)$ in $M_{2k,2l}(K)$ through the map $\iota : x \mapsto \xi = \begin{pmatrix} x_1 & \mu\bar{x}_2 \\ x_2 & \bar{x}_1 \end{pmatrix}$. In particular D^n is identified with a subspace of the space of $(2n \times 2)$ -matrices over K . Then $D \subset M_2(K)$ operates on the right through (2×2) -matrices and $M_n(D) \subset M_{2n}(K)$ operates on the left through $(2n \times 2n)$ -matrices. With the notations of Example (2.3) we have $\iota(x^*) = \text{Int}(c^{-1})(x^t)$. Any D -sesquilinear form k on D^n can be written as $k(x, y) = x^* a y$, where $a \in M_n(D)$, as in (2.3). Let $a = a_1 + \ell a_2$, $a_i \in M_n(K)$ and let

$$\alpha = \iota(a) = \begin{pmatrix} a_1 & \mu\bar{a}_2 \\ a_2 & \bar{a}_1 \end{pmatrix}.$$

Let $\eta = \iota(y)$, $y = y_1 + \ell y_2$. We have

$$k(x, y) = x^* a y = \xi^* \alpha \eta = \begin{pmatrix} x_1 & \mu\bar{x}_2 \\ x_2 & \bar{x}_1 \end{pmatrix}^* \begin{pmatrix} a_1 & \mu\bar{a}_2 \\ a_2 & \bar{a}_1 \end{pmatrix} \begin{pmatrix} y_1 & \mu\bar{y}_2 \\ y_2 & \bar{y}_1 \end{pmatrix}.$$

On the other side it follows from $h = P + \ell R$ that $R(x, y) = \xi^t \rho \eta$ with

$$\rho = \begin{pmatrix} a_2 & \bar{a}_1 \\ -a_1 & -\mu\bar{a}_2 \end{pmatrix}.$$

Assume that $\theta = [k]$, so that σ_θ corresponds to the involution $\text{Int}(\gamma^{-1}) \circ *$, where $\gamma = \alpha - \alpha^*$. Similarly $\sigma_{[R]}$ corresponds to the involution $\text{Int}(\tilde{\rho}^{-1}) \circ t$ where $\tilde{\rho} = \rho + \rho^t$. We obviously have $\rho = c\alpha$ with $c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so that $\rho^t = \alpha^t c^t = -\alpha^t c = -c\alpha^*$ and $\rho + \rho^t = c(\alpha - \alpha^*)$ or $c\gamma = \tilde{\rho}$. Now $*$ = $\text{Int}(c^{-1}) \circ t$ implies $\sigma_{[R]}|_{M_n(D)} = \sigma_\theta$. We finally check that $f_\theta = f_{[R]}|_{\text{Sym}(M_n(D), \sigma_\theta)}$. We have $f_\theta(s) = \text{Trd}_{M_n(D)}(\gamma^{-1}\alpha s)$ and $f_{[R]}(s) = \text{Trd}_{M_{2n}(K)}(\tilde{\rho}^{-1}\rho s)$, hence the claim, since $\rho = c\alpha$ and $\tilde{\rho} = c\gamma$ implies $\gamma^{-1}\alpha = \tilde{\rho}^{-1}\rho$. \square

COROLLARY 5.3. *The embedding $\text{End}_D(V) \hookrightarrow \text{End}_K(V^0)$ induces*

- 1) *an isomorphism $(\text{End}_D(V), \sigma_\theta, f_\theta) \otimes K \xrightarrow{\sim} (\text{End}_K(V^0), \sigma_{[R]}, f_{[R]})$,*
- 2) *an isomorphism $C(\text{End}_D(V), \sigma_\theta, f_\theta) \otimes K \xrightarrow{\sim} C_0(V^0, [R])$.*

In view of (2) the semilinear automorphism $T : V^0 \xrightarrow{\sim} V^0$, $Tx = x\ell$, is a semilinear similitude with multiplier $-\mu$ of the quadratic form $[R]$, such that $T^2 = \mu$.

LEMMA 5.4. *The map T induces a semilinear automorphism $C_0(T)$ of $C_0(V^0, R)$ such that*

$$C_0(T)(xy) = (-\mu)^{-1}T(x)T(y) \text{ for } x, y \in V^0$$

and $C_0(T)^2 = Id$.

Proof. This follows (for example) as in [6, (13.1)] □

PROPOSITION 5.5.

$$C(\text{End}_D(V), \sigma_\theta, f_\theta) = \{c \in C_0(V^0, R) \mid C_0(T)(c) = c\}.$$

Proof. The claim follows from the defining relations of $C(\text{End}_D(V), \sigma_\theta, f_\theta)$ and the fact that

$$\text{End}_D(V) = \{f \in \text{End}_K(V^0) \mid T^{-1}fT = f\}.$$

□

We call $C(\text{End}_D(V), \sigma_\theta, f_\theta)$ or equivalently $\text{Cl}(V, \theta)$ the *Clifford algebra of the quadratic quaternion space* (V, θ) .

Let t be a semilinear similitude of a quadratic space (U, q) of even dimension over K . Assume that $\text{disc}(q)$ is trivial, so that $C_0(U, q)$ decomposes as product of two K -algebras $C^+(U, q)$ and $C^-(U, q)$. We say that t is *proper* if $C_0(t)(C^\pm(U, q)) \subset C^\pm(U, q)$ and we say that t is *improper* if $C_0(t)(C^\pm(U, q)) \subset C^\mp(U, q)$. In general we say that t is *proper* if t is proper over some field extension of F which trivializes $\text{disc}(q)$. For any semilinear similitude t , let $d(t) = 1$ is t if proper and $d(t) = -1$ if t is improper.

LEMMA 5.6. *Let t_i be a semilinear similitude of (U_i, q_i) , $i = 1, 2$. We have $d(t_1 \perp t_2) = d(t_1)d(t_2)$.*

Proof. We assume that $\text{disc}(q_i)$, $i = 1, 2$, is trivial. Let e_i be an idempotent generating the center Z_i of $C_0(q_i)$. We have $t_i(e_i) = e_i$ if t_i is proper and $t_i(e_i) = 1 - e_i$ if t_i is improper. The idempotent $e = e_1 + e_2 - 2e_1e_2 \in C_0(q_1 \perp q_2)$ generates the center of $C_0(q_1 \perp q_2)$ (see for example [5, (2.3), Chap. IV]) and the claim follows by case checking. □

LEMMA 5.7. *Let V, θ, V^0, R and T be as above. Let $\dim_K V^0 = 2m$. Then T is proper if m is even and is improper if m is odd.*

Proof. The quadratic space (V, θ) is the orthogonal sum of 1-dimensional spaces and we get a corresponding orthogonal decomposition of $(V^0, [R])$ into subspaces (U_i, q_i) of dimension 2. In view of (5.6) it suffices to check the case $m = 1$. Let $\alpha = a = a_1 + la_2 \in D$ and $\rho = \begin{pmatrix} a_2 & \bar{a}_1 \\ -a_1 & -\mu\bar{a}_2 \end{pmatrix}$. We choose $\mu = 1$, $a_1 = j$ (j as in (2.4)), put $i = 1 - 2j$, so that $\bar{i} = -i$ and choose $a_2 = 0$. Let $x = x_1e_1 + x_2e_2 \in V^0$, so $[R](x_1, x_2) = ix_1x_2$ and $C([R])$ is generated by e_1, e_2 with the relations $e_1^2 = 0, e_2^2 = 0, e_1e_2 + e_2e_1 = i$. The element $e = i^{-1}e_1e_2$ is an idempotent generating the center. Since $T(x_1e_1 + x_2e_2) = \bar{x}_2e_1 + \bar{x}_1e_2$, we have $C_0(T)(e_1e_2) = -e_2e_1$ and $C_0(T)(e) = 1 - e$. Thus T is not proper. □

Of special interest for the next section are quadratic quaternion forms $[k]$ such that the induced quadratic forms $\pi_2([k])$ are Pfister forms. For convenience we call such forms *Pfister quadratic quaternion forms*. Hyperbolic spaces of dimension 2^n are Pfister forms, hence spaces of the form $\beta([b])$, b a hermitian form over K , are Pfister, in view of the exactness of the sequence of Lewis [7]. It is in fact easy to give explicit examples of Pfister forms using the following constructions:

EXAMPLE 5.8 (Char $F \neq 2$). Let $q = \langle \lambda_1, \dots, \lambda_n \rangle$ be a diagonal quadratic form on F^n , i.e., $q(x) = \sum \lambda_i x_i^2$. Let $[k]$ on D^n be given by the diagonal form ℓq . Then the corresponding quadratic form $[R]$ on K^{2n} is given by the diagonal form $\langle 1, -\mu \rangle \otimes q$. In particular we get the 3-Pfister form $\langle\langle a, b, \mu \rangle\rangle$ choosing for q the norm form of a quaternion algebra $(a, b)_F$.

EXAMPLE 5.9 (Char $F = 2$). Let $b = \langle \lambda_1, \dots, \lambda_n \rangle$ be a bilinear diagonal form on F^n , i.e., $b(x, y) = \sum \lambda_i x_i y_i$. Let $k = (j + \ell)b$ on D^n . Then the corresponding quadratic form $[R]$ over $K = R(j)$, $j^2 = j + \lambda$, is given by the form $[R] = b \otimes [1, \lambda]$ where $[\xi, \eta] = \xi x_1^2 + x_1 x_2 + \eta x_2^2$. In particular, for $b = \langle 1, a, c, ac \rangle$, we get the 3-Pfister form $\langle\langle a, c, \lambda \rangle\rangle$ with the notations of [6], p. xxi.

6. TRIALITY FOR SEMILINEAR SIMILITUDES

Let \mathfrak{C} be a Cayley algebra over F with conjugation $\pi : x \mapsto \bar{x}$ and norm $\mathfrak{n} : x \mapsto x\bar{x}$. The new multiplication $x \star y = \bar{x}\bar{y}$ satisfies

$$(6) \quad x \star (y \star x) = (x \star y) \star x = \mathfrak{n}(x)y$$

for $x, y \in \mathfrak{C}$. Further, the polar form $b_{\mathfrak{n}}$ is *associative* with respect to \star , in the sense that

$$b_{\mathfrak{n}}(x \star y, z) = b_{\mathfrak{n}}(x, y \star z).$$

PROPOSITION 6.1. For $x, y \in \mathfrak{C}$, let $r_x(y) = y \star x$ and $\ell_x(y) = x \star y$. The map $\mathfrak{C} \rightarrow \text{End}_F(\mathfrak{C} \oplus \mathfrak{C})$ given by

$$x \mapsto \begin{pmatrix} 0 & \ell_x \\ r_x & 0 \end{pmatrix}$$

induces isomorphisms $\alpha : (C(\mathfrak{C}, \mathfrak{n}), \tau) \xrightarrow{\sim} (\text{End}_F(\mathfrak{C} \oplus \mathfrak{C}), \sigma_{\mathfrak{n} \perp \mathfrak{n}})$ and

$$(7) \quad \alpha_0 : (C_0(\mathfrak{C}, \mathfrak{n}), \tau_0) \xrightarrow{\sim} (\text{End}_F(\mathfrak{C}), \sigma_{\mathfrak{n}}) \times (\text{End}_F(\mathfrak{C}), \sigma_{\mathfrak{n}}),$$

of algebras with involution.

Proof. We have $r_x(\ell_x(y)) = \ell_x(r_x(y)) = \mathfrak{n}(x) \cdot y$ by (6). Thus the existence of the map α follows from the universal property of the Clifford algebra. The fact that α is compatible with involutions is equivalent to

$$b_{\mathfrak{n}}(x \star (z \star y), u) = b_{\mathfrak{n}}(z, y \star (u \star x))$$

for all x, y, z, u in \mathfrak{C} . This formula follows from the associativity of $b_{\mathfrak{n}}$. Since $C(\mathfrak{C}, \mathfrak{n})$ is central simple, the map α is an isomorphism by a dimension count. \square

Assume from now on that \mathfrak{C} is defined over a field K which is quadratic Galois over F . Any proper semilinear similitude t of \mathfrak{n} induces a semilinear automorphism $C(t)$ of the even Clifford algebra $(C_0(\mathfrak{C}, \mathfrak{n}), \tau_0)$, which does not permute the two components of the center of $C_0(\mathfrak{C}, \mathfrak{n})$. Thus $\alpha_0 \circ C_0(t) \circ \alpha_0^{-1}$ is a pair of semilinear automorphisms of $(\text{End}_K(\mathfrak{C}), \sigma_{\mathfrak{n}})$. It follows as in (4.5) that, for any quadratic space (V, q) , semilinear automorphisms of $(\text{End}_K(V), \sigma_q, f_q)$ are of the form $\text{Int}(f)$, where f is a semilinear similitude of q . The following result is due to Wonenburger [12] in characteristic different from 2:

PROPOSITION 6.2. *For any proper semilinear similitude t_1 of \mathfrak{n} with multiplier μ_1 , there exist proper semilinear similitudes t_2, t_3 such that*

$$\alpha_0 \circ C_0(t_1) \circ \alpha_0^{-1} = (\text{Int}(t_2), \text{Int}(t_3))$$

and

$$(8) \quad \begin{aligned} \mu_3^{-1}t_3(x \star y) &= t_1(x) \star t_2(y), \\ \mu_1^{-1}t_1(x \star y) &= t_2(x) \star t_3(y), \\ \mu_2^{-1}t_2(x \star y) &= t_3(x) \star t_1(y). \end{aligned}$$

Let t_1 be an improper similitude with multiplier μ_1 . There exist improper similitudes t_2, t_3 such that

$$\begin{aligned} \mu_3^{-1}t_3(x \star y) &= t_1(y) \star t_2(x), \\ \mu_1^{-1}t_1(x \star y) &= t_2(y) \star t_3(x), \\ \mu_2^{-1}t_2(x \star y) &= t_3(y) \star t_1(x). \end{aligned}$$

The pair (t_2, t_3) is determined by t_1 up to a factor (λ, λ^{-1}) , $\lambda \in K^\times$, and we have $\mu_1\mu_2\mu_3 = 1$.

Furthermore, any of the formulas in (8) implies the two others.

Proof. The proof given in [6, (35.4)] for similitudes can also be used for semilinear similitudes. \square

REMARK 6.3. The class of two of the t_i , $i = 1, 2, 3$, modulo K^\times is uniquely determined by the class of the third t_i .

COROLLARY 6.4. *Let T_1 be a proper semilinear similitude of $(\mathfrak{C}, \mathfrak{n})$ such that $T_1^2 = \mu_1$, $\mu_1 \in K^\times$ and with multiplier $-\mu_1$. There exist elements $a_i \in K^\times$, $i = 1, 2, 3$, and proper semilinear similitudes T_i of $(\mathfrak{C}, \mathfrak{n})$, with $T_i^2 = \mu_i$, $\mu_i \in K^\times$ and with multiplier $-\mu_i$, $i = 2, 3$, such that $a_i\bar{a}_i\mu_i = \mu_{i+1}\mu_{i+2}$ and*

$$\begin{aligned} a_3T_3(x \star y) &= T_1(x) \star T_2(y) \\ a_1T_1(x \star y) &= T_2(x) \star T_3(y) \\ a_2T_2(x \star y) &= T_3(x) \star T_1(y) \end{aligned}$$

The class of any T_i modulo K^\times determines the two other classes and the μ_i 's are determined up to norms from K^\times . Furthermore any of the three formulas determines the two others.

Proof. Counting indices modulo 3, we have relations

$$T_i(x) \star T_{i+1}(y) = b_{i+2} T_{i+2}, \quad b_i \in K^\times$$

in view of (6.2). If we replace all T_j by $T_j \circ \rho_{\nu_j}$, $\nu_j \in K^\times$, we get new constants a_i . The claim then follows from (3.3). \square

7. TRIALITY FOR QUADRATIC QUATERNION FORMS

Let $D_1 = K \oplus \ell_1 K = [K, \mu_1]$ be a quaternion algebra over F and let (V_1, q_{θ_1}) be a quaternion quadratic space of dimension 4 over D_1 . Let $\theta_1 = [h_1]$, $h_1(x, y) = P_1(x, y) + \ell R_1(x, y)$, so that $[R_1] = \pi_2(\theta_1)$ corresponds to a 8-dimensional (classical) quadratic form on V_1^0 over K . The map $T_1 : V_1^0 \rightarrow V_1^0$, $T_1(x) = x\ell_1$, is a semilinear similitude of $(V_1^0, [R_1])$ with multiplier $-\mu_1$ and such that $T_1^2 = \mu_1$. We recall that by (3.5) it is equivalent to have a quadratic quaternion space (V_1, q_{θ_1}) or a pair $(V_1^0, [T_1])$. We assume from now on that the quadratic form $q_{[R_1]}$ is a 3-Pfister form, i.e., the norm form \mathfrak{n} of a Cayley algebra \mathfrak{C} over K . In view of (6.4) T_1 induces two semilinear similitudes T_2 , resp. T_3 , with multipliers μ_2 , resp. μ_3 , which in turn define a quaternion quadratic space (V_2, θ_2) of dimension 4 over $D_2 = [K, \mu_2]$, resp. a quaternion quadratic space (V_3, θ_3) of dimension 4 over $D_3 = [K, \mu_3]$. Let $\text{Br}(F)$ be the Brauer group of F .

PROPOSITION 7.1. 1) $[D_1][D_2][D_3] = 1 \in \text{Br}(F)$,

2) The restriction of $\alpha : C_0(\mathfrak{C}, \mathfrak{n}) \xrightarrow{\sim} \text{End}_K(\mathfrak{C}) \times \text{End}_K(\mathfrak{C})$ to $C(V_i, D_i, \theta_i)$ induces isomorphisms

$$\alpha_i : (C(V_i, D_i, \theta_i), \tau) \xrightarrow{\sim} (\text{End}_{D_{i+1}}(V_{i+1}), \sigma_{\theta_{i+1}}) \times (\text{End}_{D_{i+2}}(V_{i+2}), \sigma_{\theta_{i+2}})$$

Proof. The first claim follows from the fact that $\mu_1\mu_2 = \mu_3 \text{Nrd}_{D_3}(a_3)$ and the second is a consequence of (5.5), (3.5) and the definition of α . \square

EXAMPLE 7.2. Let \mathfrak{C}_0 be a Cayley algebra over F and let $\mathfrak{C} = \mathfrak{C}_0 \otimes_F K$. For any $c \in \mathfrak{C}_0$ such that $c^2 = \mu_1 \in F^\times$, $T_1 : \mathfrak{C} \rightarrow \mathfrak{C}$ given by $T_1(k \otimes x) = \bar{k} \otimes xc$ is a semilinear similitude with multiplier $-\mu_1$ such that $T_1^2 = \mu_1$. The Moufang identity $(cx)(yc) = c(xy)c$ in \mathfrak{C} implies that

$$(xc) \star (cy) = \bar{c}(x \star y)\bar{c}.$$

Thus $T_2(k \otimes y) = \bar{k} \otimes cy$ and $T_3(k \otimes z) = i\bar{k} \otimes \bar{c}z\bar{c}$ (where $i \in K^\times$ is such that $\bar{i} = -i$) satisfy (6.4). The corresponding triple of quaternion algebras is $([K, \mu_1], [K, \mu_1], [K, i\bar{i}\mu_1^2])$, the third algebra being split.

EXAMPLE 7.3. Let D_i , $i = 1, 2, 3$, be quaternion algebras over F such that $[D_1][D_2][D_3] = 1 \in \text{Br}(F)$. We may assume that the D_i contain a common separable quadratic field K and that $D_i = [K, \mu_i]$, $\mu_i \in F^\times$ such that $\mu_1\mu_2\mu_3 \in$

$F^{\times 2}$. In [6, (43.12)] similitudes S_i with multiplier μ_i , $i = 1, 2, 3$, of the split Cayley algebra \mathfrak{C}_s over F are given, such that 1) $\mu_3^{-1}S_3(x \star y) = S_1(x) \star S_2(y)$ and 2) $S_i^2 = \mu_i$. Let $\mathfrak{C} = K \otimes \mathfrak{C}_s$. Let $u \in K^\times$ be such that $\bar{u} = -u$. The semilinear similitudes $T_i(k \otimes x) = u\bar{k} \otimes S_i(x)$, $i = 1, 2, 3$, satisfy

$$a_3 T_3(x \star y) = T_1(x) \star T_2(y)$$

with $a_3 = u\mu_3^{-1}$ (we use the same notation \star in \mathfrak{C}_s and in \mathfrak{C}). Thus there exist a triple of quadratic quaternion forms $(\theta_1, \theta_2, \theta_3)$ corresponding to the three given quaternion algebras. We hope to describe the corresponding quadratic quaternion forms in a subsequent paper.

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