A Weak Hasse Principle for Central Simple Algebras with an Involution

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Abstract. The notions of totally indefinite and weakly isotropic algebras with involution are introduced and a proof is given of the fact that a field satisfies the Effective Diagonalization Property (ED) if and only if it satisfies the following weak Hasse principle: every totally indefinite central simple algebra with involution of the first kind over the given field is weakly isotropic. This generalizes a known result from quadratic form theory.

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1. Introduction

Let F be a field of characteristic different from two and let (A, σ) be a central simple algebra over F with involution of the first kind (i.e. $\sigma|_F = \mathbf{1}_F$). Recall that σ is called *orthogonal* (resp. *symplectic*) if σ is adjoint to a symmetric (resp. skew-symmetric) bilinear form, after scalar extension to a splitting field of A.

The connection between orthogonal involutions and quadratic forms has been a motivation for extending quadratic form theoretic concepts and theorems to the realm of algebras with involution (of any kind). For example, the classical invariants (discriminant, Clifford algebra, signature) of quadratic forms have been defined for algebras with involution (see [7]) and classification theorems à la Elman and Lam [5] have been obtained by Lewis and Tignol [14]. Some more examples include: a Cassels-Pfister theorem [19], an orthogonal sum for Morita-equivalent algebras with involution [2] and analogues of the Witt ring [12, 3].

In this paper we will examine the extension to central simple algebras with an involution of the first kind of the following weak Hasse principle for weak isotropy:

(WH): *Every totally indefinite quadratic form over F is weakly isotropic* and prove an analogue of the following theorem due to Prestel [15] and Elman et al. [4]:

Theorem 1.1. *F satisfies* (WH) $\iff$ *F satisfies the Strong Approximation Property.*

In particular, we will show that every totally indefinite central simple F-algebra with involution of the first kind is weakly isotropic if and only if F satisfies the Effective Diagonalization Property. Our result can also be re-interpreted to give a partial generalization of a theorem of Lewis [11] on sums of squares representing zero in a central simple algebra.
We mention that there is a refined (and more difficult) version of Theorem 1.1, which holds for arbitrary base fields, due to Bröcker [1, 3.9] and Prestel [16, p. 93]. It says that if ϕ is a totally indefinite quadratic form over a field F, and if for every valuation with real residue class field, at least one residue class form of ϕ is weakly isotropic, then ϕ is weakly isotropic. This statement can also be found in [18, 3.7.12]. Its converse is easily seen to be true.
All involutions on central simple algebras considered in this paper are of the first kind and all forms (quadratic, hermitian, ...) are assumed to be nonsingular. Standard references are [8] and [18] for the theory of quadratic forms, [7] for central simple algebras with an involution and [16] for real fields.

2. Weakly isotropic and totally indefinite algebras

In this section we will generalize the notions of totally indefinite and weakly isotropic quadratic forms to the setting of central simple algebras (A, σ) with an involution of the first kind over a field F of characteristic $\neq 2$. We denote the space of orderings of F by X_F and an arbitrary ordering of F by P.

Definition 2.1. Let (A, σ) be a central simple F-algebra with involution of the first kind. A right ideal I in A is called *isotropic* (with respect to the involution σ) if for all x and y in I we have that $\sigma(x)y = 0$. The algebra with involution (A, σ) is called *isotropic* if A contains a nonzero isotropic right ideal, or equivalently, if there exists an idempotent $e \neq 0$ in A such that $\sigma(e)e = 0$ (see [7, 6.A]). We also say that (A, σ) is *anisotropic* if for $x \in A$, $\sigma(x)x = 0$ implies $x = 0$.

Recall that a quadratic form q over F is weakly isotropic if there exists an $n \in \mathbb{N}$ such that $n \times q$ is isotropic.

Definition 2.2. The algebra with involution (A, σ) is called *weakly isotropic* if there exist nonzero $x_1, \ldots, x_n \in A$ such that $\sigma(x_1)x_1 + \cdots + \sigma(x_n)x_n = 0$ and *strongly anisotropic* otherwise.

Remark 2.3. In [21] an n-fold orthogonal sum $\boxplus^n(A, \sigma)$ is defined and it is shown there that $\boxplus^n(A, \sigma) \cong (M_n(F), t) \otimes_F (A, \sigma)$, where t denotes the transpose involution. This is on the one hand in accordance with Dejaiffe's [2] construction of an orthogonal sum of two Morita-equivalent algebras with involution and on the other hand what one would expect since $n \times q = \langle\underbrace{1, \dots, 1}_{n\times}\rangle \otimes q$ and $\boxplus^n(A,\sigma)$ reduces to $n \times q$ in the split case when σ is adjoint to a quadratic form q. In analogy with the quadratic form case, one could define (A,σ) to be weakly isotropic by requiring that $\boxplus^n(A,\sigma)$ is isotropic for some positive integer n and it is easy to see that this condition is equivalent with the one given in Definition 2.2.

Let (D,ϑ) be a central division algebra over F with involution of the first kind and (V,h) an ε-hermitian form over (D,ϑ), $\varepsilon = \pm 1$. Recall [7, 4.A] that the adjoint involution σ_h of h on $\operatorname{End}_D(V)$ is implicitly defined by

$$h\bigl(x, f(y)\bigr) = h\bigl(\sigma_h(f)(x), y\bigr) \quad \text{for } x, y \in V \text{ and } f \in \operatorname{End}_D(V)$$

and that σ_h is also of the first kind.

Just as for quadratic forms, we say that the ε-hermitian form h is weakly isotropic if there exists a positive integer n such that $n \times h$ is isotropic.

Lemma 2.4. *Let (D,ϑ), (V,h) and σ_h be as above. Then $(\operatorname{End}_D(V),\sigma_h)$ is weakly isotropic if and only if h is weakly isotropic. More precisely, there exist $f_1,\dots,f_n \in \operatorname{End}_D(V)$ such that $\sigma_h(f_1)f_1 + \cdots + \sigma_h(f_n)f_n = 0$ if and only if there exist $x_1,\dots,x_n \in V$ such that $h(x_1,x_1) + \cdots + h(x_n,x_n) = 0$.*

Proof. The lemma is folklore, and we only give the argument since we couldn't find a suitable reference. It suffices to show that $(\operatorname{End}_D(V),\sigma_h)$ is isotropic if and only if h is isotropic.

If σ_h is isotropic, there is $0 \neq f \in \operatorname{End}_D(V)$ with $\sigma_h(f)f = 0$. Choose $v \in V$ with $f(v) \neq 0$. Then

$$0 = h\bigl(\sigma_h(f)(f(v)),\, v\bigr) = h\bigl(f(v),\, f(v)\bigr)$$

shows that h is isotropic. Conversely, if $h(v,v) = 0$ for some $v \in V$, then $\sigma_h(f)f = 0$ for any $f \in \operatorname{End}_D(V)$ with $f(V) \subset vD$. ∎

Corollary 2.5. *Let $(\operatorname{End}_F(V),\sigma_q)$ be a split algebra with involution, adjoint to a quadratic form q on V. Then there exist $f_1,\dots,f_n \in \operatorname{End}_F(V)$ such that $\sigma_q(f_1)f_1 + \cdots + \sigma_q(f_n)f_n = 0$ if and only if there exist $x_1,\dots,x_n \in V$ such that $q(x_1) + \cdots + q(x_n) = 0$.*

Now suppose that F is a real field and that P is an ordering of F. In [13] Lewis and Tignol defined the signature of an algebra (A,σ) with involution of the first kind as

$$\operatorname{sig}_P \sigma = \sqrt{\operatorname{sig}_P T_\sigma},$$

where T_σ is the involution trace form, defined by $T_\sigma(x) := \operatorname{Trd}_A(\sigma(x)x), \forall x \in A$. If (A,σ) is split with orthogonal involution, $(A,\sigma) \cong (\operatorname{End}_F(V),\sigma_q)$, then Lewis and Tignol showed that $\operatorname{sig}_P \sigma_q = |\operatorname{sig}_P q|$.

Recall that a quadratic form q over F is called totally indefinite if it is indefinite for each ordering P of F, i.e. $|\operatorname{sig}_P q| < \dim q$ for each P.

Definition 2.6. The algebra with involution (A, σ) is called *indefinite* for the ordering P of F if $\operatorname{sig}_P \sigma < \deg A$ and *totally indefinite* if it is indefinite for each ordering P of F.

3. The weak Hasse principle

We now have all the ingredients ready to generalize (WH) to:

(WHA): *Every totally indefinite algebra with involution of the first kind over F is weakly isotropic.*

In [20, Ch. 5] Unger showed that (WHA) holds for fields with a unique ordering, algebraic number fields and $\mathbb{R}(t)$. These fields are some of the standard examples of SAP fields, as described below.

Definition 3.1. The field F satisfies the *Strong Approximation Property* (or is SAP, for short) if the following equivalent conditions hold:

- (i) Every clopen subset of X_F has the form $\{P \in X_F \mid a >_P 0\}$ for some $a \in F^\times$.
- (ii) For all $a, b \in F^\times$ the quadratic form $\langle 1, a, b, -ab \rangle$ is weakly isotropic.
- (iii) Every quadratic form q such that a power of q is weakly isotropic, is itself weakly isotropic.
- (iv) For any two disjoint closed subsets X, Y of X_F, there exists an $a \in F^\times$ such that $a >_P 0, \forall P \in X$ and $a <_P 0, \forall P \in Y$.
- (v) For every (Krull) valuation $v : F^\times \twoheadrightarrow \Gamma$ with value group Γ and real residue class field $\overline{F}_v$, either (a) or (b) holds:
  - (a) $\Gamma = 2\Gamma$;
  - (b) $|\Gamma/2\Gamma| = 2$ and $\overline{F}_v$ has a unique ordering.

Condition (iv) is the original definition of SAP fields, due to Knebusch et al. [6, Thm. 12]. The equivalence (i) $\iff$ (iv) is given in [6, Thm. 12, Cor. 13]. Prestel [15, (2.2), (3.1)] showed (ii) $\iff$ (iii) $\iff$ (v) $\iff$ F is a Pasch field, while the equivalence F is SAP $\iff$ F is Pasch can be found in [4, Thm. C]. The notion of a Pasch field was first introduced by Prestel; for a definition we refer the reader to [15]. Additional references for SAP fields are the monographs by Lam [9] and Prestel [16].

Example 3.2. Here are some examples of SAP fields:

1. Fields with only one ordering.
2. Algebraic number fields.
3. Fields of transcendence degree ≤ 1 over a real-closed field, e.g. $\mathbb{R}(t)$.
4. $F((t))$ if F has at most one ordering.

The following fields are not SAP:

5. The rational function field $\mathbb{Q}(x)$.
6. The rational function field $F(x, y)$, where F is any real field.

Based on the results in [20, Ch. 5] it was tempting to think that (WHA) would hold for all SAP fields. The Strong Approximation Property is definitely required, for if F is not SAP, we can construct a counterexample as follows:

There exist $a, b \in F^\times$ such that $q := \langle 1, a, b, -ab\rangle$ is strongly anisotropic. Hence the algebra $(A, \sigma) = (\mathrm{End}_F(F^4), \sigma_q)$ is strongly anisotropic by Corollary 2.5. However, the form $T_\sigma = q \otimes q$ is equal to

$$\begin{aligned} q \otimes q &= q \perp aq \perp bq \perp -abq \\ &= \langle 1, a, b, -ab, a, 1, ab, -b, b, ab, 1, -a, -ab, -b, -a, 1\rangle \\ &= 6 \times \langle 1, -1\rangle \perp \langle 1, 1, 1, 1\rangle, \end{aligned}$$

so T_σ is in fact isotropic and hence totally indefinite. Therefore the orthogonal involution σ is totally indefinite.

For quadratic forms, this argument was of course already known in the 1970's, as testified by Theorem 1.1. We merely presented it from the point of view of algebras with involution.

A counterexample in the symplectic case can be constructed by tensoring the previous algebra with the quaternion division algebra $(-1, -1)_F$ equipped with the canonical (symplectic) involution, which is strongly anisotropic.

As it turns out, a property stronger than SAP is needed, the Effective Diagonalization Property, first defined by Ware [22], which we will describe now.

Definition 3.3. A quadratic form $\langle a_1, \ldots, a_n\rangle$ is *effectively diagonalizable* if it is isometric to a form $\langle b_1, \ldots, b_n\rangle$ satisfying $b_i \in P \implies b_{i+1} \in P$ for all $1 \le i < n$ and all $P \in X_F$. The field F satisfies the *Effective Diagonalization Property* (or is ED, for short) if every quadratic form over F is effectively diagonalizable.

The class of ED fields is a proper subclass of the class of SAP fields.

Example 3.4. The field $\mathbb{Q}((t))$ is SAP, but not ED.

Prestel and Ware [17] proved the following characterization theorem:

Theorem 3.5. *F is* ED *if and only if for every (Krull) valuation $v : F^\times \twoheadrightarrow \Gamma$ with value group Γ and real residue class field $\overline{F}_v$, we have $|\Gamma/2\Gamma| \le 2$ and $\overline{F}_v$ is euclidean in case $|\Gamma/2\Gamma| = 2$.*

(Recall that a field is *euclidean* if it is uniquely ordered and every positive element is a square.) They also showed:

Theorem 3.6. *If F is* ED *then every 2-extension of F is also* ED. *(In particular, the pythagorean closure of F is* ED.*)*

(Recall that an extension K of F is called a 2-*extension* of F if K is contained in the quadratic closure of F.)

Remark 3.7. The ED property also played an important role in the classification theorems of Lewis and Tignol [14].

Our generalization of Theorem 1.1 reads:

Theorem 3.8. *F is* ED $\iff$ *F satisfies* (WHA).

The proof will follow from the results below (Theorems 3.11 and 3.12).

Lemma 3.9. *Let (A,σ) be a central simple algebra with involution of the first kind over F. Let $d \in F$ be a sum of squares, and let $K = F(\sqrt{d})$. Suppose that $(A \otimes_F K, \sigma_K)$ is weakly isotropic. Then (A,σ) is weakly isotropic.*

Proof. We may assume that $(A \otimes_F K, \sigma_K)$ is isotropic. Hence there exist $x, y \in A$, not both zero such that

$$\big(\sigma(x) + \sigma(y)\sqrt{d}\big)\big(x + y\sqrt{d}\big) = 0.$$

Separating, this implies $\sigma(x)x + d\sigma(y)y = 0$. Suppose $d = d_1^2 + \cdots + d_r^2$ with $d_i \in F$, then

$$\sigma(x)x + \sigma(d_1 y)(d_1 y) + \cdots + \sigma(d_r y)(d_r y) = 0,$$

i.e. (A,σ) is weakly isotropic. ∎

Lemma 3.10. *Let F be a pythagorean* SAP *field and A a central simple algebra of exponent 2 over F. Then A is Brauer-equivalent to a quaternion division algebra $(-1,f)_F$ for some $f \in F^\times$.*

Proof. By a well-known theorem of Merkurjev, A is Brauer-equivalent to a tensor product of finitely many quaternion division algebras over F. Without loss of generality, we may assume that A is Brauer-equivalent to $(a,b)_F \otimes_F (a',b')_F$ for certain $a, a', b, b' \in F^\times$, and that $(a,b)_F$ and $(a',b')_F$ do not split. Since $(a,b)_F$ is a division algebra, its norm form $\langle 1,-a,-b,ab\rangle$ is anisotropic. Hence $\langle a,b,-ab\rangle$ is anisotropic. Since F is SAP, the quadratic form $\langle 1,a,b,-ab\rangle$ is weakly isotropic, and thus isotropic, since F is pythagorean. Hence $\langle 1,a,b,-ab\rangle \simeq \langle 1,-1,c,d\rangle$ for certain $c, d \in F^\times$. Comparing determinants, we get $\langle 1,a,b,-ab\rangle \simeq \langle 1,-1,c,c\rangle$, which implies $\langle a,b,-ab\rangle \simeq \langle -1,c,c\rangle$, and thus $(a,b)_F \cong (-1,c)_F$.
Similarly, $(a',b')_F \cong (-1,c')_F$ for some $c \in F^\times$, and so A is Brauer-equivalent to $(-1,cc')_F$. ∎

Theorem 3.11. *Assume that F is* ED*, then F satisfies* (WHA).

Proof. Let (A,σ) be totally indefinite. We will show that (A,σ) is weakly isotropic.
If A is split, the theorem is true by Theorem 1.1 (when σ is orthogonal) or trivial (when σ is symplectic).
If the degree of A is odd, then A is split and σ is orthogonal. So we are done in this case. Hence we may assume that A is not split and $\deg A = n = 2m$ is even.
Since F is ED, its pythagorean closure is ED. By Lemma 3.9 we may replace F by its pythagorean closure. (The pythagorean closure F_{pyth} is in general an infinite extension of F but, for any given algebra A, we only need to pass to a finite extension of F, sitting inside F_{pyth}. Then we apply Lemma 3.9 finitely many times.) So we assume from now on that F is a pythagorean ED field.

By Lemma 3.10, A is Brauer-equivalent to a quaternion division algebra $D := (-1, f)_F$ for some $f \in F^\times$. So now we have

$$(A, \sigma) \cong (\mathrm{End}_D(D^m), \sigma_h) \cong (M_m(D), \sigma_h),$$

where σ_h is the adjoint involution of a form $h : D^m \times D^m \longrightarrow D$, which is hermitian or skew-hermitian with respect to quaternion conjugation $^-$ on D, according to whether σ is symplectic or orthogonal.

Suppose first that σ is symplectic, so that h is hermitian. By [18, 7.6.3] there exists a basis $\{e_1, \dots, e_m\}$ of D^m over D which is orthogonal with respect to h. Let $\lambda_i = h(e_i, e_i)$ for $i = 1, \dots, m$. Lewis and Tignol [13, Cor. 2] showed that $\lambda_i \in F$ for all $i = 1, \dots, m$ and that

$$T_\sigma = \langle 2 \rangle \otimes N \otimes \Lambda \otimes \Lambda$$

where N is the norm form of D and $\Lambda = \langle \lambda_1, \dots, \lambda_m \rangle$. By assumption T_σ is totally indefinite, and hence weakly isotropic. Then $N \otimes \Lambda \otimes \Lambda$ is weakly isotropic and so $N \otimes N \otimes \Lambda \otimes \Lambda$ is weakly isotropic. Since F is SAP, this implies (by Definition 3.1(iii)) that $N \otimes \Lambda$ is weakly isotropic. Since $h(x, x) = \lambda_1 N(x_1) + \cdots + \lambda_m N(x_m)$ for $x = (x_1, \dots, x_m) \in D^m$, this implies that the hermitian form h is weakly isotropic over D and hence that (A, σ) is weakly isotropic.

Suppose next that σ is orthogonal, so that h is skew-hermitian. Put $K = F(\sqrt{f})$ (note that f is not a square, since D is a division algebra). Over K, the algebra A splits. Since (A, σ) is totally indefinite, it is clear that $(A, \sigma) \otimes_F K$ is also totally indefinite. Being a 2-extension of the ED field F, the field K is SAP and, by Theorem 1.1, it follows that $(A, \sigma) \otimes_F K$ is weakly isotropic, since $A \otimes_F K$ is split. This implies that the skew-hermitian form h becomes weakly isotropic over K (i.e. as a form over $D_K \cong M_2(K)$). ¿From this we will now deduce that that the form h itself is weakly isotropic, i.e. that the algebra (A, σ) is weakly isotropic.

Replacing h by $N \times h$ for $N \gg 0$ if necessary, there are $x, y \in D^m$, not both zero, such that $h_K(x + y\sqrt{f}, x + y\sqrt{f}) = 0$. This implies $h(x, x) + f h(y, y) = 0$. If $h(y, y) = 0$, it follows that h is (weakly) isotropic and we are done. Otherwise, $u := h(y, y) \in D$ is a non-zero pure quaternion (since h is skew-hermitian) and h has a diagonalization

$$h \simeq \langle -fu, \dots \rangle.$$

Now let $d := u^2 = -\mathrm{Nrd}(u) \in F^\times$. Then $d < 0$ on $\{P \in X_F | f <_P 0\}$ and therefore $D_{F_P} \cong (d, f)_{F_P}$ for all orderings $P \in X_F$ (here F_P denotes the real closure of F with respect to P). Hence $D \cong (d, f)_F$ by Pfister's local-global principle (note that the Witt ring of F is torsion free, since F is pythagorean; see [18, 2.4.10–11]), and there exists a pure quaternion $v \in D$ with

$$v^2 = -\mathrm{Nrd}(v) = f \quad \text{and} \quad uv + vu = 0.$$

Thus $\overline{v} = \mathrm{Nrd}(v) v^{-1} = -f v^{-1}$, and so

$$\overline{v} u v = -f v^{-1} u v = -f(-u) = fu.$$

Therefore $h(yv, yv) = \overline{v}uv = fu$ and h also has a diagonalization

$$h \simeq \langle fu, \ldots \rangle.$$

This implies that $h \perp h \simeq \langle -fu, fu, \ldots \rangle$, which is isotropic. In other words, h is weakly isotropic and hence σ is weakly isotropic. We are done. ∎

Theorem 3.12. *For any non-*ED *field F, there is an algebra (A, σ) with involution of the first kind (and of either type) over F which is strongly anisotropic but totally indefinite.*

Proof. The statement is clear if the field is not SAP (there is an involution which is totally indefinite and strongly anisotropic, as explained just after Example 3.2), so we concentrate on the case of a SAP field which is not ED.
Let F be such a field. Then F has a (Krull) valuation v whose value group Γ satisfies $\Gamma/2\Gamma = \mathbb{Z}/2\mathbb{Z}$, and whose residue field $\overline{F}_v$ is real without being euclidean (this follows from Definition 3.1(v) and Theorem 3.5). Let $\pi \in F^\times$ with $v(\pi) \notin 2\Gamma$, and let $a \in F^\times$ be a v-unit whose residue class in $\overline{F}_v$ is a sum of squares but not a square. We choose a such that a is a sum of squares in F, and consider the quaternion (division) algebra $A = (a, \pi)$ over F. Let $1, i, j, k = ij$ be the standard F-basis of A, satisfying $i^2 = a$, $j^2 = \pi$ and $k^2 = -a\pi$. Let h be the (diagonal) skew-hermitian form $h = \langle j, k \rangle$ over $(A, ^-)$ (where $^-$ denotes the standard (symplectic) involution on A), and let σ be the adjoint involution of h on $M_2(A)$. We claim that σ is totally indefinite, but not weakly isotropic. To show this, let $L = F(\sqrt{a})$. We fix the splitting $\phi\colon A_L \xrightarrow{\sim} M_2(L)$ over L given by

$$i \mapsto \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix} \quad \text{and} \quad j \mapsto \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}.$$

Under ϕ, h corresponds to a similarity class of quadratic forms q (of rank 4) over L. We are going to calculate q.
For $x \in A^\times$ with $x + \overline{x} = 0$, let σ_x be the (orthogonal) involution on A given by $\sigma_x(z) = x^{-1}\overline{z}x$. Under ϕ, $\sigma_x \otimes \mathbf{1}$ corresponds to a similarity class of quadratic forms q_x over L. Writing $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ we have $q_x = J \cdot \phi(x)$. In particular, taking $x = j$ and $x = k$, we find

$$q_j = \langle -\pi, 1 \rangle \quad \text{and} \quad q_k = \langle \pi\sqrt{a}, \sqrt{a} \rangle.$$

Thus

$$q \simeq \langle 1, \sqrt{a}, -\pi, \pi\sqrt{a} \rangle.$$

The form q is totally indefinite. Since the extension L/F is totally real, the involution trace form T_σ (over F) is totally indefinite as well and hence σ is totally indefinite.
On the other hand, the residue forms of q with respect to v are $\langle 1, \sqrt{\overline{a}} \rangle$ and $\langle -1, \sqrt{\overline{a}} \rangle$ (note that $^-$ denotes taking residue classes here). Neither of them is totally indefinite. Hence q is strongly anisotropic. Therefore σ cannot be weakly isotropic.

The symplectic case can be treated again by tensoring our algebra with the quaternion division algebra $(-1,-1)_F$, equipped with quaternion conjugation. ∎

Putting everything together now yields a proof of Theorem 3.8.

4. Sums of hermitian squares

In [11], Lewis proved the following theorem, settling a conjecture of Leep et al. [10].

Theorem 4.1. *Let A be a central simple algebra over a field F of characteristic $\neq 2$. Then 0 is a nontrivial sum of squares, i.e. there exist nonzero $x_1,\dots,x_\ell \in A$ such that $0 = x_1^2+\cdots+x_\ell^2$, if and only if the trace form T_A is weakly isotropic.*

The natural adaptation of this theorem in the setting of algebras with involution of the first kind is an easy consequence of the work we have done hitherto:

Definition 4.2. Let (A,σ) be a central simple algebra with involution of the first kind over a field F and $x \in A$. Then $\sigma(x)x$ is called a *hermitian square* in A.

Theorem 4.3. *Let (A,σ) be a central simple algebra with involution of the first kind over an* ED*-field F. Then 0 is a nontrivial sum of hermitian squares, i.e. there exist nonzero $x_1,\dots,x_\ell \in A$ such that $0 = \sigma(x_1)x_1+\cdots+\sigma(x_\ell)x_\ell$, if and only if the involution trace form T_σ is weakly isotropic.*

Proof. The necessary condition follows trivially (and does not require ED) by simply taking the reduced trace of both sides of $0=\sum_{i=1}^{\ell}\sigma(x_i)x_i$.
For the sufficient condition, suppose that T_σ is weakly isotropic. Then T_σ, and hence σ, is totally indefinite. Therefore σ is weakly isotropic (since F is ED), i.e. there exist nonzero $x_1,\dots,x_\ell \in A$ such that $\sigma(x_1)x_1+\cdots+\sigma(x_\ell)x_\ell = 0$. ∎

Remark 4.4. For several special classes of algebras with involution of the first kind, the condition on F can be relaxed and the conclusion of Theorem 4.3 will still hold. This happens for example

(1) when (A,σ) is an algebra of index 2 with symplectic involution over a SAP field F;
(2) when (Q,σ) is a quaternion algebra with involution of the first kind over a field F of characteristic not 2;
(3) when $(A,\sigma)\cong(Q_1,\sigma_1)\otimes_F\cdots\otimes_F(Q_\ell,\sigma_\ell)$ is a multi-quaternion algebra over a field F of characteristic not 2 and each σ_i is an arbitrary involution of the first kind.

For proofs, see [20, Ch. 5].

Finally, we obtain a version of Springer's theorem for strongly anisotropic involutions:

Corollary 4.5. *Let (A, σ) be a central simple algebra with involution of the first kind over an* ED*-field F and let K/F be any finite extension of odd degree. If (A, σ) is strongly anisotropic, then $(A \otimes_F K, \sigma_K)$ is (strongly) anisotropic.*

Proof. Since σ is strongly anisotropic, T_σ is strongly anisotropic by Theorem 4.3. By Springer's theorem (see e.g. [18, 2.5.3]), $(T_\sigma)_K = T_{\sigma_K}$ is strongly anisotropic over K. Hence σ_K is strongly anisotropic by contraposition of the trivial direction of Theorem 4.3. ∎

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