# Ramification of Local Fields with Imperfect Residue Fields II

Dedicated to Kazuya Kato
on the occasion of his 50th birthday

Ahmed Abbes and Takeshi Saito

Received: October 10, 2002
Revised: August 6, 2003

Abstract. In [1], a filtration by ramification groups and its logarithmic version are defined on the absolute Galois group of a complete discrete valuation field without assuming that the residue field is perfect. In this paper, we study the graded pieces of these filtrations and show that they are abelian except possibly in the absolutely unramified and non-logarithmic case.

2000 Mathematics Subject Classification: primary 11S15, secondary 14G22
Keywords and Phrases: local fields, wild ramification, log structure, affinoid variety.

In the previous paper [1], a filtration by ramification groups and its logarithmic version are defined on the absolute Galois group $G_K$ of a complete discrete valuation field $K$ without assuming that the residue field is perfect. In this paper, we study the graded pieces of these filtrations and show that they are abelian except possibly in the absolutely unramified and non-logarithmic case. Let $G_K^j$ $(j > 0, \in \mathbb{Q})$ denote the decreasing filtration by ramification groups and $G_{K,\log}^j$ $(j > 0, \in \mathbb{Q})$ be its logarithmic variant. We put $G_K^{j+} = \overline{\bigcup_{j'>j} G_K^{j'}}$ and $G_{K,\log}^{j+} = \overline{\bigcup_{j'>j} G_{K,\log}^{j'}}$. In [1], we show that the wild inertia subgroup $P \subset G_K$ is equal to $G_K^{1+} = G_{K,\log}^{0+}$. The main result is the following.

Theorem 1 *Let $K$ be a complete discrete valuation field.*
*1.* (see Theorem 2.15) *Assume either $K$ has equal characteristics $p > 0$ or $K$ has mixed characteristic and $p$ is not a prime element. Then, for a rational number $j > 1$, the graded piece $Gr^j G_K = G_K^j / G_K^{j+}$ is abelian and is a subgroup of the center of the pro-$p$-group $G_K^{1+}/G_K^{j+}$.*
*2.* (see Theorem 5.12) *For a rational number $j > 0$, the graded piece $Gr^j_{\log} G_K = G^j_{K,\log}/G^{j+}_{K,\log}$ is abelian and is a subgroup of the center of the pro-$p$-group $G^{0+}_{K,\log}/G^{j+}_{K,\log}$.*

The idea of the proof of 1 is the following. Under some finiteness assumption, denoted by (F), we define a functor $\bar{X}^j$ from the category of finite étale $K$-algebras with ramification bounded by $j+$ to the category of finite étale schemes over a certain tangent space $\Theta^j$ with continuous semi-linear action of $G_K$. For a finite Galois extension $L$ of $K$ with ramification bounded by $j+$, the image $\bar{X}^j(L)$ has two mutually commuting actions of $G = \mathrm{Gal}(L/K)$ and $G_K$. The arithmetic action of $G_K$ comes from the definition of the functor $\bar{X}^j$ and the geometric action of $G$ is defined by functoriality. Using these two commuting actions, we prove the assertion. The assumption that $p$ is not a prime element is necessary in the construction of the functor $\bar{X}^j$.
In Section 1, for a rational number $j > 0$ and a smooth embedding of a finite flat $O_K$-algebra, we define its $j$-th tubular neighborhood as an affinoid variety. We also define its $j$-th twisted reduced normal cone.
We recall the definition of the filtration by ramification groups in Section 2.1 using the notions introduced in Section 1. In the equal characteristic case, under the assumption (F), we define a functor $\bar{X}^j$ mentioned above in Section 2.2 using $j$-th tubular neighborhoods. In the mixed characteristic case, we give a similar but subtler construction using the twisted normal cones, assuming further that the residue characteristic $p$ is not a prime element of $K$ in Section 2.3. Then, we prove Theorem 2.15 in Section 2.4. We also define a canonical surjection $\pi_1^{\mathrm{ab}}(\Theta^j) \to Gr^j G_K$ under the assumption (F).
After some preparations on generalities of log structures in Section 3, we study a logarithmic analogue in Sections 4 and 5. We define a canonical surjection $\pi_1^{\mathrm{ab}}(\Theta^j_{\log}) \to Gr^j_{\log} G_K$ under the assumption (F) and prove the logarithmic part, Theorem 5.12, of the main result in Section 5.2. Among other results, we compare the construction with the logarithmic construction given in [1] in Lemma 4.10. We also prove in Corollary 4.12 a logarithmic version of [1] Theorem 7.2 (see also Corollary 1.16).
In Section 6, assuming the residue field is perfect, we show that the surjection $\pi_1^{\mathrm{ab}}(\Theta^j_{\log}) \to Gr^j_{\log} G_K$ induces an isomorphism $\pi_1^{\mathrm{ab,gp}}(\Theta^j_{\log}) \to Gr^j_{\log} G_K$ where $\pi_1^{\mathrm{ab,gp}}(\Theta^j_{\log})$ denotes the quotient classifying the étale isogenies to $\Theta^j_{\log}$ regarded as an algebraic group.
When one of the authors (T.S.) started studing mathematics, Kazuya Kato, who was his adviser, suggested to read [13] and to study how to generalize it when the residue field is no longer assumed perfect. This paper is a partial

answer to his suggestion. The authors are very happy to dedicate this paper to him for his 51st anniversary.

Notation. Let $K$ be a complete discrete valuation field, $O_K$ be its valuation ring and $F$ be its residue field of characteristic $p > 0$. Let $\bar{K}$ be a separable closure of $K$, $O_{\bar{K}}$ be the integral closure of $O_K$ in $\bar{K}$, $\bar{F}$ be the residue field of $O_{\bar{K}}$, and $G_K = \mathrm{Gal}(\bar{K}/K)$ be the Galois group of $\bar{K}$ over $K$. Let $\pi$ be a uniformizer of $O_K$ and ord be the valuation of $K$ normalized by $\mathrm{ord}\pi = 1$. We denote also by ord the unique extension of ord to $\bar{K}$.

## 1 Tubular neighborhoods for finite flat algebras

For a semi-local ring $R$, let $\mathfrak{m}_R$ denote the radical of $R$. We say that an $O_K$-algebra $R$ is formally of finite type over $O_K$ if $R$ is semi-local, $\mathfrak{m}_R$-adically complete, Noetherian and the quotient $R/\mathfrak{m}_R$ is finite over $F$. An $O_K$-algebra $R$ formally of finite type over $O_K$ is formally smooth over $O_K$ if and only if its factors are formally smooth. We say that an $O_K$-algebra $R$ is topologically of finite type over $O_K$ if $R$ is $\pi$-adically complete, Noetherian and the quotient $R/\pi R$ is of finite type over $F$. For an $O_K$-algebra $R$ formally of finite type over $O_K$, we put $\hat{\Omega}_{R/O_K} = \varprojlim_n \Omega_{(R/\mathfrak{m}_R^n)/O_K}$. For an $O_K$-algebra $R$ topologically of finite type over $O_K$, we put $\hat{\Omega}_{R/O_K} = \varprojlim_n \Omega_{(R/\pi^n R)/O_K}$. Here and in the following, $\Omega$ denotes the module of differential 1-forms. For a surjection $R \to R'$ of rings, its formal completion is defined to be the projective limit $R^\wedge = \varprojlim_n R/(\mathrm{Ker}(R \to R'))^n$.
In this section, $A$ will denote a finite flat $O_K$-algebra.

### 1.1 Embeddings of finite flat algebras

Definition 1.1 *1. Let $A$ be a finite flat $O_K$-algebra and $\mathbf{A}$ be an $O_K$-algebra formally of finite type and formally smooth over $O_K$. We say that a surjection $\mathbf{A} \to A$ of $O_K$-algebras is an embedding if it induces an isomorphism $\mathbf{A}/\mathfrak{m}_\mathbf{A} \to A/\mathfrak{m}_A$.*
*2. We define $\mathcal{E}mb_{O_K}$ to be the category whose objects and morphisms are as follows. An object of $\mathcal{E}mb_{O_K}$ is a triple $(\mathbf{A} \to A)$ where:*

- *$A$ is a finite flat $O_K$-algebra.*
- *$\mathbf{A}$ is an $O_K$-algebra formally of finite type and formally smooth over $O_K$.*
- *$\mathbf{A} \to A$ is an embedding.*

*A morphism $(f, \mathbf{f}) : (\mathbf{A} \to A) \to (\mathbf{B} \to B)$ of $\mathcal{E}mb_{O_K}$ is a pair of $O_K$-homomorphisms $f : A \to B$ and $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ such that the diagram*

$$\begin{array}{ccc} \mathbf{A} & \longrightarrow & A \\ {\scriptstyle \mathbf{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{B} & \longrightarrow & B \end{array}$$

*is commutative.*
*3. For a finite flat $O_K$-algebra $A$, let $\mathcal{E}mb_{O_K}(A)$ be the subcategory of $\mathcal{E}mb_{O_K}$ whose objects are of the form $(\mathbf{A} \to A)$ and morphisms are of the form $(\mathrm{id}_A, \mathbf{f})$.*
*4. We say that a morphism $(f, \mathbf{f}) : (\mathbf{A} \to A) \to (\mathbf{B} \to B)$ of $\mathcal{E}mb_{O_K}$ is finite flat if $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ is finite and flat and if the map $\mathbf{B} \otimes_{\mathbf{A}} A \to B$ is an isomorphism.*

If $(\mathbf{A} \to A)$ is an embedding, the $\mathbf{A}$-module $\hat{\Omega}_{\mathbf{A}/O_K}$ is locally free of finite rank.

Lemma 1.2 *1. For a finite flat $O_K$-algebra $A$, the category $\mathcal{E}mb_{O_K}(A)$ is non-empty.*
*2. For a morphism $f : A \to B$ of finite flat $O_K$-algebras and for embeddings $(\mathbf{A} \to A)$ and $(\mathbf{B} \to B)$, there exists a morphism $(f, \mathbf{f}) : (\mathbf{A} \to A) \to (\mathbf{B} \to B)$ lifting $f$.*
*3. For a morphism $f : A \to B$ of finite flat $O_K$-algebras, the following conditions are equivalent.*
*(1) The map $f : A \to B$ is flat and locally of complete intersection.*
*(2) Their exists a finite flat morphism $(f, \mathbf{f}) : (\mathbf{A} \to A) \to (\mathbf{B} \to B)$ of embeddings.*

*Proof.* 1. Take a finite system of generators $t_1, \ldots, t_n$ of $A$ over $O_K$ and define a surjection $O_K[T_1, \ldots, T_n] \to A$ by $T_i \mapsto t_i$. Then the formal completion $\mathbf{A} \to A$ of $O_K[T_1, \ldots, T_n] \to A$, where $\mathbf{A} = \varprojlim_m O_K[T_1, \ldots, T_n]/(\mathrm{Ker}(O_K[T_1, \ldots, T_n] \to A))^m$, is an embedding.
2. Since $\mathbf{A}$ is formally smooth over $O_K$ and $\mathbf{B} = \varprojlim_n \mathbf{B}/I^n$ where $I = \mathrm{Ker}(\mathbf{B} \to B)$, the assertion follows.
3. (1)⇒(2). We may assume $A$ and $B$ are local. By 1 and 2, there exists a morphism $(f, \mathbf{f}) : (\mathbf{A} \to A) \to (\mathbf{B} \to B)$ lifting $f$. Replacing $\mathbf{B} \to B$ by the projective limit $\varprojlim_n (\mathbf{A}/\mathfrak{m}_{\mathbf{A}}^n \otimes_{O_K} \mathbf{B}/\mathfrak{m}_{\mathbf{B}}^n)^\wedge \to B/\mathfrak{m}_B^n$ of the formal completion $(\mathbf{A}/\mathfrak{m}_{\mathbf{A}}^n \otimes_{O_K} \mathbf{B}/\mathfrak{m}_{\mathbf{B}}^n)^\wedge \to B/\mathfrak{m}_B^n$ of the surjections $\mathbf{A}/\mathfrak{m}_{\mathbf{A}}^n \otimes_{O_K} \mathbf{B}/\mathfrak{m}_{\mathbf{B}}^n \to B/\mathfrak{m}_B^n$, we may assume that the map $\mathbf{A} \to \mathbf{B}$ is formally smooth. Since $A \to B$ is locally of complete intersection, the kernel of the surjection $\mathbf{B} \otimes_{\mathbf{A}} A \to B$ is generated by a regular sequence $(t_1, \ldots, t_n)$. Take a lifting $(\tilde{t}_1, \ldots \tilde{t}_n)$ in $\mathbf{B}$ and define a map $\mathbf{A}[[T_1, \ldots, T_n]] \to \mathbf{B}$ by $T_i \mapsto t_i$. We consider an embedding $\mathbf{A}[[T_1, \ldots, T_n]] \to A$ defined by the composition $\mathbf{A}[[T_1, \ldots, T_n]] \to \mathbf{A} \to A$ sending $T_i$ to 0. Replacing $\mathbf{A}$ by $\mathbf{A}[[T_1, \ldots, T_n]]$, we obtain a map $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ such that the map $\mathbf{B} \otimes_{\mathbf{A}} A \to B$ is an isomorphism and $\dim \mathbf{A} = \dim \mathbf{B}$. By Nakayama's lemma, the map $\mathbf{A} \to \mathbf{B}$ is finite. Hence the map $\mathbf{A} \to \mathbf{B}$ is flat by EGA Chap $0_{\mathrm{IV}}$ Corollaire (17.3.5) (ii).
(2)⇒(1). Since $\mathbf{A}$ and $\mathbf{B}$ are regular, $\mathbf{B}$ is locally of complete intersection over $\mathbf{A}$. Since $\mathbf{B}$ is flat over $\mathbf{A}$, $B$ is also flat and locally of complete intersection over $A$. □

The base change of an embedding by an extension of complete discrete valuation fields is defined as follows.

Lemma 1.3 *Let $K'$ be a complete discrete valuation field and $K \to K'$ be a morphism of fields inducing a local homomorphism $O_K \to O_{K'}$. Let $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb_{O_K}$. We define $\mathbf{A}\hat{\otimes}_{O_K}O_{K'}$ to be the projective limit $\varprojlim_n(\mathbf{A}/\mathfrak{m}_{\mathbf{A}}^n \otimes_{O_K} O_{K'})$. Then the $O_{K'}$-algebra $\mathbf{A}\hat{\otimes}_{O_K}O_{K'}$ is formally of finite type and formally smooth over $O_{K'}$. The natural surjection $\mathbf{A}\hat{\otimes}_{O_K}O_{K'} \to A\hat{\otimes}_{O_K}O_{K'}$ defines an object $(\mathbf{A}\hat{\otimes}_{O_K}O_{K'} \to A \otimes_{O_K} O_{K'})$ of $\mathcal{E}mb_{O_{K'}}$.*

*Proof.* The $O_K$-algebra $\mathbf{A}$ is finite over the power series ring $O_K[[T_1, \ldots, T_n]]$ for some $n \geq 0$. Hence the $O_{K'}$-algebra $\mathbf{A}\hat{\otimes}_{O_K}O_{K'}$ is finite over $O_{K'}[[T_1, \ldots, T_n]]$ and is formally of finite type over $O_{K'}$. The formal smoothness is clear from the definition. The rest is clear. □.

For an object $(\mathbf{A} \to A)$ of $\mathcal{E}mb_{O_K}$, we let the object $(\mathbf{A}\hat{\otimes}_{O_K}O_{K'} \to A\otimes_{O_K}O_{K'})$ of $\mathcal{E}mb_{O_{K'}}$ defined in Lemma 1.3 denoted by $(\mathbf{A} \to A)\hat{\otimes}_{O_K}O_{K'}$. By sending $(\mathbf{A} \to A)$ to $(\mathbf{A} \to A)\hat{\otimes}_{O_K}O_{K'}$, we obtain a functor $\hat{\otimes}_{O_K}O_{K'} : \mathcal{E}mb_{O_K} \to \mathcal{E}mb_{O_{K'}}$. If $K'$ is a finite extension of $K$, we have $\mathbf{A}\hat{\otimes}_{O_K}O_{K'} = \mathbf{A} \otimes_{O_K} O_{K'}$.

## 1.2 Tubular neighborhoods for embbedings

Let $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb_{O_K}$ and $I$ be the kernel of the surjection $\mathbf{A} \to A$. Mimicing [3] Chapter 7, for a pair of positive integers $m, n > 0$, we define an $O_K$-algebra $\mathcal{A}^{m/n}$ topologically of finite type as follows. Let $\mathbf{A}[I^n/\pi^m]$ be the subring of $\mathbf{A} \otimes_{O_K} K$ generated by $\mathbf{A}$ and the elements $f/\pi^m$ for $f \in I^n$ and let $\mathcal{A}^{m/n}$ be its $\pi$-adic completion. For two pairs of positive integers $m, n$ and $m', n'$, if $m'$ is a multiple of $m$ and if $m'/n' \leq m/n$, we have an inclusion $\mathbf{A}[I^{n'}/\pi^{m'}] \subset \mathbf{A}[I^n/\pi^m]$. It induces a continuous homomorphism $\mathcal{A}^{m'/n'} \to \mathcal{A}^{m/n}$. Then we have the following.

Lemma 1.4 *Let $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb_{O_K}$ and $m, n > 0$ be a pair of positive integers. Then,*
*1. The $O_K$-algebra $\mathcal{A}^{m/n}$ is topologically of finite type over $O_K$. The tensor product $\mathcal{A}_K^{m/n} = \mathcal{A}^{m/n} \otimes_{O_K} K$ is an affinoid algebra over $K$.*
*2. The map $\mathbf{A} \to \mathcal{A}^{m/n}$ is continuous with respect to the $\mathfrak{m}_{\mathbf{A}}$-adic topology on $\mathbf{A}$ and the $\pi$-adic topology on $\mathcal{A}^{m/n}$.*
*3. Let $m', n'$ be another pair of positive integers and assume that $m'$ is a multiple of $m$ and $j' = m'/n' \leq j = m/n$. Then, by the map $X^{m/n} = \operatorname{Sp} \mathcal{A}_K^{m/n} \to X^{m'/n'} = \operatorname{Sp} \mathcal{A}_K^{m'/n'}$ induced by the inclusion $\mathbf{A}[I^{n'}/\pi^{m'}] \subset \mathbf{A}[I^n/\pi^m]$, the affinoid variety $X^{m/n}$ is identified with a rational subdomain of $X^{m'/n'}$.*
*4. The affinoid variety $X^{m/n} = \operatorname{Sp} \mathcal{A}_K^{m/n}$ depends only on the ratio $j = m/n$.*

The proof is similar to that of [3] Lemma 7.1.2.

*Proof.* 1. Since the $O_K$-algebra $\mathcal{A}^{m/n}$ is $\pi$-adically complete, it is sufficient to show that the quotient $\mathbf{A}[I^n/\pi^m]/(\pi)$ is of finite type over $F$. Since it is finitely generated over $\mathbf{A}/(\pi, I^n)$ and $\mathbf{A}/(\pi, I) = A/(\pi)$ is finite over $F$, the assertion follows.

2. Since $A/\pi = \mathbf{A}/(\pi, I)$ is of finite length, a power of $\mathfrak{m}_{\mathbf{A}}$ is in $(\pi^m, I^n)$. Since the image of $(\pi^m, I^n)$ in $\mathcal{A}^{m/n}$ is in $\pi^m\mathcal{A}^{m/n}$, the assertion follows.
3. Take a system of generators $f_1, \ldots, f_N$ of $I^n$ and define a surjection $\mathbf{A}[I^{n'}/\pi^{m'}][T_1, \ldots, T_N]/(\pi^m T_i - f_i) \to \mathbf{A}[I^n/\pi^m]$ by sending $T_i$ to $f_i/\pi^m$. Since it induces an isomorphism after tensoring with $K$, its kernel is annihilated by a power of $\pi$. Hence it induces an isomorphism $\mathcal{A}_K^{m'/n'}\langle T_1, \ldots, T_N\rangle/(\pi^m T_i - f_i, i = 1, \ldots, N) \to \mathcal{A}_K^{m/n}$.
4. Further assume $m/n = m'/n'$ and put $k = m'/m$. Let $f_1, \ldots, f_N \in I^n$ be a system of generators of $I^n$ as above. Then $\mathbf{A}[I^n/\pi^m]$ is generated by $(f_1/\pi^m)^{k_1} \cdots (f_N/\pi^m)^{k_N}, 0 \leq k_i < k$ as an $\mathbf{A}[I^{n'}/\pi^{m'}]$-module. Hence the cokernel of the inclusion $\mathcal{A}^{m'/n'} \to \mathcal{A}^{m/n}$ is annihilated by a power of $\pi$ and the assertion follows. □

If $\mathbf{A} = O_K[[T_1, \ldots, T_N]]$ and $I = (T_1, \ldots, T_N)$, the ring $\mathcal{A}^{m/1}$ is isomorphic to the $\pi$-adic completion of $O_K[T_1/\pi^m, \ldots, T_N/\pi^m]$ and is denoted by $O_K\langle T_1/\pi^m, \ldots, T_N/\pi^m\rangle$. By Lemma 1.4.4, the integral closure $\mathcal{A}^j$ of $\mathcal{A}^{m/n}$ in the affinoid algebra $\mathcal{A}^{m/n} \otimes_{O_K} K$ depends only on $j = m/n$.

Definition 1.5 *Let $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb_{O_K}$ and $j > 0$ be a rational number. We define $\mathcal{A}^j$ to be the integral closure of $\mathcal{A}^{m/n}$ for $j = m/n$ in the affinoid algebra $\mathcal{A}^{m/n} \otimes_{O_K} K$ and define the $j$-th tubular neighborhood $X^j(\mathbf{A} \to A)$ to be the affinoid variety* $\mathrm{Sp}\ \mathcal{A}_K^j$.

In the case $\mathbf{A} = O_K[[T_1, \ldots, T_n]]$ and the map $\mathbf{A} \to A = O_K$ is defined by sending $T_i$ to 0, the affinoid variety $X^j(\mathbf{A} \to A)$ is the $n$-dimensional polydisk $D(0, \pi^j)^n$ of center 0 and of radius $\pi^j$. For each positive rational number $j > 0$, the construction attaching the $j$-th tubular neighboorhood $X^j(\mathbf{A} \to A)$ to an object $(\mathbf{A} \to A)$ of $\mathcal{E}mb_{O_K}$ defines a functor

$$X^j : \mathcal{E}mb_{O_K} \to (\mathrm{Affinoid}/K)$$

to the category of affinoid varieties over $K$. For $j' \leq j$, we have a natural morphism $X^j \to X^{j'}$ of functors. A finite flat morphism of embeddings induces a finite flat morphism of affinoid varieties.

Lemma 1.6 *Let $j > 0$ be a positive rational number and $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ be a finite and flat morphism in $\mathcal{E}mb_{O_K}$. Then, the induced map $f^j : X^j(\mathbf{B} \to B) \to X^j(\mathbf{A} \to A)$ is a finite flat map of affinoid varieties.*

*Proof.* Let $I$ and $J = IB$ be the kernels of the surjections $\mathbf{A} \to A$ and $\mathbf{B} \to B$. Since the map $\mathbf{A} \to \mathbf{B}$ is flat, it induces isomorphisms $\mathbf{B} \otimes_{\mathbf{A}} \mathbf{A}[I^n/\pi^m] \to \mathbf{B}[J^n/\pi^m]$ and $\mathbf{B} \otimes_{\mathbf{A}} \mathcal{A}_K^j \to \mathcal{B}_K^j$. The assertion follows from this immediately. □

For an extension $K'$ of complete discrete valuation field $K$, the construction of $j$-th tubular neighborhoods commutes with the base change. More precisely, we have the following. Let $K'$ be a complete discrete valuation field and $K \to K'$ be a morphism of fields inducing a local homomorphism $O_K \to O_{K'}$. Then by sending an affinoid variety Sp $\mathcal{A}_K$ over $K$ to the affinoid variety Sp $\mathcal{A}_K \hat{\otimes}_K K'$ over $K'$, we obtain a functor $\hat{\otimes}_K K'$ : (Affinoid$/K$) $\to$ (Affinoid$/K'$) (see [2] 9.3.6). Let $e$ be the ramification index $e_{K'/K}$ and $j > 0$ be a positive rational number. Then the canonical map $\mathbf{A} \to \mathbf{A}\hat{\otimes}_{O_K} O_{K'}$ induces an isomorphism $X^j(\mathbf{A} \to A)\hat{\otimes}_K K' \to X^{ej}((\mathbf{A} \to A)\hat{\otimes}_{O_K} O_{K'})$ of affinoid varieties over $K'$. In other words, we have a commutative diagram of functors

$$\begin{array}{ccc} X^j : \mathcal{E}mb_{O_K} & \longrightarrow & (\text{Affinoid}/K) \\ {\scriptstyle \hat{\otimes}_{O_K} O_{K'}} \big\downarrow & & \big\downarrow {\scriptstyle \hat{\otimes}_K K'} \\ X^{ej} : \mathcal{E}mb_{O_K} & \longrightarrow & (\text{Affinoid}/K'). \end{array}$$

LEMMA 1.7 *For a rational number $j > 0$, the affinoid algebra $\mathcal{A}^j_K$ is smooth over $K$.*

*Proof.* By the commutative diagram above, it is sufficient to show that there is a finite separable extension $K'$ of $K$ such that the base change $X^j(\mathbf{A} \to A) \otimes_K K' = X^j(\mathbf{A} \otimes_{O_K} O_{K'} \to A \otimes_{O_K} O_{K'})$ is smooth over $K'$. Replacing $K$ by $K'$ and separating the factors of $A$, we may assume $A/\mathfrak{m}_A = F$. Then we also have $\mathbf{A}/\mathfrak{m}_{\mathbf{A}} = F$ and an isomorphism $O_K[[T_1, \ldots, T_n]] \to \mathbf{A}$. We define an object $(\mathbf{A} \to O_K)$ of $\mathcal{E}mb_{O_K}$ by sending $T_i \in \mathbf{A}$ to 0. Let $I$ and $I'$ be the kernel of $\mathbf{A} \to A$ and $\mathbf{A} \to O_K$ respectively and put $j = m/n$. Since $\mathbf{A}/(\pi^m, I^n)$ is of finite length, there is an integer $n' > 0$ such that $I'^{n'} \subset (\pi^m, I^n)$. Then we have an inclusion $\mathbf{A}[I'^{n'}/\pi^m] \to \mathbf{A}[I^n/\pi^m]$ and hence a map $X^{m/n}(\mathbf{A} \to A) \to X^{m/n'}(\mathbf{A} \to O_K)$. By the similar argument as in the proof of Lemma 1.4.3, the affinoid variety $X^{m/n}(\mathbf{A} \to A)$ is identified with a rational subdomain of $X^{m/n'}(\mathbf{A} \to O_K)$. Since the affinoid variety $X^{m/n'}(\mathbf{A} \to O_K)$ is a polydisk, the assertion follows. □

By Lemma 1.7, the $j$-th tubular neighborhoods in fact define a functor

$$X^j : \mathcal{E}mb_{O_K} \longrightarrow (\text{smoooth Affinoid}/K)$$

to the category of smooth affinoid varieties over $K$. Also by Lemma 1.7, $\hat{\Omega}_{\mathcal{A}^j/O_K} \otimes K$ is a locally free $\mathcal{A}^j_K$-module.

An idea behind the definition of the $j$-th tubular neighborhood is the following description of the valued points. Let $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb_{O_K}$ and $j > 0$ be a rational number. Let $\mathcal{A}^j_K$ be the affinoid algebra defining the affinoid variety $X^j(\mathbf{A} \to A)$ and let $X^j(\mathbf{A} \to A)(\bar{K})$ be the set of $\bar{K}$-valued points. Since a continuous homomorphism $\mathcal{A}^j_K \to \bar{K}$ is determined by the induced map $\mathbf{A} \to O_{\bar{K}}$, we have a natural injection $X^j(\mathbf{A} \to A)(\bar{K}) \to Hom_{\text{cont}.O_K\text{-alg}}(\mathbf{A}, O_{\bar{K}})$. The surjection $\mathbf{A} \to A$ induces an injection

$$Hom_{O_K\text{-alg}}(A, O_{\bar{K}}) \longrightarrow X^j(\mathbf{A} \to A)(\bar{K}). \tag{1.8.0}$$

For a rational number $j > 0$, let $\mathfrak{m}^j$ denote the ideal $\mathfrak{m}^j = \{x \in \bar{K}; \text{ord}x \geq j\}$. We naturally identify the set $Hom_{O_K\text{-alg}}(A, O_{\bar{K}}/\mathfrak{m}^j)$ of $O_K$-algebra homomorphisms with a subset of the set $Hom_{\mathrm{cont}.O_K\text{-alg}}(\mathbf{A}, O_{\bar{K}}/\mathfrak{m}^j)$ of continuous $O_K$-algebra homomorphisms.

LEMMA 1.8 *Let $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb_{O_K}$ and $j > 0$ be a rational number. Then by the injection $X^j(\mathbf{A} \to A)(\bar{K}) \to Hom_{\mathrm{cont}.O_K\text{-}alg}(\mathbf{A}, O_{\bar{K}})$ above, the set $X^j(\mathbf{A} \to A)(\bar{K})$ is identified with the inverse image of the subset $Hom_{O_K\text{-}alg}(A, O_{\bar{K}}/\mathfrak{m}^j)$ by the projection $Hom_{cont.O_K\text{-}alg}(\mathbf{A}, O_{\bar{K}}) \to Hom_{cont.O_K\text{-}alg}(\mathbf{A}, O_{\bar{K}}/\mathfrak{m}^j)$. In other words, we have a cartesian diagram*

$$\begin{array}{ccc} X^j(\mathbf{A} \to A)(\bar{K}) & \longrightarrow & Hom_{\mathrm{cont}.O_K\text{-alg}}(\mathbf{A}, O_{\bar{K}}) \\ \downarrow & & \downarrow \\ Hom_{O_K\text{-alg}}(A, O_{\bar{K}}/\mathfrak{m}^j) & \longrightarrow & Hom_{\mathrm{cont}.O_K\text{-alg}}(\mathbf{A}, O_{\bar{K}}/\mathfrak{m}^j). \end{array} \tag{1.8.1}$$

*The arrows are compatible with the natural $G_K$-action.*

*Proof.* Let $j = m/n$. By the definition of $\mathcal{A}^{m/n}$, a continuous morphism $\mathbf{A} \to O_{\bar{K}}$ is extended to $\mathcal{A}^j_K \to \bar{K}$, if and only if the image of $I^n$ is contained in the ideal $(\pi^m)$. Hence the assertion follows. □

For an affinoid variety $X$ over $K$, let $\pi_0(X_{\bar{K}})$ denote the set $\varprojlim_{K'/K} \pi_0(X_{K'})$ of geometric connected components, where $K'$ runs over finite extensions of $K$ in $\bar{K}$. The set $\pi_0(X_{\bar{K}})$ is finite and carries a natural continuous right action of the absolute Galois group $G_K$. To get a left action, we let $\sigma \in G_K$ act on $X_{\bar{K}}$ by $\sigma^{-1}$. The natural map $X^j(\mathbf{A} \to A)(\bar{K}) \to \pi_0(X_{\bar{K}})$ is compatible with this left $G_K$-action. Let $G_K$-(Finite Sets) denote the category of finite sets with a continuous left action of $G_K$ and let (Finite Flat$/O_K$) be the category of finite flat $O_K$-algebras. Then, for a rational number $j > 0$, we obtain a sequence of functors

$$\text{(Finite Flat}/O_K) \longleftarrow \mathcal{E}mb_{O_K} \xrightarrow{X^j} \text{(smooth Affinoid}/K) \xrightarrow{X \mapsto \pi_0(X_{\bar{K}})} G_K\text{-(Finite Sets)}.$$

We show that the composition $\mathcal{E}mb_{O_K} \to G_K$-(Finite Sets) induces a functor (Finite Flat$/O_K$) $\to G_K$-(Finite Sets).

LEMMA 1.9 *Let $j > 0$ be a positive rational number.*

*1. Let $(\mathbf{A} \to A)$ be an embedding. Then, the map $X^j(\mathbf{A} \to A)(\bar{K}) \to Hom_{O_K\text{-alg}}(A, O_{\bar{K}}/\mathfrak{m}^j)$ (1.8.1) induces a surjection*

$$Hom_{O_K\text{-alg}}(A, O_{\bar{K}}/\mathfrak{m}^j) \longrightarrow \pi_0(X^j(\mathbf{A} \to A)_{\bar{K}}). \tag{1.9.1}$$

*2. Let $(\mathbf{A} \to A)$ and $(\mathbf{A}' \to A)$ be embeddings. Then, there exists a unique bijection $\pi_0(X^j(\mathbf{A} \to A)_{\bar{K}}) \to \pi_0(X^j(\mathbf{A}' \to A)_{\bar{K}})$ such that the diagram*

$$\begin{array}{ccc} Hom_{O_K\text{-alg}}(A, O_{\bar{K}}/\mathfrak{m}^j) & \longrightarrow & \pi_0(X^j(\mathbf{A} \to A)_{\bar{K}}) \\ \| & & \downarrow \\ Hom_{O_K\text{-alg}}(A, O_{\bar{K}}/\mathfrak{m}^j) & \longrightarrow & \pi_0(X^j(\mathbf{A}' \to A)_{\bar{K}}) \end{array} \tag{1.9.2}$$

*is commutative.*
*3. Let* $(f,\mathbf{f}) : (\mathbf{A} \to A) \to (\mathbf{B} \to B)$ *be a morphism of* $\mathcal{E}mb_{O_K}$*. Then, the induced map* $\pi_0(X^j(\mathbf{B} \to B)_{\bar{K}}) \to \pi_0(X^j(\mathbf{A} \to A)_{\bar{K}})$ *depends only on* $f$*.*
*4. Let* $(f,\mathbf{f}) : (\mathbf{A} \to O_K) \to (\mathbf{B} \to B)$ *be a finite flat morphism of* $\mathcal{E}mb_{O_K}$*. Then the map* (1.8.0) *induces a surjection*

$$Hom_{O_K\text{-}alg}(B, O_{\bar{K}}) \longrightarrow \pi_0(X^j(\mathbf{B} \to B)_{\bar{K}}). \tag{1.9.3}$$

*Proof.* 1. The fibers of the map $Hom_{\text{cont}.O_K\text{-alg}}(\mathbf{A}, O_{\bar{K}}) \to Hom_{\text{cont}.O_K\text{-alg}}(\mathbf{A}, O_{\bar{K}}/\mathfrak{m}^j)$ are $\bar{K}$-valued points of polydisks. Hence the surjection $X^j(\mathbf{A} \to A)(\bar{K}) \to Hom_{\text{cont}.O_K\text{-alg}}(A, O_{\bar{K}}/\mathfrak{m}^j)$ induces a surjection $Hom_{\text{cont}.O_K\text{-alg}}(A, O_{\bar{K}}/\mathfrak{m}^j) \to \pi_0(X^j(\mathbf{A} \to A)_{\bar{K}})$ by Lemma 1.8.
2. By 1 and Lemma 1.2.2, there exists a unique surjection $\pi_0(X^j(\mathbf{A} \to A)_{\bar{K}}) \to \pi_0(X^j(\mathbf{A}' \to A)_{\bar{K}})$ such that the diagram (1.9.2) is commutative. Switching $\mathbf{A} \to A$ and $\mathbf{A}' \to A$, we obtain the assertion.
3. In the commutative diagram

$$\begin{array}{ccc}
Hom_{\text{cont}.O_K\text{-alg}}(B, O_{\bar{K}}/\mathfrak{m}^j) & \longrightarrow & \pi_0(X^j(\mathbf{B} \to B)_{\bar{K}}) \\
{\scriptstyle f^*}\downarrow & & \downarrow \\
Hom_{\text{cont}.O_K\text{-alg}}(A, O_{\bar{K}}/\mathfrak{m}^j) & \longrightarrow & \pi_0(X^j(\mathbf{A} \to A)_{\bar{K}}),
\end{array}$$

the horizontal arrows are surjective by 1. Hence the assertion follows.
4. The map $f^j : X^j(\mathbf{B} \to B) \to X^j(\mathbf{A} \to O_K)$ is finite and flat by Lemma 1.6. Let $y : X^j(\mathbf{A} \to O_K)(\bar{K})$ be the point corresponding to the map $\mathbf{A} \to O_K$. Then the fiber $(f^j)^{-1}(y)$ is identified with the set $Hom_{O_K\text{-alg}}(B, O_{\bar{K}})$. Since $X^j(\mathbf{A} \to O_K)_{\bar{K}}$ is isomorphic to a disk and is connected, the assertion follows. □

For a rational number $j > 0$ and a finite flat $O_K$-algebra $A$, we put

$$\Psi^j(A) = \varprojlim_{(\mathbf{A} \to A) \in \mathcal{E}mb_{O_K}(A)} \pi_0(X^j(\mathbf{A} \to A)_{\bar{K}}).$$

By Lemmas 1.2.1 and 1.9.2, the projective system in the right is constant. Further by Lemma 1.9.3, we obtain a functor

$$\Psi^j : (\text{Finite Flat}/O_K) \longrightarrow G_K\text{-(Finite Sets)}$$

sending a finite flat $O_K$-algebra $A$ to $\Psi^j(A)$. Let $\Psi : (\text{Finite Flat}/O_K) \to G_K$-(Finite Sets) be the functor defined by $\Psi(A) = Hom_{O_K\text{-alg}}(A, \bar{K})$. Then, the map (1.9.1) induces a map $\Psi \to \Psi^j$ of functors.

## 1.3 Stable normalized integral models and their closed fibers

We briefly recall the stable normalized integral model of an affinoid variety and its closed fiber (cf. [1] Section 4). It is based on the finiteness theorem of Grauert-Remmert.

Theorem 1.10 (Finiteness theorem of Grauert-Remmert, [1] Theorem 4.2) *Let $\mathcal{A}$ be an $O_K$-algebra topologically of finite type. Assume that the generic fiber $\mathcal{A}_K = \mathcal{A} \otimes_{O_K} K$ is geometrically reduced. Then,*
*1. There exists a finite separable extension $K'$ of $K$ such that the geometric closed fiber $\mathcal{A}_{O_{K'}} \otimes_{O_{K'}} \bar{F}$ of the integral closure $\mathcal{A}_{O_{K'}}$ of $\mathcal{A}$ in $\mathcal{A} \otimes_{O_K} K'$ is reduced.*
*2. Assume further that $\mathcal{A}$ is flat over $O_K$ and that the geometric closed fiber $\mathcal{A} \otimes_{O_K} \bar{F}$ is reduced. Let $K'$ be an extension of complete discrete valuation field over $K$ and $\pi'$ be a prime element of $K'$. Then the $\pi'$-adic completion of the base change $\mathcal{A} \otimes_{O_K} O_{K'}$ is integrally closed in $\mathcal{A} \otimes_{O_K} K'$.*

Let $\mathcal{A}$ be an $O_K$-algebra topologically of finite type such that $\mathcal{A}_K$ is smooth. If a finite separable extension $K'$ satisfies the condition in Theorem 1.10.1, we say that the integral closure $\mathcal{A}_{O_{K'}}$ of $\mathcal{A}$ in $\mathcal{A}_{K'}$ is a stable normalized integral model of the affinoid variety $X_K = \mathrm{Sp}\ \mathcal{A}_K$ and that the stable normalized integral model is defined over $K'$. The geometric closed fiber $\bar{X} = \mathrm{Spec}\ \mathcal{A}_{O_{K'}} \otimes_{O_{K'}} \bar{F}$ of a stable normalized integral model is independent of the choice of an extension $K'$ over which a stable normalized integral model is defined, by Theorem 1.10.2. Hence, the scheme $\bar{X}$ carries a natural continuous action of the absolute Galois group $G_K = \mathrm{Gal}(\bar{K}/K)$ compatible with its action on $\bar{F}$.
The construction above defines a functor as follows. Let $G_K$-$(\mathrm{Aff}/\bar{F})$ denote the category of affine schemes of finite type over $\bar{F}$ with a semi-linear continuous action of the absolute Galois group $G_K$. More precisely, an object is an affine scheme $Y$ over $\bar{F}$ with an action of $G_K$ compatible with the action of $G_K$ on $\bar{F}$ satisfying the following property: There exist a finite Galois extension $K'$ of $K$ in $\bar{K}$, an affine scheme $Y_{K'}$ of finite type over the residue field $F'$ of $K'$, an action of $\mathrm{Gal}(K'/K)$ on $Y_{K'}$ compatible with the action of $\mathrm{Gal}(K'/K)$ on $F'$ and a $G_K$-equivariant isomorphism $Y_{K'} \otimes_{F'} \bar{F} \to Y$. Then Theorem 1.10 implies that the geometric closed fiber of a stable normalized integral model defines a functor

$$(\text{smooth Affinoid}/K) \to G_K\text{-}(\mathrm{Aff}/\bar{F}) : X \mapsto \bar{X}.$$

Corollary 1.11 *Let $\mathcal{A}$ be an $O_K$-algebra topologically of finite type such that the generic fiber $\mathcal{A}_K$ is geometrically reduced as in Theorem 1.10. Let $X_K = \mathrm{Sp}\ \mathcal{A}_K$ be the affinoid variety and $X_{\bar{F}}$ be the geometric closed fiber of the stable normalized integral model. Then the natural map $\pi_0(X_{\bar{F}}) \to \pi_0(X_{\bar{K}})$ is a bijection.*

*Proof.* Replacing $\mathcal{A}$ by its image in $\mathcal{A}_K$, we may assume $\mathcal{A}$ is flat over $O_K$. Let $K'$ be a finite separable extension of $K$ in $\bar{K}$ such that the stable normalized integral model $\mathcal{A}_{O_{K'}}$ is defined over $K'$. Then since $\mathcal{A}_{O_{K'}}$ is $\pi$-adically complete, the canonical maps $\pi_0(\mathrm{Spec}\mathcal{A}_{O_{K'}}) \to \pi_0(\mathrm{Spec}(\mathcal{A}_{O_{K'}} \otimes_{O_{K'}} F'))$ is bijective. Since the idempotents of $\mathcal{A}_{K'}$ are in $\mathcal{A}_{O_{K'}}$, the canonical maps $\pi_0(\mathrm{Spec}\mathcal{A}_{O_{K'}}) \to \pi_0(\mathrm{Spec}\mathcal{A}_{K'})$ is also bijective. By taking the limit, we obtain the assertion. □

By Corollary 1.11, the functor (smooth Affinoid/$K$) $\to G_K$-(Finite Sets) sending a smooth affinoid variety $X$ to $\pi_0(X_{\bar K})$ may be also regarded as the composition of the functors

$$\text{(smooth Affinoid}/K) \xrightarrow{X\mapsto \bar X} G_K\text{-}(\text{Aff}/\bar F) \xrightarrow{\pi_0} G_K\text{-(Finite Sets)}\cdot$$

Lemma 1.12 *Let $j>0$ be a positive rational number and $(f,\mathbf{f}):(\mathbf{A}\to O_K)\to(\mathbf{B}\to B)$ be a finite flat morphism of $\mathcal{E}mb_{O_K}$. Let $f^j: X^j(\mathbf{B}\to B)\to X^j(\mathbf{A}\to O_K)$ be the induced map and $\bar f^j:\bar X^j(\mathbf{B}\to B)\to \bar X^j(\mathbf{A}\to O_K)$ be its reduction. Let $y\in X^j(\mathbf{A}\to O_K)(\bar K)$ be the point corresponding to $\mathbf{A}\to A=O_K\to \bar K$ and $\bar y\in \bar X^j(\mathbf{A}\to O_K)$ be its specialization. Then the surjections $(f^j)^{-1}(y)=Hom_{O_K\text{-}alg}(B,O_{\bar K})\to \pi_0(X^{j'}(\mathbf{B}\to B)_{\bar K})$ (1.9.3) and the specialization map $(f^j)^{-1}(y)\to(\bar f^j)^{-1}(\bar y)$ induces a bijection*

$$\varinjlim_{j'>j}\pi_0(X^{j'}(\mathbf{B}\to B)_{\bar K})\longrightarrow(\bar f^j)^{-1}(\bar y). \tag{1.12.1}$$

*Proof.* The map $(f^j)^{-1}(y)\to\pi_0(X^{j'}(\mathbf{B}\to B)_{\bar K})$ is a surjection of finite sets by Lemma 1.9.4. Hence there exists a rational number $j'>j$ such that the surjection $\pi_0(X^{j'}(\mathbf{B}\to B)_{\bar K})\to\varinjlim_{j''>j}\pi_0(X^{j''}(\mathbf{B}\to B)_{\bar K})$ is a bijection. Let $K'$ be a finite separable extension such that the surjection $\pi_0(X^{j'}(\mathbf{B}\to B)_{\bar K})\to\pi_0(X^{j'}(\mathbf{B}\to B)_{K'})$ is a bijection and that the stable normalized integral models $\mathcal{B}^j_{O_{K'}}$ of $X^j(\mathbf{B}\to B)$ is defined over $K'$. Enlarging $K'$ further if necessary, we assume that $e'j$ is an integer where $e'=e_{K'/K}$ is the ramification index. Then the integral model $\mathcal{A}^j_{O_{K'}}$ of $X^j(\mathbf{A}\to O_K)$ is also defined over $K'$. If $O_K[[T_1,\ldots,T_n]]\to\mathbf{A}$ is an isomorphism such that the kernel of $\mathbf{A}\to O_K$ is generated by $T_1,\ldots,T_n$ and $\pi'$ is a prime element of $K'$, it induces an isomorphism $O_{K'}\langle T_1/\pi'^{e'j},\ldots,T_n/\pi'^{e'j}\rangle\to\mathcal{A}^j_{O_{K'}}$. Let $\mathcal{A}^j_{O_{K'}}\to O_{K'}$ be the map induced by $\mathbf{A}\to O_K$ and $\mathbf{A}^j_{O_{K'}}$ be the formal completion respect to the surjection $\mathcal{A}^j_{O_{K'}}\to O_{K'}$. If $O_K[[T_1,\ldots,T_n]]\to\mathbf{A}$ is an isomorphism as above, it induces an isomorphism $O_{K'}[[T_1/\pi'^{e'j},\ldots,T_n/\pi'^{e'j}]]\to\mathbf{A}^j_{O_{K'}}$. We put $\mathbf{B}^j_{O_{K'}}=\mathcal{B}^j_{O_{K'}}\otimes_{\mathcal{A}^j_{O_{K'}}}\mathbf{A}^j_{O_{K'}}$. The ring $\mathbf{B}^j_{O_{K'}}$ is finite over $\mathbf{A}^j_{O_{K'}}$ since $\mathcal{B}^j_{O_{K'}}$ is finite over $\mathcal{A}^j_{O_{K'}}$. Enlarging $K'$ further if necessary, we assume that the canonical map $(\bar f^j)^{-1}(\bar y)\to\pi_0(\text{Spec }\mathbf{B}^j_{O_{K'}})$ is a bijection.

We show that the surjection $\pi_0(X^{j'}(\mathbf{B}\to B)_{K'})\to\pi_0(\text{Spec }\mathbf{B}^j_{O_{K'}})$ is a bijection. For a rational number $j'>0$, let $\mathcal{A}^{j'}_K$ and $\mathcal{B}^{j'}_K$ denote the affinoid $K$-algebras defining $X^{j'}(\mathbf{A}\to O_K)$ and $X^{j'}(\mathbf{B}\to B)$. We have $\mathcal{B}^{j'}_K=\mathbf{B}\otimes_{\mathbf{A}}\mathcal{A}^{j'}_K$. Since $\pi_0(X^{j'}(\mathbf{B}\to B)_{\bar K})\to\varinjlim_{j''>j}\pi_0(X^{j''}(\mathbf{B}\to B)_{\bar K})$ is a bijection, the injection $\mathcal{B}^{j''}_{\bar K}\to\mathcal{B}^{j'}_{\bar K}$ induce a bijection of idempotents for $j<j''<j'$. Since $\pi_0(X^{j'}(\mathbf{B}\to B)_{\bar K})\to\pi_0(X^{j'}(\mathbf{B}\to B)_{K'})$ is a bijection, the idempotents of $\mathcal{B}^{j'}_{\bar K}$ are in $\mathcal{B}^{j'}_{K'}$. Hence, for $j<j''<j'$, the map $\mathcal{B}^{j''}_{K'}\to\mathcal{B}^{j'}_{K'}$ induces a bijection of idempotents for $j<j''<j'$. Therefore, the map $\mathbf{B}^j_{O_{K'}}\to\mathcal{B}^{j'}_{K'}$

induces a bijection of idempotents by [3] 7.3.6 Proposition. Thus, the map $\pi_0(X^{j'}(\mathbf{B} \to B)_{K'}) \to \pi_0(\mathrm{Spec}\ \mathbf{B}^j_{O_{K'}})$ is a bijection as required. □

For later use in the proof of the commutativity in the logarithmic case, we give a more formal description of the functor (smooth Affinoid/$K$) $\to G_K$-(Aff/$\bar{F}$) : $X \mapsto \bar{X}$. For this purpose, we introduce a category $\varinjlim_{K'/K}$(Aff/$F'$) and an equivalence $\varinjlim_{K'/K}$(Aff/$F'$) $\to G_K$-(Aff/$\bar{F}$) of categories. More generally, we define a category $\varinjlim_{K'/K} \mathcal{V}(K')$ in the following setting. Suppose we are given a category $\mathcal{V}(K')$ for each finite separable extension $K'$ of $K$ and a functor $f^* : \mathcal{V}(K'') \to \mathcal{V}(K')$ for each morphism $f : K' \to K''$ of finite separable extension of $K$ satisfying $(f \circ g)^* = g^* \circ f^*$ and $\mathrm{id}^*_{K'} = \mathrm{id}_{\mathcal{V}(K')}$. In the application here, we will take $\mathcal{V}(K')$ to be (Aff/$F'$) for the residue field $F'$. In Section 4, we will take $\mathcal{V}(K')$ to be $\mathcal{E}mb_{O_{K'}}$. We say that a full subcategory $\mathcal{C}$ of the category (Ext/$K$) of finite separable extensions in $\bar{K}$ is cofinal if $\mathcal{C}$ is non empty and a finite extension $K''$ of an extension $K'$ in $\mathcal{C}$ is also in $\mathcal{C}$. We define $\varinjlim_{K'/K} \mathcal{V}(K')$ to be the category whose objects and morphisms are as follows. An object of $\varinjlim_{K'/K} \mathcal{V}(K')$ is a system $((X_{K'})_{K' \in ob(\mathcal{C})}, (\varphi_f)_{f:K' \to K'' \in mor(\mathcal{C})})$ where $\mathcal{C}$ is some cofinal full subcategory of (Ext/$K$), $X_{K'}$ is an object of $\mathcal{V}(K')$ for each object $K'$ in $\mathcal{C}$ and $\varphi_f : X_{K''} \to f^*(X_{K'})$ is an isomorphism in $\mathcal{V}(K'')$ for each morphism $f : K' \to K''$ in $\mathcal{C}$ satisfying $\varphi_{f \circ f'} = f^*(\varphi_{f'}) \circ \varphi_f$ for morphisms $f' : K' \to K''$ and $f : K'' \to K'''$ in $\mathcal{C}$. For objects $X = ((X_{K'})_{K' \in ob(\mathcal{C})}, (\varphi_f)_{f:K' \to K'' \in mor(\mathcal{C})})$ and $Y = ((Y_{K'})_{K' \in ob(\mathcal{C}')}, (\psi_f)_{f:K' \to K'' \in mor(\mathcal{C}')})$ of the category $\varinjlim_{K'/K} \mathcal{V}(K')$, a morphism $g : X \to Y$ is a system $(g_{K'})_{K' \in ob(\mathcal{C}'')}$, where $\mathcal{C}''$ is some cofinal full subcategory of $\mathcal{C} \cap \mathcal{C}'$ and $g_{K'} : X_{K'} \to Y_{K'}$ is a morphism in $\mathcal{V}(\mathcal{K}')$ such that the diagram

$$\begin{array}{ccc} X_{K''} & \xrightarrow{g_{K'}} & Y_{K''} \\ {\scriptstyle \varphi_f}\downarrow & & \downarrow{\scriptstyle \psi_f} \\ f^*X_{K'} & \xrightarrow{g_{K''}} & f^*Y_{K'} \end{array}$$

is commutative for each morphism $f : K' \to K''$ in $\mathcal{C}''$.

Applying the general construction above, we define a category $\varinjlim_{K'/K}$(Aff/$F'$). An equivalence $\varinjlim_{K'/K}$(Aff/$F'$) $\to G_K$-(Aff/$\bar{F}$) of categories is defined as follows. Let $X = ((X_{K'})_{K' \in ob(\mathcal{C})}, (f^*)_{f:K' \to K'' \in mor(\mathcal{C})})$ be an object of $\varinjlim_{K'/K}$(Aff/$F'$). Let $\mathcal{C}_{\bar{K}}$ be the category of finite extensions of $K$ in $\bar{K}$ which are in $\mathcal{C}$. Then, $X_{\bar{K}} = \varprojlim_{K' \in \mathcal{C}_{\bar{K}}} X_{K'}$ is an affine scheme over $\bar{F}$ and has a natural continuous semi-linear action of the Galois group $G_K$. By sending $X$ to $X_{\bar{K}}$, we obtain a functor $\varinjlim_{K'/K}$(Aff/$F'$) $\to G_K$-(Aff/$\bar{F}$). We can easily verify that this functor gives an equivalence of categories.

The reduced geometic closed fiber defines a functor (smooth Affinoid/$K$) $\to$ $\varinjlim_{K'/K}$(Aff/$F'$) as follows. Let $X$ be a smooth affinoid variety over $K$. Let $\mathcal{C}_X$

be the full subcategory of $(\mathrm{Ext}/K)$ consisting of finite extensions $K'$ such that a stable normalized integral model $\mathcal{A}_{O_{K'}}$ is defined over $K'$. By Theorem 1.10.1, the subcategory $\mathcal{C}_X$ is cofinal. Further, by Theorem 1.10.2, the system $\bar{X} = (\mathrm{Spec}\ \mathcal{A}_{O_{K'}} \otimes_{O_{K'}} F')_{K' \in ob\mathcal{C}_X}$ defines an object of $\varinjlim_{K'/K}(\mathrm{Aff}/F')$. Thus, by sending $X$ to $\bar{X}$, we obtain a functor (smooth Affinoid$/K$) $\to \varinjlim_{K'/K}(\mathrm{Aff}/\bar{F}')$. By taking the composition with the equivalence of categories, we recover the functor (smooth Affinoid$/K$) $\to G_K$-$(\mathrm{Aff}/\bar{F})$.

### 1.4 Twisted normal cones

Let $(\mathbf{A} \to A)$ be an object in $\mathcal{E}mb_{O_K}$ and $j > 0$ be a positive rational number. We define $\bar{X}^j(\mathbf{A} \to A)$ to be the geometric closed fiber of the stable normalized integral model of $X^j(\mathbf{A} \to A)$. We will also define a twisted normal cone $\bar{C}^j(\mathbf{A} \to A)$ as a scheme over $A_{\bar{F},\mathrm{red}} = (A \otimes_{O_K} \bar{F})_{\mathrm{red}}$ and a canonical map $\bar{X}^j(\mathbf{A} \to A) \to \bar{C}^j(\mathbf{A} \to A)$.

Let $I$ be the kernel of the surjection $\mathbf{A} \to A$. Then the normal cone $C_{A/\mathbf{A}}$ of Spec $A$ in Spec $\mathbf{A}$ is defined to be the spectrum of the graded $A$-algebra $\bigoplus_{n=0}^{\infty} I^n/I^{n+1}$. We say that a surjection $R \to R'$ of Noetherian rings is regular if the immersion $\mathrm{Spec}R' \to \mathrm{Spec}R$ is a regular immersion. If the surjection $\mathbf{A} \to A$ is regular, the conormal sheaf $N_{A/\mathbf{A}} = I/I^2$ is locally free and the normal cone $C_{A/\mathbf{A}}$ is equal to the normal bundle, namely the covariant vector bundle over $\mathrm{Spec}A$ defined by the locally free $A$-module $Hom_A(N_{A/\mathbf{A}}, A)$.

For a rational number $j$, let $\mathfrak{m}^j$ be the fractional ideal $\mathfrak{m}^j = \{x \in O_{\bar{K}}; \mathrm{ord}(x) \geq j\}$ and put $N^j = \mathfrak{m}^j \otimes_{O_{\bar{K}}} \bar{F}$.

Definition 1.13 *Let $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb_{O_K}$ and $j > 0$ be a rational number. We define the $j$-th twisted normal cone $\bar{C}^j(\mathbf{A} \to A)$ to be the reduced part*

$$\left(\mathrm{Spec} \bigoplus_{n=0}^{\infty} (I^n/I^{n+1} \otimes_{O_K} N^{-jn})\right)_{\mathrm{red}}$$

*of the spectrum of the $A \otimes_{O_K} \bar{F}$-algebra $\bigoplus_{n=0}^{\infty}(I^n/I^{n+1} \otimes_{O_K} N^{-jn})$.*

It is a reduced affine scheme over Spec $A_{\bar{F},\mathrm{red}}$ non-canonically isomorphic to the reduced part of the base change $C_{A/\mathbf{A}} \otimes_{O_K} \bar{F}$. It has a natural continuous semi-linear action of $G_K$ via $N^{-jn}$. The restriction to the wild inertia subgroup $P$ is trivial and the $G_K$-action induces an action of the tame quotient $G_K^{\mathrm{tame}} = G_K/P$. If the surjection $\mathbf{A} \to A$ is regular, the scheme $\bar{C}^j(\mathbf{A} \to A)$ is the covariant vector bundle over Spec $A_{\bar{F},\mathrm{red}}$ defined by the $A_{\bar{F},\mathrm{red}}$-module $(Hom_A(I/I^2, A) \otimes_{O_K} N^j) \otimes_{A \otimes_{O_K} \bar{F}} A_{\bar{F},\mathrm{red}}$.

A canonical map $\bar{X}^j(\mathbf{A} \to A) \to \bar{C}^j(\mathbf{A} \to A)$ is defined as follows. Let $K'$ be a finite separable extension of $K$ such that the stable normalized integral model $\mathcal{A}^j_{O_{K'}}$ is defined over $K'$ and that the product $je$ with the ramification index

$e = e_{K'/K}$ is an integer. Then, we have a natural ring homomorphism

$$\bigoplus_{n\geq 0} I^n \otimes_{O_K} \mathfrak{m}_{K'}^{-jen} \longrightarrow \mathcal{A}^j_{O_{K'}} : f \otimes a \mapsto af.$$

Since $I\mathcal{A}^j_{O_{K'}} \subset \mathfrak{m}^{je}_{K'}\mathcal{A}^j_{O_{K'}}$, it induces a map $\bigoplus_n I^n/I^{n+1} \otimes_{O_K} \mathfrak{m}_{K'}^{-jen} \to \mathcal{A}^j_{O_{K'}}/\mathfrak{m}_{K'}\mathcal{A}^j_{O_{K'}}$. Let $F'$ be the residue field of $K'$. Then by extending the scalar, we obtain a map $\bigoplus_{n=0}^{\infty}(I^n/I^{n+1} \otimes_{O_K} N^{-jn}) \to \mathcal{A}^j_{O_{K'}}/\mathfrak{m}_{K'}\mathcal{A}^j_{O_{K'}} \otimes_{F'} \bar{F}$. By the assumption that $\mathcal{A}^j_{O_{K'}}$ is a stable normalized integral model, we have $\bar{X}^j(\mathbf{A} \to A) = \mathrm{Spec}\ (\mathcal{A}^j_{O_{K'}}/\mathfrak{m}_{K'}\mathcal{A}^j_{O_{K'}} \otimes_{F'} \bar{F})$. Since $\bar{X}^j(\mathbf{A} \to A)$ is a reduced scheme over $\bar{F}$, we obtain a map $\bar{X}^j(\mathbf{A} \to A) \to \bar{C}^j(\mathbf{A} \to A)$ of schemes over $\bar{F}$.

For a positive rational number $j > 0$, the constructions above define a functor $\bar{C}^j : \mathcal{E}mb_{O_K} \to G_K\text{-}(\mathrm{Aff}/\bar{F})$ and a morphism of functors $\bar{X}^j \to \bar{C}^j$.

Lemma 1.14 *Let $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb_{O_K}$ and $j > 0$ be a rational number. Then, we have the following.*
*1. The canonical map $\bar{X}^j(\mathbf{A} \to A) \to \bar{C}^j(\mathbf{A} \to A)$ is finite.*
*2. Let $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ be a morphism in $\mathcal{E}mb_{O_K}$. Then, the canonical maps form a commutative diagram*

$$\begin{array}{ccccc}
\bar{X}^j(\mathbf{B} \to B) & \longrightarrow & \bar{C}^j(\mathbf{B} \to B) & \longrightarrow & \mathrm{Spec}\ B_{\bar{F},\mathrm{red}} \\
\downarrow & & \downarrow & & \downarrow \\
\bar{X}^j(\mathbf{A} \to A) & \longrightarrow & \bar{C}^j(\mathbf{A} \to A) & \longrightarrow & \mathrm{Spec}\ A_{\bar{F},\mathrm{red}}.
\end{array}$$

*If the morphism $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ is finite flat, then the right square in the commutative diagram is cartesian.*
*3. Assume $A = O_K$. Then the surjection $\mathbf{A} \to A$ is regular and the canonical map $N_{A/\mathbf{A}} \to \hat{\Omega}_{\mathbf{A}/O_K} \otimes_{\mathbf{A}} A$ is an isomorphism. The twisted normal cone $\bar{C}^j(\mathbf{A} \to A)$ is equal to the $\bar{F}$-vector space $Hom_{\bar{F}}(\hat{\Omega}_{\mathbf{A}/O_K} \otimes_{\mathbf{A}} \bar{F}, N^j)$. The canonical map $\bar{X}^j(\mathbf{A} \to A) \to \bar{C}^j(\mathbf{A} \to A)$ is an isomorphism.*

*Proof.* 1. Let $K'$ be a finite extension such that the stable normalized integral model $\mathcal{A}^j_{O_{K'}}$ is defined. Let $\mathcal{A}'$ denote the $\pi'$-adic completion of the image of the map $\bigoplus_{n\geq 0} I^n \otimes_{O_K} \mathfrak{m}_{K'}^{-jen} \to \mathbf{A} \otimes_{O_K} K'$. Then by the definition and by Lemma 1.3, $\mathcal{A}^j_{O_{K'}}$ is the integral closure of $\mathcal{A}'$ in $\mathcal{A}'_K$. Hence $\mathcal{A}^j_{O_{K'}}/m_{K'}\mathcal{A}^j_{O_{K'}}$ is finite over $\bigoplus_n I^n/I^{n+1} \otimes_{O_K} \mathfrak{m}_{K'}^{-jen}$. Thus the assertion follows.
2. Clear from the definitions.
3. If $A = O_K$, there is an isomorphism $O_K[[T_1, \ldots, T_n]] \to \mathbf{A}$ for some $n$ such that the composition $O_K[[T_1, \ldots, T_n]] \to A$ maps $T_i$ to 0. Then the assertions are clear. □

### 1.5 Étale covering of tubular neighborhoods

Let $A$ and $B$ be the integer rings of finite étale $K$-algebras. For a finite flat morphism $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ of embeddings, we study conditions for the induced finite morphism $X^j(\mathbf{A} \to A) \to X^j(\mathbf{B} \to B)$ to be étale.
Let $X = \text{Sp } \mathcal{B}_K$ and $Y = \text{Sp } \mathcal{A}_K$ be geometrically reduced affinoid varieties and $\mathcal{A}$ and $\mathcal{B}$ be the maximum integral models. Then a finite map $f : X \to Y$ of affinoid varieties is uniquelly extended to a finite map $\mathcal{A} \to \mathcal{B}$ of integral models.

Proposition 1.15 *Let $A$ and $B = O_L$ be the integer rings of finite separable extensions of $K$ and $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ be a finite flat morphism of embeddings. Let $j > 1$ be a rational number, $\pi_L$ a prime element of $L$ and $e = \text{ord}\pi_L$ be the ramification index.*
*1.* ([1] Proposition 7.3) *Assume $A = O_K$. Suppose that, for each $j' > j$, there exists a finite separable extension $K'$ of $K$ such that the base change $X^{j'}(\mathbf{B} \to B)_{K'}$ is isomorphic to the disjoint union of finitely many copies of $X^{j'}(\mathbf{A} \to A)_{K'}$ as an affinoid variety over $X^{j'}(\mathbf{A} \to A)$. Then there is an integer $0 \leq n < ej$ such that $\pi_L^n$ annihilates $\Omega_{B/A}$.*
*2.* ([1] Proposition 7.5) *If there is an integer $0 \leq n < ej$ such that $\pi_L^n$ annihilates $\Omega_{B/A}$, then the finite flat map $X^j(\mathbf{B} \to B) \to X^j(\mathbf{A} \to A)$ is étale.*

Corollary 1.16 ([1] Theorem 7.2) *Let $A = O_K$ and let $B$ be the integer ring of a finite étale $K$-algebra. Let $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ be a finite flat morphism of embeddings. Let $j > 1$ be a rational number. Suppose that, for each $j' > j$, there exists a finite separable extension $K'$ of $K$ such that the base change $X^{j'}(\mathbf{B} \to B)_{K'}$ is isomorphic to the disjoint union of finitely many copies of $X^{j'}(\mathbf{A} \to A)_{K'}$ as in Proposition 1.15.1. Let $I$ be the kernel of the surjection $\mathbf{B} \to B$ and let $N_{B/\mathbf{B}}$ be the $B$-module $I/I^2$. Then, we have the following.*
*1. The finite map $X^j(\mathbf{B} \to B) \to X^j(\mathbf{A} \to A)$ is étale and is extended to a finite étale map of stable normalized integral models.*
*2. The finite map $\bar{X}^j(\mathbf{B} \to B) \to \bar{X}^j(\mathbf{A} \to A)$ is étale.*
*3. The twisted normal cone $\bar{C}^j(\mathbf{B} \to B)$ is canonically isomorphic to the covariant vector bundle defined by the $B_{\bar{F},\text{red}}$-module $(Hom_B(N_{B/\mathbf{B}}, B) \otimes_{O_K} N^j) \otimes_{B_{\bar{F}}} B_{\bar{F},\text{red}}$ and the finite map $\bar{X}^j(\mathbf{B} \to B) \to \bar{C}^j(\mathbf{B} \to B)$ is étale.*

Though these statements except Corollary 1.16.3 are proved in [1] Section 7, we present here slightly modified proofs in order to compare with the proofs of the corresponding statements in the logarithmic setting given in Section 4.3. To prove Proposition 1.15, we use the following.

Lemma 1.17 *Let $A = O_L$ be the integer ring of a finite separable extension $L$, $\mathbf{A} \to A$ be an embedding and let $\mathbf{M}$ be an $\mathbf{A}$-module of finite type. Let $j > 1$ be a rational number and $K'$ be a finite separable extension of $K$ such that the stable normalized integral model $\mathcal{A}^j_{O_{K'}}$ of $X^j(\mathbf{A} \to A)$ is defined over $K'$. Let $e$ and $e'$ be the ramification indices of $L$ and of $K'$ over $K$ and $\pi_L$ and $\pi'$ be*

*prime elements of $L$ and $K'$. Assume that $e'/e$ and $e'j$ are integers. Then, the following conditions are equivalent.*
*(1) There exists an integer $0 \leq n < ej$ such that the $A$-module $M = \mathbf{M} \otimes_{\mathbf{A}} A$ is annihilated by $\pi_L^n$.*
*(2) The $\mathcal{A}^j_{O_{K'}}$-module $\mathcal{M}^j = \mathbf{M} \otimes_{\mathbf{A}} \mathcal{A}^j_{O_{K'}}$ is annihilated by $\pi'^{e'j-1}$.*

*Proof of Lemma 1.17.* The image of an element in the kernel $I$ of the surjection $\mathbf{A} \to A$ in $\mathcal{A}^j_{O_{K'}}$ is divisible by $\pi'^{e'j}$. Hence we have a commutative diagram

$$\begin{array}{ccc} \mathbf{A} & \longrightarrow & \mathcal{A}^j_{O_{K'}} \\ \downarrow & & \downarrow \\ A & \longrightarrow & \mathcal{A}^j_{O_{K'}}/(\pi'^{e'j}). \end{array}$$

We show that the ideals of $\mathcal{A}^j_{O_{K'}}/(\pi'^{e'j})$ generated by the image of $\pi_L \in A$ and by the image of $\pi'^{e'/e} \in \mathcal{A}^j_{O_{K'}}$ are equal. Take a lifting $a \in \mathbf{A}$ of $\pi_L \in A$. Then, the image of $a^e$ is a unit times $\pi$ and hence is a unit times $\pi'^{e'}$ in $\mathcal{A}^j_{O_{K'}}/(\pi'^{e'j})$. Since $\mathcal{A}^j_{O_{K'}}$ is $\pi'$-adically complete, we have $a^e = u\pi'^{e'} + v\pi'^{e'j}$ for some $u \in \mathcal{A}^{j\times}_{O_{K'}}$ and $v \in \mathcal{A}^j_{O_{K'}}$. Since $j > 1$ and $\mathcal{A}^j_{O_{K'}}$ is $\pi'$-adically complete, we have $(a/\pi'^{e'/e})^e = u + v\pi'^{e'(j-1)}$ is a unit in $\mathcal{A}^j_{O_{K'}}$. Since $\mathcal{A}^j_{O_{K'}}$ is normal, we have $a/\pi'^{e'/e} \in \mathcal{A}^{j\times}_{O_{K'}}$ and the claim follows.
Assume that the $A$-module $M$ is isomorphic to $A^r \oplus \bigoplus_{i=1}^s A/(\pi_L^{n_i})$ for integers $0 < n_1 \leq \ldots \leq n_s$. Then, by the commutative diagram above and by the equality $(\pi_L) = (\pi'^{e'/e})$ of the ideals of $\mathcal{A}^j_{O_{K'}}/(\pi'^{e'j})$ proved above, the $\mathcal{A}^j_{O_{K'}}/(\pi'^{e'j})$-module $\mathcal{M}^j/\pi'^{e'j}\mathcal{M}^j$ is isomorphic to $(\mathcal{A}^j_{O_{K'}}/(\pi'^{e'j}))^r \oplus \bigoplus_{i=1}^s \mathcal{A}^j_{O_{K'}}/(\pi'^{\min(e'j, e'n_i/e)})$. The condition (1) is clearly equivalent to that $r = 0$ and $n_s < ej$. We see that the condition (2) is also equivalent to this condition by taking the localization at a prime ideal $\mathcal{A}^j_{O_{K'}}$ of height 1 containing $\pi'$. □

*Proof of Proposition 1.15.* 1. Since $A = O_K$, there is an isomorphism $O_K[[T_1, \ldots, T_n]] \to \mathbf{A}$ such that the composition $O_K[[T_1, \ldots, T_n]] \to A$ maps $T_i$ to 0. For $j > 0$, the affinoid variety $X^j(\mathbf{A} \to A)$ is a polydisk. By the proof of Lemma 1.7, there exist a finite separable extension $K'$ of $K$ of ramification index $e'$, an embedding $(\mathbf{B} \otimes_{O_K} O_{K'} \to B')$ in $\mathcal{E}mb_{O_{K'}}$ isomorphic to the embbedding $(O_{K'}[[S_1, \ldots, S_n]]^N \to O_{K'}^N)$ sending $S_i$ to 0 for some $N > 0$, a positive rational number $\epsilon < j$ and an open immersion $X^j(\mathbf{B} \to B) \otimes_K K' \to X^{e'\epsilon}(\mathbf{B} \otimes_{O_K} O_{K'} \to B')$ as a rational subdomain. The affinoid variety $X^{e'\epsilon}(\mathbf{B} \otimes_{O_K} O_{K'} \to B')$ is the disjoint union of finitely many copies of polydisks. Enlarging $K'$ if necessary, we may assume that $e'j$ and $e'\epsilon$ are integers. We may further assume that there is a rational number $j < j' < j + \epsilon$ such that $e'j'$ is an integer, that the stable normalized integral models $\mathcal{B}^{j'}_{O_{K'}}$ and $\mathcal{B}'^{e'\epsilon}_{O_{K'}}$ of $X^{j'}(\mathbf{B} \to B)$ and of $X^{e'\epsilon}(\mathbf{B} \otimes_{O_K} O_{K'} \to B')$ are defined over $K'$ and that $X^{j'}(\mathbf{B} \to B)_{K'}$ is isomorphic to the disjoint union of copies of $X^{j'}(\mathbf{A} \to A)_{K'}$. Since $e'j'$ is an integer, the stable normalized integral model $\mathcal{A}^{j'}_{O_{K'}}$ of $X^{j'}(\mathbf{A} \to A)$ is also defined over $K'$. Then we have a commutative diagram

$$\begin{array}{ccccc} \mathbf{A} & & \longrightarrow & & \mathcal{A}^{j'}_{O_{K'}} \\ \downarrow & & & & \downarrow \\ \mathbf{B} & \longrightarrow & \mathcal{B}'^{e'\epsilon}_{O_{K'}} & \longrightarrow & \mathcal{B}^{j'}_{O_{K'}}. \end{array}$$

We consider the modules $\hat{\Omega}_{\mathbf{A}/O_K} = \varprojlim_n \Omega_{(\mathbf{A}/\mathfrak{m}^n_{\mathbf{A}})/O_K)}, \hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}} = \varprojlim_n \Omega_{(\mathcal{A}^{j'}_{O_{K'}}/\pi'^n\mathcal{A}^{j'}_{O_{K'}})/O_{K'}}$ etc as defined in the beginning of Section 1.1. By Lemma 1.4.2, we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K} & & \longrightarrow & & \mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathcal{A}^{j'}_{O_{K'}}} \hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}} \\ \downarrow & & & & \downarrow \\ \mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K} & \longrightarrow & \mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathcal{B}'^{e'\epsilon}_{O_{K'}}} \hat{\Omega}_{\mathcal{B}'^{e'\epsilon}_{O_{K'}}/O_{K'}} & \longrightarrow & \hat{\Omega}_{\mathcal{B}^{j'}_{O_{K'}}/O_{K'}}. \end{array}$$

We show that the five $\mathcal{B}^{j'}_{O_{K'}}$-modules are free of rank $n$ and that the five maps are injective. We also show that by identifying the modules with their images in $\hat{\Omega}_{\mathcal{B}^{j'}_{O_{K'}}/O_{K'}}$, we have an inclusion $\pi'^{e'j'}\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K} \subset \pi'^{e'\epsilon}\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K}$ of submodules of $\hat{\Omega}_{\mathcal{B}^{j'}_{O_{K'}}/O_{K'}}$. By the assumption on the covering $X^{j'}(\mathbf{B} \to B)_{K'} \to X^{j'}(\mathbf{A} \to A)_{K'}$, the $\mathcal{A}^{j'}_{O_{K'}}$-algebra $\mathcal{B}^{j'}_{O_{K'}}$ is isomorphic to the product of finitely many copies of $\mathcal{A}^{j'}_{O_{K'}}$. Hence the right vertical map $\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathcal{A}^{j'}_{O_{K'}}} \hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}} \to \hat{\Omega}_{\mathcal{B}^{j'}_{O_{K'}}/O_{K'}}$ is an isomorphism. The isomorphism $O_K[[T_1, \ldots, T_n]] \to \mathbf{A}$ in the beginning of the proof induces an isomorphism $O_{K'}\langle T_1/\pi'^{e'j}, \ldots, T_n/\pi'^{e'j}\rangle \to \mathcal{A}^{j'}_{O_{K'}}$ and we see that the $\mathbf{A}$-module $\hat{\Omega}_{\mathbf{A}/O_K}$ and the $\mathcal{A}^{j'}_{O_{K'}}$-module $\hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}}$ are free of rank $n$. Hence $\hat{\Omega}_{\mathcal{B}^{j'}_{O_{K'}}/O_{K'}}$ is also a free $\mathcal{B}^{j'}_{O_{K'}}$-module of rank $n$. Further by the canonical maps $\mathcal{A}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K} \to \hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}}$, the module $\mathcal{A}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K}$ is identified with the submodule $\pi'^{e'j'}\hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}}$. Similarly, the $\mathbf{B}$-module $\hat{\Omega}_{\mathbf{B}/O_K}$ and the $\mathcal{B}'^{e'\epsilon}_{O_{K'}}$-module $\hat{\Omega}_{\mathcal{B}'^{e'\epsilon}_{O_{K'}}/O_{K'}}$ are free of rank $n$ and $\mathcal{B}'^{e'\epsilon}_{O_{K'}} \otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K}$ is identified with the submodule $\pi'^{e'\epsilon}\hat{\Omega}_{\mathcal{B}'^{e'\epsilon}_{O_{K'}}/O_{K'}}$. Since $X^j(\mathbf{B} \to B) \otimes_K K'$ is a rational subdomain of $X^{e'\epsilon}(\mathbf{B} \otimes_{O_K} O_{K'} \to B')$, the map $\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathcal{B}'^{e'\epsilon}_{O_{K'}}} \hat{\Omega}_{\mathcal{B}'^{e'\epsilon}_{O_{K'}}/O_{K'}} \to \hat{\Omega}_{\mathcal{B}^{j'}_{O_{K'}}/O_{K'}}$ is an injection. Thus, we obtain an inclusion $\pi'^{e'j'}\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K} \subset \pi'^{e'\epsilon}\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K}$ as submodules of $\hat{\Omega}_{\mathcal{B}^{j'}_{O_{K'}}/O_{K'}}$.

Thus the $\mathcal{B}^{j'}_{O_{K'}}$-module $\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{B}} \Omega_{\mathbf{B}/\mathbf{A}} = \mathrm{Coker}(\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K} \to \mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K})$ is annihilated by $\pi'^{e'(j'-\epsilon)}$. Since $0 < j - \epsilon < j' - \epsilon < j$, it suffices to apply Lemma 1.17 (2)⇒(1).

2. Let $K'$ be a finite separable extension such that $e'j$ is an integer and the stable normalized integral models $\mathcal{A}^j_{O_{K'}}$ and $\mathcal{B}^j_{O_{K'}}$ are defined over $K'$. By the proof of Lemma 1.9.2, we have $\mathcal{B}^j_{O_{K'}} \otimes_{O_{K'}} K' = \mathbf{B} \otimes_{\mathbf{A}} \mathcal{A}^j_{O_{K'}} \otimes_{O_{K'}} K'$ and the map $\mathcal{A}^j_{O_{K'}} \otimes_{O_{K'}} K' \to \mathcal{B}^j_{O_{K'}} \otimes_{O_{K'}} K'$ is finite flat. By Lemma 1.17 (1)⇒(2), the $\mathcal{B}^j_{O_{K'}}$-module $\mathcal{B}^j_{O_{K'}} \otimes_{\mathbf{B}} \Omega_{\mathbf{B}/\mathbf{A}}$ is annihilated by $\pi'^{n'}$ for an integer $0 \le n' < e'j$. Hence the map $\mathcal{A}^j_{O_{K'}} \otimes_{O_{K'}} K' \to \mathcal{B}^j_{O_{K'}} \otimes_{O_{K'}} K'$ is étale. □

*Proof of Corollary 1.16.* 1. It follows from Proposition 1.15 that the map $X^j(\mathbf{B} \to B) \to X^j(\mathbf{A} \to A)$ is finite étale. By Lemma 1.12, the fiber $(\bar{f}^j)^{-1}(\bar{y})$ has the same cardinality as the degree of the map $X^j(\mathbf{B} \to B) \to X^j(\mathbf{A} \to A)$ in the notation there. Hence the finite map $X^j(\mathbf{B} \to B)_{O_{K'}} \to X^j(\mathbf{A} \to A)_{O_{K'}}$ of the normalized integral models is étale at a point of $X^j(\mathbf{A} \to A)_{O_{K'}}$ in the closed fiber. Since $X^j(\mathbf{A} \to A)_{O_{K'}}$ is a regular Noetherian scheme, the assertion follows by the purity of branch locus.

2. Clear from 1.

3. Since the surjection $\mathbf{B} \to B$ is regular, the twisted normal cone $\bar{C}^j(\mathbf{B} \to B)$ is canonically isomorphic to the covariant vector bundle defined by the $B_{\bar{F},\mathrm{red}}$-module $(Hom_B(I/I^2, B) \otimes_{O_K} N^j) \otimes_{B_{\bar{F}}} B_{\bar{F},\mathrm{red}}$. We consider the commutative diagram

$$\begin{array}{ccccc}
\bar{X}^j(\mathbf{B} \to B) & \longrightarrow & \bar{C}^j(\mathbf{B} \to B) & \longrightarrow & \mathrm{Spec}\ B_{\bar{F},\mathrm{red}} \\
\downarrow & & \downarrow & & \downarrow \\
\bar{X}^j(\mathbf{A} \to O_K) & \longrightarrow & \bar{C}^j(\mathbf{A} \to O_K) & \longrightarrow & \mathrm{Spec}\ \bar{F}
\end{array}$$

in Lemma 1.14.2. Since the map $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ is finite and flat, the right square is cartesian. Hence the middle vertical arrow is étale. Since $A = O_K$, the lower left horizontal arrow is an isomorphism by Lemma 1.14.3. By 2, the left vertial arrow is finite étale. Thus the assertion is proved. □

## 2 Filtration by ramification groups: the non-logarithmic case

### 2.1 Construction

In this subsection, we rephrase the definition of the filtration by ramification groups given in the previous paper [1] by using the construction in Section 1. The main purpose is to emphasize the parallelism between the non-logarithmic construction recalled here and the logarithmic construction to be recalled in Section 5.1.

Let $\Phi$ : (Finite Étale/$K$) $\to$ $G_K$-(Finite Sets) denote the fiber functor sending a finite étale $K$-algebra $L$ to the finite set $\Phi(L) = Hom_{K\text{-alg}}(L, \bar{K})$ with the continuous $G_K$-action. For a rational number $j > 0$, we define a functor $\Phi^j$ : (Finite Étale/$K$) $\to$ $G_K$-(Finite Sets) as the composition of the functor (Finite Étale/$K$) $\to$ (Finite Flat/$O_K$) sending a finite étale $K$-algebra $L$ to the integral closure $O_L$ of $O_K$ in $L$ and the functor $\Psi^j$ : (Finite Flat/$O_K$) $\to$ $G_K$-(Finite Sets) defined at the end of Section 1.2. The map (1.9.3) defines a surjection $\Phi \to \Phi^j$ of functors. In [1], we define the filtration by ramification groups on $G_K$ by using the family of surjections $(\Phi \to \Phi^j)_{j>0,\in\mathbb{Q}}$ of functors. The filtration by the ramification groups $G_K^j \subset G_K, j > 0, \in \mathbb{Q}$ is characterized by the condition that the canonical map $\Phi(L) \to \Phi^j(L)$ induces a bijection $\Phi(L)/G_K^j \to \Phi^j(L)$ for each finite étale algebra $L$ over $K$.
The functor $\Phi^j$ is defined by the commutativity of the diagram

$$\begin{tikzcd}
\text{(Finite Étale}/K) \arrow[r, "\Phi^j"] \arrow[d] & G_K\text{-(Finite Sets)} \\
\text{(Finite Flat}/O_K) \arrow[ur, "\Psi^j"'] & G_K\text{-}(\mathrm{Aff}/\bar{F}) \arrow[u, "\pi_0"'] \\
\mathcal{E}mb_{O_K} \arrow[u] \arrow[r, "X^j"'] & \text{(smooth Affinoid}/K) \arrow[u, "X\mapsto\bar{X}"']
\end{tikzcd}$$

We briefly recall how the other arrows in the diagram are defined. The forgetful functor $\mathcal{E}mb_{O_K} \to$ (Finite Flat/$O_K$) sends $(\mathbf{A} \to A)$ to $A$. The functor $X^j$ : $\mathcal{E}mb_{O_K} \to$ (smooth Affinoid/$K$) is defined by the $j$-th tubular neighborhood. The functor (smooth Affinoid/$K$) $\to$ $G_K$-(Aff/$\bar{F}$) sends $X$ to the geometric closed fiber $\bar{X}$ of the stable normalized integral model. The functor $\pi_0$ : $G_K$-(Aff/$\bar{F}$) $\to$ $G_K$-(Finite Sets) is defined by the set of connected components. They induce a functor $\Psi^j$ : (Finite Flat/$O_K$) $\to$ $G_K$-(Finite Sets) by Lemma 1.9. The functor $\Phi^j$ is defined as the composition of $\Psi^j$ with the functor (Finite Étale/$K$) $\to$ (Finite Flat/$O_K$) sending a finite étale algebra $L$ over $K$ to the integral closure $O_L$ in $L$ of $O_K$. More concretely, we have

$$\Phi^j(L) = \varprojlim_{(\mathbf{A}\to O_L)\in\mathcal{E}mb_{O_K}(O_L)} \pi_0(\bar{X}^j(\mathbf{A} \to O_L))$$

for a finite étale $K$-algebra $L$.
For a rational number $j \geq 0$, we define a functor $\Phi^{j+}$ : (Finite Étale/$K$) $\to\to$ $G_K$-(Finite Sets) by $\Phi^{j+}(L) = \varinjlim_{j'>j} \Phi^{j'}(L)$ for a finite étale $K$-algebra $L$. We define a closed normal subgroup $G_K^{j+}$ to be $\overline{\cup_{j'>j} G_K^{j'}}$. Then we have $\Phi^{j+}(L) = \Phi(L)/G_K^{j+}$. The finite set $\Phi^{j+}(L)$ has the following geometric description.

Lemma 2.1 *Let $B$ be the integer ring of a finite étale algebra $L$ over $K$ and $j > 0$ be a rational number. Let $(f, \mathbf{f}) : (\mathbf{A} \to O_K) \to (\mathbf{B} \to B)$ be a finite* *flat morphism of embeddings. Let* $f^j : X^j(\mathbf{B} \to B) \to X^j(\mathbf{A} \to O_K)$ *and* $\bar{f}^j : \bar{X}^j(\mathbf{B} \to B) \to \bar{X}^j(\mathbf{A} \to O_K)$ *be the canonical maps. Let* $0 \in X^j(\mathbf{A} \to O_K)$ *be the point corresponding to the map* $\mathbf{A} \to O_K$ *and* $\bar{0} \in \bar{X}^j(\mathbf{A} \to O_K)$ *be its specialization. Then the maps* (1.8.0), (1.12.1) *and the specialization map form a commutative diagram*

$$
\begin{array}{ccccc}
\Phi(L) & \longrightarrow & \Phi^{j+}(L) & \longrightarrow & \Phi^j(L) \\
\downarrow & & \downarrow & & \downarrow \\
(f^j)^{-1}(0) & \longrightarrow & (\bar{f}^j)^{-1}(0) & \longrightarrow & \pi_0(\bar{X}^j(\mathbf{B} \to B))
\end{array}
\tag{2.1.1}
$$

*and the vertical arrows are bijections.*

*Proof.* Since the map $(\mathbf{A} \to O_K) \to (\mathbf{B} \to B)$ is finite flat, the map $\mathbf{B} \to B$ induces an isomorphism $\mathcal{B}^j_K \otimes_{\mathcal{A}^j_K} K \to L$. Hence we obtain a bijection $\Phi(L) = Hom_{K\text{-alg}}(L, \bar{K}) \to (f^j)^{-1}(0)$. By Lemma 1.12 and the definition of $\Phi^{j+}(L)$, we have a bijection $\Phi^{j+}(L) \to (\bar{f}^j)^{-1}(0)$. The bijection $\Phi^j(L) \to \pi_0(\bar{X}^j(\mathbf{B} \to B))$ is clear from the definition of $\Phi^j(L)$. The commutativity is clear. □

For a finite étale algebra $L$ over $K$ and a rational number $j > 0$, we say that the ramification of $L$ is bounded by $j$ if the canonical map $\Phi(L) \to \Phi^j(L)$ is a bijection. Let $A = O_K$ and let $B = O_L$ be the integer ring of a finite étale $K$-algebra $L$ and $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ be a finite flat morphism of embeddings. Then, since the map $X^j(\mathbf{B} \to B) \to X^j(\mathbf{A} \to A)$ is finite flat of degree $[L : K]$, the ramification of $L$ is bounded by $j$ if and only if there exists a finite separable extension $K'$ of $K$ such that the affinoid variety $X^j(\mathbf{B} \to B)_{K'}$ is isomorphic to the disjoint union of finitely many copies of $X^j(\mathbf{A} \to A)_{K'}$ over $X^j(\mathbf{A} \to A)_{K'}$. We say that the ramification of $L$ is bounded by $j+$ if the ramification of $L$ is bounded by every rational number $j' > j$. The ramification of $L$ is bounded by $j+$ if and only if the canonical map $\Phi(L) \to \Phi^{j+}(L)$ is a bijection.

Lemma 2.2 *Let* $K \to K'$ *be a map of complete discrete valuation fields inducing a local homomorphism* $O_K \to O_{K'}$ *of integer rings. Assume that a prime element of* $K$ *goes to a prime element of* $K'$ *and that the residue field* $F'$ *of* $K'$ *is a separable extension of the residue field* $F$ *of* $K$. *Then, for a rational number* $j > 0$, *the map* $G_{K'} \to G_K$ *induces a surjection* $G^j_{K'} \to G^j_K$.

*Proof.* Let $A$ be the integer ring of a finite étale $K$-algebra $L$ and $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb_{O_K}$. By the assumption, the tensor product $A \otimes_{O_K} O_{K'}$ is the integer ring of $L \otimes_K K'$. By the isomorphism $X^j(\mathbf{A} \to A)\hat{\otimes}_K K' \to X^j(\mathbf{A}\hat{\otimes}_{O_K} O_{K'} \to A \otimes_{O_K} O_{K'})$ in Section 1.2 and Theorem 1.10, the natural map $\Phi^j(L \otimes_K K') \to \Phi^j(L)$ is a bijection. Hence the assertion follows. □

*Example.* Let $K = \mathbb{F}_p(x, y)((\pi))$ and put $L = K[t]/(t^p - t - \frac{x}{\pi^{p^2}}), M = L[t_1, t_2]/(t_1^p - t_1 - \frac{x}{\pi^{p^3}}, t_2^p - t_2 - \frac{y}{\pi^{p^3}})$ and $G = \mathrm{Gal}(M/K) \simeq \mathbb{F}_p^3$. Then we have $G^j = G$ for $j \leq p^2$, $G^j = H = \mathrm{Gal}(M/L) \simeq \mathbb{F}_p^2$ for $p^2 < j \leq p^3$ and $G^j = 1$ for $p^3 < j$.

We put $z = \pi^p t$. Then we have $O_L = O_K[z]/(z^p - \pi^{p(p-1)}z - x)$ and $L = \mathbb{F}_p(z,y)((\pi))$. By putting $s = t_1 - \frac{z}{\pi^{p^2}}$, we also have $M = L[s,t_2]/(s^p - s - \frac{z(-1+\pi^{p(p-1)^2})}{\pi^{p(p^2-p+1)}}, t_2^p - t_2 - \frac{x}{\pi^{p^3}})$. We put $M_1 = L(s) \subset M$. Then we have $H^j = H$ for $j \leq p(p^2-p+1)$, $H^j = \mathrm{Gal}(M/M_1) \simeq \mathbb{F}_p$ for $p(p^2-p+1) < j \leq p^3$ and $H^j = 1$ for $p^3 < j$.
This example shows that the filtration on the subgroup $H$ induced from the filtration by ramification groups on $G$ is not the filtration by ramification groups on $H$ even after renumbering. It also shows that the "lower numbering" filtration is not equal to the upper numbering filtration defined here even after renumbering.

## 2.2 Functoriality of the closed fibers of tubular neighborhoods: An equal characteristic case

For a positive rational number $j > 0$, let (Finite Étale$/K)^{\leq j+}$ denote the full subcategory of (Finite Étale$/K$) consisting of étale $K$-algebras whose ramification is bounded by $j+$. In this subsection and the following one, we assume the following condition (F) is satisfied.

(F) There exists a perfect subfield $F_0$ of $F$ such that $F$ is finitely generated over $F_0$.

Further assuming that $p$ is not a uniformizer of $K$, we will define a twisted tangent space $\Theta^j$ and show that the functor $\bar{X}^j : \mathcal{E}mb_{O_K} \to G_K\text{-}(\mathrm{Aff}/\bar{F})$ induces a functor

$$\bar{X}^j : (\text{Finite Étale}/K)^{\leq j+} \to G_K\text{-}(\text{Finite Étale}/\Theta^j).$$

In this subsection, we study the easier case where $K$ is of characteristic $p$.
Let $F_0$ be a perfect subfield of $F$ such that $F$ is finitely generated over $F_0$. We assume $K$ is of characteristic $p$. Then, $F_0$ is naturally identified with a subfield of $K$. We first define a functor

$$(\text{Finite Étale}/K) \to \mathcal{E}mb_{O_K}.$$

In this subsection, $A$ denotes the integer ring of a finite étale $K$-algebra.

Lemma 2.3 *Let $A$ be the integer ring of a finite étale $K$-algebra.*
*1. Let $(A/\mathfrak{m}_A^n \otimes_{F_0} O_K)^\wedge$ denote the formal completion of $A/\mathfrak{m}_A^n \otimes_{F_0} O_K$ of the surjection $A/\mathfrak{m}_A^n \otimes_{F_0} O_K \to A/\mathfrak{m}_A^n$ sending $a \otimes b$ to $ab$. Then the projective limit*

$$(A\hat{\otimes}_{F_0} O_K)^\wedge = \varprojlim_n (A/\mathfrak{m}_A^n \otimes_{F_0} O_K)^\wedge$$

*is an $O_K$-algebra formally of finite type and formally smooth over $O_K$.*
*2. Let $(A\hat{\otimes}_{F_0} O_K)^\wedge \to A$ be the limit of the surjections $(A/\mathfrak{m}_A^n \otimes_{F_0} O_K)^\wedge \to A/\mathfrak{m}_A^n$. Then $((A\hat{\otimes}_{F_0} O_K)^\wedge \to A)$ is an object of $\mathcal{E}mb_{O_K}$.*

*3. Let $A \to B$ be a morphism of the integer rings of finite étale $K$-algebras. Then it induces a finite flat morphism $((A\hat{\otimes}_{F_0}O_K)^\wedge \to A) \to ((B\hat{\otimes}_{F_0}O_K)^\wedge \to B)$ of $\mathcal{E}mb_{O_K}$.*

*Proof.* 1. We may assume $A$ is local. Let $E$ be the residue field of $A$ and take a transcendental basis $(\bar{t}_1, \dots, \bar{t}_m)$ of $E$ over the perfect subfield $F_0$ such that $E$ is a finite separable extension of $F_0(\bar{t}_1, \dots, \bar{t}_m)$. Take a lifting $(t_1, \dots, t_m)$ in $A$ of $(\bar{t}_1, \dots, \bar{t}_m)$ and a prime element $t_0 \in A$. We define a map $F_0[T_0, \dots, T_m] \to A$ by sending $T_i$ to $t_i$. Then $A$ is finite étale over the completion of the local ring of $F_0[T_0, \dots, T_m]$ at the prime ideal $(T_0)$. Hence there exist an étale scheme $X$ over $\mathbb{A}^{m+1}_{F_0}$, a point $\xi$ of $X$ above $(T_0)$ and an $F_0$-isomorphism $\varphi : \hat{O}_{X,\xi} \to A$. Let $i : \text{Spec } A \to X \otimes_{F_0} O_K$ be the map defined by $\varphi$ and $O_K \to A$. Then $(A\hat{\otimes}_{F_0}O_K)^\wedge$ is isomorphic to the coordinate ring of the formal completion of $X \otimes_{F_0} O_K$ along the closed immersion $i : \text{Spec } A \to X \otimes_{F_0} O_K$. Hence $(A\hat{\otimes}_{F_0}O_K)^\wedge$ is formally of finite type and formally smooth over $O_K$.
2. Since the map $(A\hat{\otimes}_{F_0}O_K)^\wedge \to A$ is surjective, the assertion follows from 1.
3. Since $(B\hat{\otimes}_{F_0}O_K)^\wedge = B \otimes_A (A\hat{\otimes}_{F_0}O_K)^\wedge$, the assertion follows. □

Thus, we obtain a functor (Finite Étale$/K$) $\to \mathcal{E}mb_{O_K}$ sending a finite étale $K$-algebra $L$ to $((O_L\hat{\otimes}_{F_0}O_K)^\wedge \to O_L)$. For a rational number $j > 0$, we have a sequence of functors

$$(\text{Finite Étale}/K) \longrightarrow \mathcal{E}mb_{O_K} \longrightarrow (\text{smooth Affinoid}/K) \longrightarrow G_K\text{-}(\text{Aff}/\bar{F}).$$

We also let $\bar{X}^j$ denote the composite functor (Finite Étale$/K$) $\to G_K$-(Aff$/\bar{F}$). For a finite étale $K$-algebra $L$, we have

$$\bar{X}^j(L) = \bar{X}^j((O_L\hat{\otimes}_{F_0}O_K)^\wedge \to O_L).$$

We define an object $\Theta^j$ of $G_K$-(Aff$/\bar{F}$) to be the $\bar{F}$-vector space $\Theta^j = Hom_F(\hat{\Omega}_{O_K/F_0} \otimes_{O_K} F, N^j)$ regarded as an affine scheme over $\bar{F}$ with a natural $G_K$-action. Let $G_K$-(Finite Étale$/\Theta^j$) denote the subcategory of $G_K$-(Aff$/\bar{F}$) whose objects are finite étale schemes over $\Theta^j$ and morphisms are over $\Theta^j$.

Lemma 2.4 *For a rational number $j > 1$, the functor $\bar{X}^j$ : (Finite Étale$/K$) $\to$ $G_K$-(Aff$/\bar{F}$) induces a functor $\bar{X}^j$ : (Finite Étale$/K)^{\leq j+} \to G_K$-(Finite Étale$/\Theta^j$).*

*Proof.* The canonical map $\hat{\Omega}_{O_K/F_0} \otimes_{O_K} (O_K\hat{\otimes}_{F_0}O_K)^\wedge \to \hat{\Omega}_{(O_K\hat{\otimes}_{F_0}O_K)^\wedge/O_K}$ is an isomorphism by the definition of $(O_K\hat{\otimes}_{F_0}O_K)^\wedge$. Hence, we obtain isomorphisms $\bar{X}^j(K) \to \bar{C}^j((O_K\hat{\otimes}_{F_0}O_K)^\wedge \to O_K) \to \Theta^j$ by Lemma 1.14.3. We identify $\bar{X}^j(K)$ with $\Theta^j$ by this isomorphism. Let $L$ be a finite étale $K$-algebras whose ramification is bounded by $j+$. Then, by Corollary 1.16, the map $\bar{X}^j(L) \to \bar{X}^j(K) = \Theta^j$ is finite and étale. Thus the assertion is proved. □

The construction in this subsection is independent of the choice of perfect subfield $F_0 \subset F$ by the following Lemma.

Lemma 2.5 *Let $K$ be a complete discrete valuation field of characteristic $p>0$ satisfying the condition (F). Let $F_0$ and $F'_0$ be perfect subfields of $F$ such that $F$ is finitely generated over $F_0$ and $F'_0$.*
*1. There exists a perfect subfield $F''_0$ of $F$ containing $F_0$ and $F'_0$.*
*2. Assume $F_0 \subset F'_0$. Then $F'_0$ is a finite separable extension of $F_0$. For the integer ring $A$ of a finite étale algebra over $K$, the canonical map $(A\hat{\otimes}_{F_0}O_K)^\wedge \to (A\hat{\otimes}_{F'_0}O_K)^\wedge$ is an isomorphism.*

*Proof.* 1. The maximum perfect subfield $\bigcap_n F^{p^n}$ of $F$ contains $F_0$ and $F'_0$ as subfields.
2. Since $F'_0$ is a perfect subfield of a finitely generated field $F$ over $F_0$, it is a finite extension of $F_0$. Since the canonical map $(A\hat{\otimes}_{F_0}O_K)^\wedge \to (A\hat{\otimes}_{F'_0}O_K)^\wedge$ is finite étale and the induced map $(A\hat{\otimes}_{F_0}O_K)^\wedge/\mathfrak{m}_{(A\hat{\otimes}_{F_0}O_K)^\wedge} \to (A\hat{\otimes}_{F'_0}O_K)^\wedge/\mathfrak{m}_{(A\hat{\otimes}_{F'_0}O_K)^\wedge}$ is an isomorphism, the assertion follows. □

## 2.3 Functoriality of the closed fibers of tubular neighborhoods: A mixed characteristic case

In this subsection, we keep the assumption:

(F) There exists a perfect subfield $F_0$ of $F$ such that $F$ is finitely generated over $F_0$.

We do not assume that the characterisic of $K$ is $p$. Under the assumption (F), there exists a subfield $K_0$ of $K$ such that $O_{K_0} = O_K \cap K_0$ is a complete discrete valuation ring with residue field $F_0$. If $K$ is of characteristic 0, the fraction field $K_0$ of the ring of the Witt vectors $W(F_0) = O_{K_0}$ regarded as a subfield of $K$ satisfies the conditions. If $K$ is of characteristic $p$, we naturally identify $F_0$ as a subfield of $K$ and the subfield $F_0((t))$ for any non-zero element $t \in \mathfrak{m}_K$ satisfies the conditions. In this subsection, we take a subfield $K_0$ of $K$ such that $O_{K_0} = O_K \cap K_0$ is a complete discrete valuation ring with residue field $F_0$. Here, we do *not* define a functor (Finite Étale$/K) \to \mathcal{E}mb_{O_K}$. Instead, we introduce a new category $\mathcal{E}mb_{K,O_{K_0}}$ and a functor

$$\mathcal{E}mb_{K,O_{K_0}} \to \mathcal{E}mb_{O_K}.$$

In this subsection, $A$ denotes the integer ring of a finite étale $K$-algebra and $\pi_0$ denotes a prime element of the subfield $K_0 \subset K$. For a complete Noetherian local $O_{K_0}$-algebra $R$ formally smooth over $O_{K_0}$, we define its relative dimension over $O_{K_0}$ to be the sum $\mathrm{tr.deg}(E/k) + \dim_E \mathfrak{m}_R/(\pi_0, \mathfrak{m}_R^2)$ of the transcendental degree of $E = R/\mathfrak{m}_R$ over $k$ and the dimension $\dim_E \mathfrak{m}_R/(\pi_0, \mathfrak{m}_R^2)$.

Definition 2.6 *Let $K$ be a complete discrete valuation field and $K_0$ be a subfield of $K$ such that $O_{K_0} = O_K \cap K_0$ is a complete discrete valuation ring with perfect residue field $F_0$ and that $F$ is finitely generated over $F_0$.*
*1. We define $\mathcal{E}mb_{K,O_{K_0}}$ to be the category whose objects and morphisms are as follows. An object of $\mathcal{E}mb_{K,O_{K_0}}$ is a triple $(\mathbf{A}_0 \to A)$ where:*

- *$A$ is the integer ring of a finite étale $K$-algebra.*
- *$\mathbf{A}_0$ is a complete semi-local Noetherian $O_{K_0}$-algebra formally smooth of relative dimension $\mathrm{tr.deg}(F/F_0)+1$ over $O_{K_0}$.*
- *$\mathbf{A}_0 \to A$ is a regular surjection of codimension 1 of $O_{K_0}$-algebras inducing an isomorphism $\mathbf{A}_0/\mathfrak{m}_{\mathbf{A}_0} \to A/\mathfrak{m}_A$.*

*A morphism $(f, \mathbf{f}) : (\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ is a pair of an $O_K$-homomorphism $f : A \to B$ and an $O_{K_0}$-homomorphism $\mathbf{f} : \mathbf{A}_0 \to \mathbf{B}_0$ such that the diagram*

$$\begin{array}{ccc} \mathbf{A}_0 & \longrightarrow & A \\ {\scriptstyle \mathbf{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{B}_0 & \longrightarrow & B \end{array}$$

*is commutative.*
*2. For the integer ring $A$ of a finite étale $K$-algebra, we define $\mathcal{E}mb_{K,O_{K_0}}(A)$ to be the subcategory of $\mathcal{E}mb_{K,O_{K_0}}$ whose objects are of the form $(\mathbf{A}_0 \to A)$ and morphisms are of the form $(\mathrm{id}_A, \mathbf{f})$.*
*3. We say that a morphism $(\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ is finite flat if $\mathbf{A}_0 \to \mathbf{B}_0$ is finite flat and the map $\mathbf{B}_0 \otimes_{\mathbf{A}_0} A \to B$ is an isomorphism.*

Lemma 2.7 *1. If $A$ is the integer ring of a finite étale $K$-algebra, then the category $\mathcal{E}mb_{K,O_{K_0}}(A)$ is non-empty.*
*2. Let $(\mathbf{A}_0 \to A)$ and $(\mathbf{B}_0 \to B)$ be objects of $\mathcal{E}mb_{K,O_{K_0}}$ and $A \to B$ be an $O_K$-homomorphism. Then there exists a homomorphism $(\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ in $\mathcal{E}mb_{K,O_{K_0}}$ extending $A \to B$.*
*3. Let $(\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ be a morphism of $\mathcal{E}mb_{K,O_{K_0}}$. If a prime element $\pi_0$ of $K_0$ is not a prime element of any factor of $A$, then the map $(\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ is finite and flat.*

*Proof.* 1. We may assume $A$ is local. Take a transcendental basis $(\bar{t}_1, \ldots, \bar{t}_m)$ of the residue field $E$ of $A$ over $k$ such that $E$ is a finite separable extension of $k(\bar{t}_1, \ldots, \bar{t}_m)$. Take a lifting $(t_1, \ldots, t_m)$ in $O_K$ of $(\bar{t}_1, \ldots, \bar{t}_m)$ and a prime element $t_0$ of $A$. Then $A$ is unramified over the completion of the local ring of $O_{K_0}[T_0, \ldots, T_m]$ at the prime ideal $(\pi_0, T_0)$ by the map sending $T_i$ to $t_i$. Hence there are an étale scheme $X$ over $\mathbb{A}^{m+1}_{O_{K_0}}$, a point $\xi$ of $X$ above $(\pi_0, T_0)$ and a regular surjection $\varphi : \hat{O}_{X,\xi} \to A$ of codimension 1. Let $\mathbf{A}_0$ be the $O_{K_0}$-algebra $\hat{O}_{X,\xi}$. Then $(\mathbf{A}_0 \to A)$ is an object of $\mathcal{E}mb_{K,O_{K_0}}$.
2. Since $\mathbf{A}_0$ is formally smooth over $O_{K_0}$, it follows from that $\mathbf{B}_0$ is the formal completion of itself with respect to the surjection $\mathbf{B}_0 \to B$.
3. We may assume $A$ and $B$ are local. We show that the map $\mathbf{B}_0 \otimes_{\mathbf{A}_0} A \to B$ is an isomorphism. Let $f$ be a generator of the kernel of $\mathbf{A}_0 \to A$ and consider the class of $f$ in $\mathfrak{m}_{\mathbf{A}_0}/\mathfrak{m}^2_{\mathbf{A}_0}$. We show that the image of the class of $f$ in $\mathfrak{m}_{\mathbf{B}_0}/\mathfrak{m}^2_{\mathbf{B}_0}$ is not 0. Let $t_0 \in \mathbf{A}_0$ and $t'_0 \in \mathbf{B}_0$ be liftings of prime

elements of $A$ and $B$ respectively. By the assumption that $\pi_0$ is not a prime element, the surjection $\hat{\Omega}_{\mathbf{A}_0/O_{K_0}} \to \hat{\Omega}_{A/O_{K_0}}$ induces an isomorphism $\hat{\Omega}_{\mathbf{A}_0/O_{K_0}} \otimes_{\mathbf{A}_0} A/\mathfrak{m}_A \to \hat{\Omega}_{A/O_{K_0}} \otimes_A A/\mathfrak{m}_A$. Hence the image of $dt_0$ is a basis of the kernel of $\hat{\Omega}_{\mathbf{A}_0/O_{K_0}} \otimes_{\mathbf{A}_0} A/\mathfrak{m}_A \to \Omega_{(A/\mathfrak{m}_A)/k}$. Therefore, $(\pi_0, t_0)$ is a basis of $\mathfrak{m}_{\mathbf{A}_0}/\mathfrak{m}^2_{\mathbf{A}_0}$. Further, by the assumption that $\pi_0$ is not a prime element, the kernel of the map $\mathfrak{m}_{\mathbf{A}_0}/\mathfrak{m}^2_{\mathbf{A}_0} \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is generated by the class of $\pi_0$. Hence the class of $f$ is a non-zero multiplie of the class of $\pi_0$. Similarly $(\pi_0, t'_0)$ is a basis of $\mathfrak{m}_{\mathbf{B}_0}/\mathfrak{m}^2_{\mathbf{B}_0}$. Thus the image of $f$ in $\mathfrak{m}_{\mathbf{B}_0}/\mathfrak{m}^2_{\mathbf{B}_0}$ is not zero as is claimed. Hence the kernel of $\mathbf{B}_0 \to B$ is also generated by the image of $f$ and the map $\mathbf{B}_0 \otimes_{\mathbf{A}_0} A \to B$ is an isomorphism. Since $B$ is finite over $A$, $\mathbf{B}_0$ is also finite over $\mathbf{A}_0$ by Nakayama's lemma. Since $\dim \mathbf{A}_0 = \dim \mathbf{B}_0 = 2$, the assertion follows by EGA Chap $0_{\mathrm{IV}}$ Corollaire (17.3.5) (ii). $\square$

Corollary 2.8 *Let $A$ be the integer ring of a finite étale $K$-algebra. If a prime element $\pi_0$ of $K_0$ is not a prime element of any factor of $A$, then every morphism of $\mathcal{E}mb_{K,O_{K_0}}(A)$ is an isomorphism.*

*Proof.* If $(\mathbf{A}_0 \to A) \to (\mathbf{A}'_0 \to A)$ is a map, the map $\mathbf{A}_0 \to \mathbf{A}'_0$ is finite flat of degree 1 by Lemma 2.7.3. Hence it is an isomorphism. $\square$

We define a functor $\mathcal{E}mb_{K,O_{K_0}} \to \mathcal{E}mb_{O_K}$.

Lemma 2.9 *Let $(\mathbf{A}_0 \to A)$ be an object of $\mathcal{E}mb_{K,O_{K_0}}$.*

*1. Let $(\mathbf{A}_0/\mathfrak{m}^n_{\mathbf{A}_0} \otimes_{O_{K_0}} O_K)^\wedge$ denote the formal completion of $\mathbf{A}_0/\mathfrak{m}^n_{\mathbf{A}_0} \otimes_{O_{K_0}} O_K$ of the surjection $\mathbf{A}_0/\mathfrak{m}^n_{\mathbf{A}_0} \otimes_{O_{K_0}} O_K \to A/\mathfrak{m}^n_A$ sending $a \otimes b$ to $ab$. Then the projective limit*

$$(\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge = \varprojlim_n (\mathbf{A}_0/\mathfrak{m}^n_{\mathbf{A}_0} \otimes_{O_{K_0}} O_K)^\wedge$$

*is an $O_K$-algebra formally of finite type and formally smooth over $O_K$.*

*2. Let $(\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge \to A$ be the limit of the surjections $(\mathbf{A}_0/\mathfrak{m}^n_{\mathbf{A}_0} \otimes_{O_{K_0}} O_K)^\wedge \to A/\mathfrak{m}^n_A$. Then $((\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge \to A)$ is an object of $\mathcal{E}mb_{O_K}$.*

*3. Let $(\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ be a morphism of $\mathcal{E}mb_{K,O_{K_0}}$. Then it induces a morphism $((\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge \to A) \to ((\mathbf{B}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge \to B)$ of $\mathcal{E}mb_{O_K}$.*

*Proof.* 1. We may assume $A$ and hence $\mathbf{A}_0$ are local. Let $E$ be the residue field of $A$ and take a transcendental basis $(\bar{t}_1, \ldots, \bar{t}_m)$ of $E$ over $k$ such that $E$ is a finite separable extension of $k(\bar{t}_1, \ldots, \bar{t}_m)$. Take a lifting $(t_1, \ldots, t_m)$ in $\mathbf{A}_0$ of $(\bar{t}_1, \ldots, \bar{t}_m)$. By our assumption, the quotient ring $\mathbf{A}_0/\pi_0\mathbf{A}_0$ is a regular local ring of dimension 1 and hence is a discrete valuation ring. Take a lifting $t_0 \in \mathbf{A}_0$ of a prime element of $\mathbf{A}_0/\pi_0\mathbf{A}_0$. We define a map $O_{K_0}[T_0, \ldots, T_m] \to \mathbf{A}_0$ by sending $T_i$ to $t_i$. Then $\mathbf{A}_0$ is finite étale over the completion of the local ring of $O_{K_0}[T_0, \ldots, T_m]$ at the prime ideal $(T_0, \pi_0)$. Hence there exist an étale scheme $X$ over $\mathbb{A}^{m+1}_{O_{K_0}}$, a point $\xi$ of $X$ above $(T_0, \pi_0)$ and a $O_{K_0}$-isomorphism $\varphi : \hat{O}_{X,\xi} \to \mathbf{A}_0$. Let $i : \operatorname{Spec} A \to X \otimes_{O_{K_0}} O_K$ be the map defined by $\varphi$ and $O_K \to A$. Then $(\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge$ is isomorphic to the coordinate ring of the formal completion

of $X \otimes_{O_{K_0}} O_K$ along the closed immersion $i : \mathrm{Spec}\ A \to X \otimes_{O_{K_0}} O_K$. Hence $(\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge$ is formally of finite type and formally smooth over $O_K$.
2. Since the map $(\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge \to A$ is surjective, the assertion follows from 1.
3. Clear. □

In the rest of this subsection, we put $\mathbf{A} = (\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge$ for an object $(\mathbf{A}_0 \to A)$ of $\mathcal{E}mb_{K,O_{K_0}}$. By Lemma 2.9, we obtain a functor $\mathcal{E}mb_{K,O_{K_0}} \to \mathcal{E}mb_{O_K}$ sending $(\mathbf{A}_0 \to A)$ to $(\mathbf{A} \to A)$. For a rational number $j > 0$, we have a sequence of functors

$$\mathcal{E}mb_{K,O_{K_0}} \longrightarrow \mathcal{E}mb_{O_K} \xrightarrow{X^j} (\text{smooth Affinoid}/K) \longrightarrow G_K\text{-}(\mathrm{Aff}/\bar{F}).$$

We also let $\bar{X}^j$ denote the composite functor $\mathcal{E}mb_{K,O_{K_0}} \to G_K\text{-}(\mathrm{Aff}/\bar{F})$. For an object $(\mathbf{A}_0 \to A)$ of $\mathcal{E}mb_{K,O_{K_0}}$, we have $\bar{X}^j(\mathbf{A}_0 \to A) = \bar{X}^j((\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge \to A)$.

We study the dependence of the construction on the choice of a subfield $K_0 \subset K$, assuming the characteristic of $K$ is 0.

Lemma 2.10 *Let $K$ be a complete discrete valuation field of mixed characteristic satisfying the condition (F). Let $K_0$ and $K_0'$ be subfields of $K$ such that $O_{K_0} = O_K \cap K_0$ and $O_{K_0'} = O_K \cap K_0'$ are complete discrete valuation rings with perfect residue field $F_0$ and $F_0'$ and that $F$ is finitely generated over $F_0$ and $F_0'$.*
*1. There exists a subfield $K_0''$ of $K$ such that $O_{K_0''} = O_K \cap K_0''$ is a complete discrete valuation ring with perfect residue field and that $K_0''$ contains $K_0$ and $K_0'$ as subfields.*
*2. Assume $K_0 \subset K_0'$. Then $K_0'$ is a finite extension of $K_0$. For an object $(\mathbf{A}_0 \to A)$ of $\mathcal{E}mb_{K,O_{K_0}}$, the formal completion $\mathbf{A}_0' \to A$ of the surjection $\mathbf{A}_0 \otimes_{O_{K_0}} O_{K_0'} \to A$ defines an object $(\mathbf{A}_0' \to A)$ of $\mathcal{E}mb_{K,O_{K_0'}}$. Further, we have a canonical isomorphism $((\mathbf{A}_0' \hat{\otimes}_{O_{K_0'}} O_K)^\wedge \to A) \to ((\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge \to A)$ in $\mathcal{E}mb_{O_K}$.*

*Proof.* 1. By Lemma 2.5, we may assume the residue fields $F_0$ and $F_0'$ are the maximum perfect subfields of $F$. Then both of $K_0$ and $K_0'$ are finite over the fraction field of $W(F_0)$ regarded as a subfield of $K$. Hence it is sufficient to take the composition field.
2. By Lemma 2.5.2, the extension $K_0'$ is finite over $K_0$. The rest is clear from the construction. □

If $K$ is of characteristic $p$, the construction in this subsection is related to that in the last subsection as follows. Let $K_0$ be a subfield of $K$ such that $O_{K_0} = O_K \cap K_0$ is a complete discrete valuation ring with perfect residue field $F_0$ and that $F$ is finitely generated over $F_0$. Then, if $\pi_0$ is a prime element of $K_0$, we have an isomorphism $F_0((t)) \to K_0$ sending $t$ to $\pi_0$. For the integer ring $A$ of a finite étale algebra over $K$, let $(A \hat{\otimes}_{F_0} O_{K_0})^\wedge$ denote the projective limit of the formal completions $(A/\mathfrak{m}_A^n \otimes_{F_0} O_{K_0})^\wedge$ of the surjections $A/\mathfrak{m}_A^n \otimes_{F_0} O_{K_0} \to A/\mathfrak{m}_A^n$. The surjection $(A \hat{\otimes}_{F_0} O_{K_0})^\wedge \to A$ defines an object

$((A\hat{\otimes}_{F_0}O_{K_0})^\wedge \to A)$ of $\mathcal{E}mb_{K,O_{K_0}}$. Further, we have a canonical isomorphism $((A\hat{\otimes}_{F_0}O_{K_0})^\wedge\hat{\otimes}_{O_{K_0}}O_K)^\wedge \to A) \to ((A\hat{\otimes}_{F_0}O_K)^\wedge \to A)$ in $\mathcal{E}mb_{O_K}$.

In order to define a functor similar to the functor (Finite Étale/$K$)$^{\leq j+}$ $\to$ (Finite Étale/$\Theta^j$) in Section 2.2, we assume that $\pi_0$ is *not* a prime element of $K$ in the rest of this subsection. Note that if $p$ is not a prime element of $K$ and if the condition (F) is satisfied, there exists a subfield $K_0 \subset K$ with residue field $F_0$ such that a prime element of $K_0$ is not a prime element of $K$.

We compute the twisted normal cone $\bar{C}^j((\mathbf{A}_0\hat{\otimes}_{O_{K_0}}O_K)^\wedge \to A)$ for an object $(\mathbf{A}_0 \to A)$ of $\mathcal{E}mb_{K,O_{K_0}}$. Let $N_{A/\mathbf{A}} = I/I^2$ be the conormal module where $I$ is the kernel of the surjection $\mathbf{A} \to A$. We put $\hat{\Omega}_{O_K/O_{K_0}} = \varprojlim_n \Omega_{(O_K/\mathfrak{m}_K^n)/O_{K_0}}$ and let $\tilde{\Omega}_F$ be the $F$-vector space $\hat{\Omega}_{O_K/O_{K_0}} \otimes_{O_K} F$. Similarly, we put $\hat{\Omega}_{\mathbf{A}/\mathbf{A}_0} = \varprojlim_n \Omega_{(\mathbf{A}/\mathfrak{m}_{\mathbf{A}}^n)/\mathbf{A}_0}$. We also consider the canonical maps $N_{A/\mathbf{A}} \to \hat{\Omega}_{\mathbf{A}/\mathbf{A}_0} \otimes_{\mathbf{A}} A$ and $\hat{\Omega}_{O_K/O_{K_0}} \otimes_{O_K} A \to \hat{\Omega}_{\mathbf{A}/\mathbf{A}_0} \otimes_{\mathbf{A}} A$.

Lemma 2.11 *Assume $\pi_0$ is not a prime element of $K$ and let $m$ be the transcendental dimension of $F$ over $k$. Let $(\mathbf{A}_0 \to A)$ be an object of $\mathcal{E}mb_{K,O_{K_0}}$. Then,*

*1. The dimension of the $F$-vector space $\tilde{\Omega}_F$ is $m+1$.*

*2. The map $N_{A/\mathbf{A}} \to \hat{\Omega}_{\mathbf{A}/\mathbf{A}_0} \otimes_{\mathbf{A}} A$ is a surjection and the map $\hat{\Omega}_{O_K/O_{K_0}} \otimes_{O_K} A \to \hat{\Omega}_{\mathbf{A}/\mathbf{A}_0} \otimes_{\mathbf{A}} A$ is an isomorphism. They induce an isomorphism $N_{A/\mathbf{A}} \otimes_A A/\mathfrak{m}_A \to \tilde{\Omega}_F \otimes_F A/\mathfrak{m}_A$.*

*3. Let $(\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ be a morphism of $\mathcal{E}mb_{K,O_{K_0}}$ and put $\mathbf{B} = (\mathbf{B}_0\hat{\otimes}_{O_{K_0}}O_K)^\wedge$. Then, the diagram*

$$\begin{array}{ccc}
N_{A/\mathbf{A}} \otimes_A A/\mathfrak{m}_A & \longrightarrow & \tilde{\Omega}_F \otimes_F A/\mathfrak{m}_A \\
\downarrow & & \downarrow \\
N_{B/\mathbf{B}} \otimes_B B/\mathfrak{m}_B & \longrightarrow & \tilde{\Omega}_F \otimes_F B/\mathfrak{m}_B
\end{array}$$

*is commutative.*

*Proof.* 1. By the assumption that $\pi_0$ is not a prime element of $K$, we have an exact sequence $0 \to \mathfrak{m}_K/\mathfrak{m}_K^2 \to \tilde{\Omega}_F \to \Omega_{F/k} \to 0$. Since the $F$-vector space $\Omega_{F/k}$ is of dimension $m$, the assertion follows.

2. Since the cokernel of the map $N_{A/\mathbf{A}} \to \hat{\Omega}_{\mathbf{A}/\mathbf{A}_0} \otimes_{\mathbf{A}} A$ is $\Omega_{A/\mathbf{A}_0} = 0$, it is a surjection. By the definition of $\mathbf{A}$, the map $\hat{\Omega}_{O_K/O_{K_0}} \otimes_{O_K} \mathbf{A} \to \Omega_{\mathbf{A}/\mathbf{A}_0}$ is an isomorphism. Hence the map $\hat{\Omega}_{O_K/O_{K_0}} \otimes_{O_K} A \to \hat{\Omega}_{\mathbf{A}/\mathbf{A}_0} \otimes_{\mathbf{A}} A$ is also an isomorphism. Then the codimension of the regular surjection $\mathbf{A} \to A$ is $m+1$ and hence $N_{A/\mathbf{A}}$ is free of rank $m+1$. Since the induced map $N_{A/\mathbf{A}} \otimes_A A/\mathfrak{m}_A \to \tilde{\Omega}_F \otimes_F A/\mathfrak{m}_A$ is a surjection of free $A/\mathfrak{m}_A$-modules of rank $m+1$, it is an isomorphism.

3. By the assumption that $\pi_0$ is not a prime element of $K$, every map in $\mathcal{E}mb_{K,O_{K_0}}$ is finite flat by Lemma 2.7.3. Hence the assertion follows. □

For a rational number $j > 0$, let $\Theta^j$ be the $\bar{F}$-vector space $\Theta^j = Hom_F(\tilde{\Omega}_F, N^j)$ regarded as an affine scheme over $\bar{F}$.

COROLLARY 2.12 *Assume that $\pi_0$ is not a prime element of $K$. Let $(\mathbf{A}_0 \to A)$ be an object of $\mathcal{E}mb_{K,O_{K_0}}$ and let $(\mathbf{A} \to A)$ be the image in $\mathcal{E}mb_{K,O_{K_0}}$. Let $j > 0$ be a rational number.*
*1. The isomorphism in Lemma 2.11.2 induces an isomorphism $\bar{C}^j(\mathbf{A} \to A) \to \Theta^j \otimes_{\bar{F}} A_{\bar{F},\mathrm{red}}$.*
*2. Let $(\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ be a morphism of $\mathcal{E}mb_{K,O_{K_0}}$. Then the diagram*

$$\begin{array}{ccccc} \bar{X}^j(\mathbf{B} \to B) & \longrightarrow & \bar{C}^j(\mathbf{B} \to B) & \longrightarrow & \Theta^j \otimes_{\bar{F}} B_{\bar{F},\mathrm{red}} \\ \downarrow & & \downarrow & & \downarrow \\ \bar{X}^j(\mathbf{A} \to A) & \longrightarrow & \bar{C}^j(\mathbf{A} \to A) & \longrightarrow & \Theta^j \otimes_{\bar{F}} A_{\bar{F},\mathrm{red}} \end{array}$$

*is commutative.*
*3. If the ramification of $A \otimes_{O_K} K$ is bounded by $j+$ and $j > 1$, then the composition $\bar{X}^j(\mathbf{A} \to A) \to \bar{C}^j(\mathbf{A} \to A) \to \Theta^j \otimes_{\bar{F}} A_{\bar{F},\mathrm{red}} \to \Theta^j$ is finite and étale.*

*Proof.* 1. Since the surjection $\mathbf{A} \to A$ is regular, the assertion follows from the isomorphism in Lemma 2.11.2.
2. The left square is commutative by the construction. The commutativity of the right square is a consequence of Lemma 2.11.3.
3. By Lemma 2.7, there exist an embedding $(\mathbf{A}'_0 \to O_K)$ in $\mathcal{E}mb_{K,O_{K_0}}(O_K)$ and a finite flat morphism $(\mathbf{A}'_0 \to O_K) \to (\mathbf{A}_0 \to A)$. Since the ramification is bounded by $j+$, the finite map $\bar{X}^j(\mathbf{A} \to A) \to \bar{C}^j(\mathbf{A} \to A)$ is étale by Corollary 1.16.3. Since $A_{\bar{F},\mathrm{red}}$ is étale over $\bar{F}$, the assertion follows from 1 and 2. □.

For a rational number $j > 0$, we regard $\Theta^j$ as an object of $G_K$-(Aff$/\bar{F}$) with the natural $G_K$-action. Let $G_K$-(Finite Étale$/\Theta^j$) denote the subcategory of $G_K$-(Aff$/\bar{F}$) whose objects are finite étale schemes over $\Theta^j$ and morphisms are over $\Theta^j$. Let $\mathcal{E}mb^{\leq j+}_{K,O_{K_0}}$ denote the full subcategory of $\mathcal{E}mb_{K,O_{K_0}}$ consisting of the objects $(\mathbf{A}_0 \to A)$ such that the ramifications of $A \otimes_{O_K} K$ are bounded by $j+$. By Corollary 2.12, the functor $\bar{X}^j : \mathcal{E}mb_{K,O_{K_0}} \to G_K$-(Aff$/\bar{F}$) induces a functor $\bar{X}^j : \mathcal{E}mb^{\leq j+}_{K,O_{K_0}} \to G_K$-(Finite Étale$/\Theta^j$).

We show that the functor $\bar{X}^j : \mathcal{E}mb^{\leq j+}_{K,O_{K_0}} \to G_K$-(Finite Étale$/\Theta^j$) further induces a functor (Finite Étale$/K)^{\leq j+} \to G_K$-(Finite Étale$/\Theta^j$).

LEMMA 2.13 *Assume $\pi_0$ is not a prime element of $K$. Let $(f, \mathbf{f}), (g, \mathbf{g}) : (\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ be maps in $\mathcal{E}mb_{K,O_{K_0}}$ and $j > 1$ be a rational number. If the ramifications of $A \otimes_{O_K} K$ and $B \otimes_{O_K} K$ are bounded by $j+$ and if $f = g$, then the induced maps*

$$(f, \mathbf{f})_*, (g, \mathbf{g})_* : \bar{X}^j(\mathbf{A}_0 \to A) \longrightarrow \bar{X}^j(\mathbf{B}_0 \to B)$$

*are equal.*

*Proof.* By Corollary 2.12, the schemes $\bar{X}^j(\mathbf{A}_0 \to A)$ and $\bar{X}^j(\mathbf{B}_0 \to B)$ are finite étale over $\Theta^j$ and the maps $(f, \mathbf{f})_*, (g, \mathbf{g})_* : \bar{X}^j(\mathbf{A}_0 \to A) \to \bar{X}^j(\mathbf{B}_0 \to B)$ are maps over $\Theta^j$. Hence they are determined by the restrictions on the inverse images of a point. The inverse images of the origin $0 \in \Theta^j$ are canonically identified with the sets $Hom_{O_K}(A, \bar{K})$ and $Hom_{O_K}(B, \bar{K})$ respectively by Lemma 2.1. Hence the assertion follows. □

Corollary 2.14 *Assume $\pi_0$ is not a prime element of $K$. Let $j > 1$ be a rational number.*
*1. Let $L$ be a finite étale $K$-algebra with ramification bounded by $j+$. Then the system $\bar{X}^j(\mathbf{A}_0 \to O_L)$ parametrized by the objects $(\mathbf{A}_0 \to O_L)$ of $\mathcal{E}mb_{K,O_{K_0}}(O_L)$ is constant and the limit*

$$\bar{X}^j(L) = \varprojlim_{(\mathbf{A}_0 \to O_L) \in \mathcal{E}mb_{K,O_{K_0}}(O_L)} \bar{X}^j(\mathbf{A}_0 \to O_L)$$

*is a finite étale scheme over $\Theta^j$.*
*2. The functor $\bar{X}^j : \mathcal{E}mb^{\leq j+}_{K,O_{K_0}} \to G_K$-(Finite Étale/$\Theta^j$) induces a functor*

$$\bar{X}^j : (\text{Finite Étale}/K)^{\leq j+} \longrightarrow G_K\text{-(Finite Étale}/\Theta^j).$$

*Proof.* 1. By Corollary 2.8 and by the assumption that $\pi_0$ is not a prime element, every map in $\mathcal{E}mb_{K,O_{K_0}}(O_L)$ induces an isomorphism. By Lemma 2.7.1, the category $\mathcal{E}mb_{K,O_{K_0}}(O_L)$ is connected. To see that the system is constant, it suffices to apply Lemma 2.13 for $f = g = \mathrm{id}_{O_L}$. The map $\bar{X}^j(L) \to \Theta^j$ is finite étale by Corollary 2.12.3.
2. It is also an immediate consequence of Lemma 2.13. □

By Lemma 2.10 and the canonical isomorphism $\hat{\Omega}_{O_K/O_{K_0}} \otimes_{O_K} F \to \hat{\Omega}_{O_K/O_{K'_0}} \otimes_{O_K} F$, the functor $\bar{X}^j : (\text{Finite Étale}/K)^{\leq j+} \to G_K\text{-(Finite Étale}/\Theta^j)$ is independent of the choice of subfield $K_0$ if the characteristic of $K$ is 0. If the characteristic of $K$ is $p$, it is the same as that defined in Section 2.2.

## 2.4 Proof of commutativity

Now we are ready to prove the main result. For an integer $m$ prime to $p$, let $I_m$ be the unique open subgroup of the inertia subgroup $I \subset G_K$ of index $m$.

Theorem 2.15 *Let $K$ be a complete discrete valuation field. Let $j > 1$ be a rational number and $m$ be the prime-to-$p$ part of the denominator of $j$. Assume either $K$ has equal characteristics $p > 0$ or $K$ has mixed characteristic and $p$ is not a prime element. Then we have the following.*
*1. The graded piece $Gr^j G_K = G_K^j / G_K^{j+}$ is abelian.*
*2. The commutator $[I_m, G_K^j]$ is a subgroup of $G_K^{j+}$. In particular, $Gr^j G_K$ is a subgroup of the center of the pro-$p$-group $G_K^{1+}/G_K^{j+}$.*

*Proof.* We first prove the case where the condition
(F) There exists a perfect subfield $F_0$ of $F$ such that $F$ is finitely generated over $F_0$.
is satisfied. We use the functor $\bar{X}^j$ : (Finite Étale$/K)^{\leq j+} \to G_K$-(Finite Étale$/\Theta^j$) defined in Sections 2.2 and 2.3.
Let $L$ be a finite Galois extension of $K$ of ramification bounded by $j+$ and put $G = \mathrm{Gal}(L/K)$. To prove 1, it is sufficient to show that $G^j$ is commutative. By the definition of the functor, the image $\bar{X}^j(L)$ is a finite étale covering of $\Theta^j$ with a left action of $G_K$. We call this action of $G_K$ on $\bar{X}^j(L)$ the arithmetic action. On the other hand, by functoriality, we have a right action of $G$ on $\bar{X}^j(L)$, which commutes with the arithmetic action of $G_K$. We call this action of $G$ on $\bar{X}^j(L)$ the geometric action. We identify the inverse image in $\bar{X}^j(L)$ of the origin of $\Theta^j$ with $\Phi(L)$ as in Lemma 2.1. The arithmetic action of $\sigma \in G_K$ on $\Phi(L) = Hom_K(L, \bar{K})$ is given by $f \mapsto \sigma \circ f$ and the geometric action of $\tau \in G$ is given by $f \mapsto f \circ \tau$. Hence $\Phi(L)$ is a $G$-torsor and the étale covering $\bar{X}^j(L)$ is also a $G$-torsor over $\Theta^j$.
The stabilizer in $G_K$ of each connected component of $\bar{X}^j(L)$ with respect to the arithmeric action is equal to $G_K^j$ since $\Phi^j(L)$ is identified with $\pi_0(\bar{X}^j(L))$. Take a connected component $\bar{X}^j(L)_0$ of $\bar{X}^j(L)$. Then, the stabilizer of the intersection $\bar{X}^j(L)_0 \cap \Phi(L)$ in $G$, with respect to the geometric action, is equal to $G^j$. Hence the stabilizer of the component $\bar{X}^j(L)_0$ in $G$, with respect to the geometric action, is also equal to $G^j$ and $\bar{X}^j(L)_0$ is a connected $G^j$-torsor over $\Theta^j$. Therefore the map $G^j \to \mathrm{Aut}(\bar{X}^j(L)_0/\Theta^j)$ is an isomorphism.
On the other hand, by the assumption that $j > 1$, the group $G_K^j$ is a subgroup of the wild inertia subgroup $G_K^{1+} = P$. Hence the restriction to $G_K^j$ of the arithmetic action on $\Theta^j$ is trivial and we get a map $G_K^j \to \mathrm{Aut}(\bar{X}^j(L)_0/\Theta^j)$. Since $G_K^j$ acts on the intersection $\bar{X}^j(L)_0 \cap \Phi(L)$ transitively, the map $G_K^j \to \mathrm{Aut}(\bar{X}^j(L)_0/\Theta^j)$ is surjective. Since the geometric action of $G^j$ and the arithmetic action of $G_K^j$ on $\bar{X}^j(L)_0$ are commutative to each other, the group $G^j \simeq \mathrm{Aut}(\bar{X}^j(L)_0/\Theta^j)$ is commutative. Thus assertion 1 is proved in this case.
We prove assertion 2 assuming the condition (F). We define a canonical map $\pi_1^{\mathrm{ab}}(\Theta^j) \to Gr^j G_K$ as follows. By 1, the image of the functor $\bar{X}^j$ : (Finite Étale$/K)^{\leq j+} \to G_K$-(Finite Étale$/\Theta^j$) is in the full subcategory consisting of abelian coverings. Taking the Galois groups, we obtain a map $\pi_1^{\mathrm{ab}}(\Theta^j) \to G_K/G_K^{j+}$ inducing a surjection

$$\pi_1^{\mathrm{ab}}(\Theta^j) \longrightarrow Gr^j G_K.$$

The canonical map $\pi_1^{\mathrm{ab}}(\Theta^j) \to Gr^j G_K$ is compatible with the actions of $G_K$. The action of $G_K$ on $\pi_1^{\mathrm{ab}}(\Theta^j)$ is induced by that on $\Theta^j$ and the action on $Gr^j G_K$ is by conjugation. Since the subgroup $I_m$ acts trivially on $\Theta^j$, it also acts trivially on $\pi_1^{\mathrm{ab}}(\Theta^j)$. Hence, assertion 2 follows in this case by the compatibility of the surjection $\pi_1^{\mathrm{ab}}(\Theta^j) \to Gr^j G_K$ with the $G_K$-action.
To reduce the general case to the special case proved above, we show the following Lemma.

Lemma 2.16 *Let $K$ be a complete discrete valuation field and $K_0$ be a subfield of $K$ such that $O_{K_0} = O_K \cap K$ is a complete discrete valuation ring with perfect residue field $F_0$. Then there exist a filtered family of subextensions $K_\mu \subset K, \mu \in M$ of $K_0$ satisfying the following conditions:*
*For each $\mu \in M$, the intersection $O_{K_\mu} = O_K \cap K_\mu$ is a complete discrete valuation ring and the residue field $F_\mu$ is finitely generated over $F_0$, the residue field $F$ is a separable extension of $F_\mu$ and a prime element of $K_\mu$ is a prime element of $K$. The residue field $F$ is equal to the union $\varinjlim_{\mu \in M} F_\mu$.*

*Proof.* Let $\pi_0$ be a prime element of $K_0$. Take a transcendental basis $(\bar t_\lambda)_{\lambda\in\Lambda}$ of $F$ over $F_0$ such that $F$ is separable over $F_0(\bar t_\lambda, \lambda \in \Lambda)$. We take liftings $t_\lambda \in O_K, \lambda \in \Lambda$ of $\bar t_\lambda$. For a finite subset $\sigma \subset \Lambda$, let $K_{0,\sigma}$ be the fraction field of the completion of the local ring at the prime ideal $(\pi_0)$ of the ring $O_{K_0}[T_\lambda, \lambda \in \sigma]$ and regard it as a subfield of $K$. Let $K_{0,\mu} \subset K, \mu \in M_0$ be the family of finite unramified subextensions of $K_{0,\sigma}, \sigma \subset \Lambda$. Let $K'_0$ be the completion of the union $\varinjlim_{\mu\in M_0} K_{0,\mu}$. Then $K$ is a finite totally ramified extension of $K'_0$. Hence there is an index $\mu_0 \in M_0$ and a finite totally ramified extension $K_{\mu_0}$ of $K_{0,\mu_0}$ such that $K$ is the composite of $K_{\mu_0}$ and $K'_0$. We put $M = \{\mu \in M_0 : K_{0,\mu_0} \subset K_{0,\mu}\}$. Then the family $K_\mu = K_{\mu_0} K_{0,\mu}, \mu \in M$ satisfies the conditions. □

We complete the proof of Theorem. It is sufficient to show assertion 2. Let $F_0 = \bigcap_n F^{p^n}$ be the maximum perfect subfield of the residue field $F$. If the characteristic of $K$ is positive, we take a element $\pi_0 \in \mathfrak{m}_K^2, \neq 0$ of $K$ and put $K_0 = F_0((\pi_0)) \subset K$. If the characteristic of $K$ is 0, let $K_0$ be the fraction field of $W(F_0)$ and regard it as a subfield of $K$. By the assumption that $p$ is not a prime element of $K$, a prime element of $K_0$ is not a prime element of $K$. Let $K_\mu, \mu \in M$ be a family of subfields of $K$ as in Lemma 2.16. Since $K_0$ is a subfield of $K_\mu$ satisfying the condition (F) and a prime element of $K_0$ is not a prime element of $K_\mu$, we have $[I_{m,K_\mu}, G^j_{K_\mu}] \subset G^{j+}_{K_\mu}$ for $\mu \in M$.

Since $K' = \varprojlim_{\mu\in M} K_\mu$ is a Henselian discrete valuation field and $K$ is the completion of $K'$, the canonical maps $G_K \to G_{K'} \to \varprojlim_{\mu\in M} G_{K_\mu}$ are isomorphisms. It induces an isomorphism $I_{m,K} \to \varprojlim_\mu I_{m.K_\mu}$. By Lemma 2.2 and by the assumption that the residue field $F$ is separable over $F_\mu$ and a prime element of $K_\mu$ is a prime element of $K$, the map $G^j_K \to G^j_{K_\mu}$ is surjective. Hence we have isomorphisms $G^j_K \to \varprojlim_{\mu\in M} G^j_{K_\mu}$ and $G^{j+}_K \to \varprojlim_{\mu\in M} G^{j+}_{K_\mu}$. By taking the limit of $[I_{m,K_\mu}, G^j_{K_\mu}] \subset G^{j+}_{K_\mu}$, we obtain $[I_{m,K}, G^j_K] \subset G^{j+}_K$. □

## 3 Some generalities on log structures

To study the logarithmic filtration in later sections, we recall and establish some generalities on log structures. More systematic account of a part is given in [10] Section 4. For the basic definitions on log schemes, we refer to [6]. In

this paper, a log structure $M_X \to O_X$ on a scheme $X$ means a Zariski fs-log structure.
We prepare some basic terminologies on log schemes. We call a pair $(X,P)$ of a log scheme $X$ and a chart $P$ on $X$ a charted log scheme. For charted log schemes $(X,P)$ and $(S,N)$, we call a pair $(f,\varphi)$ of a map $f : X \to S$ of log schemes and a map $N \to P$ of fs-monoid a map $(X,P) \to (S,N)$ of charted log schemes if the diagram

$$\begin{array}{ccc} N & \longrightarrow & \Gamma(S, M_S) \\ \downarrow & & \downarrow \\ P & \longrightarrow & \Gamma(X, M_X) \end{array} \tag{1}$$

is commutative.
For an fs-monoid $P$, we regard Spec $\mathbb{Z}[P]$ as a log scheme with the log structure defined by the chart $P \to \mathbb{Z}[P]$. For maps $X \to S$ and $Y \to S$ of log schemes, let $X \times_S^{\log} Y$ denote the fibered product in the category of fs-log schemes. If $S = \operatorname{Spec} A$, $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} C$ are affine, $N \to A, P \to B$ and $Q \to C$ are charts and if $(f,\varphi) : (X,P) \to (S,N)$ and $(g,\psi) : (Y,Q) \to (S,N)$ are morphisms of charted log schemes, we have $X \times_S^{\log} Y = \operatorname{Spec} B \otimes_A^{\log} C$ where $B \otimes_A^{\log} C = (B \otimes_A C) \otimes_{\mathbb{Z}[P+Q]} \otimes \mathbb{Z}[P +_N^{\rm sat} Q]$ and $P +_N^{\rm sat} Q$ is the saturation of the image of $P + Q$ in the fibered sum $P^{\rm gp} \oplus_{N^{\rm gp}} Q^{\rm gp} = \operatorname{Coker}(\varphi - \psi : N^{\rm gp} \to P^{\rm gp} \oplus Q^{\rm gp})$.

Definition 3.1 *Let $X \to S$ be a morphism of log schemes.*
*1.* (cf. [7], [11] Theorem 4.6 (iv)) *We say that $X \to S$ is log flat if the following conditions are satisfied:*
*For each $x$, there exist a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array}$$

*of log schemes, charts $P$ on $U$ and $N$ on $V$ and morphism $(U,P) \to (V,N)$ of charted log schemes such that the underlying map $U \to X$ is a flat surjection to an open neighborhood of $x$, the underlying map $V \to S$ is flat, the map $N \to P$ is injective and the underlying map $U \to V \otimes_{\mathbb{Z}[N]} \mathbb{Z}[P]$ is flat.*
*2. We say that $X \to S$ is log locally of complete intersection if the following conditions are satisfied:*
*For each $x$, there exist a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array}$$

*of log schemes such that $U$ is an open neighborhood of $x$, the map $V \to S$ is log smooth and $U \to V$ is an exact and regular immersion.*
*3. We say that $X \to S$ is log syntomic if it is log flat and log locally of complete intersection.*

For the log syntomic morphisms, the definition here is slightly different from that in [9] (2.5). We introduce the new definition because it is a special case of the general definition due to Illusie and Olsson [5], [11] Definition 4.1 by Lemma 3.3 below. An equivalent statement of Lemma 3.2 in the resp. cases is proved in [6], and in the log flat case in [11] Theorem 4.6. Another proof is given in [10] Section 4.4.

LEMMA 3.2 (cf. [11] Theorem 4.6) *For a morphism $X \to S$ of log schemes, the following conditions are equivalent.*
*(1) The map $X \to S$ is log flat (resp. log smooth, log étale).*
*(2) Let*

$$\begin{array}{ccc} X' & \longrightarrow & S' \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & S \end{array}$$

*be a commutative diagram of log schemes such that $X' \to X \times_S^{\log} S'$ is log étale and $X' \to S'$ is strict. Then the underlying map $X' \to S'$ is flat (resp. smooth, étale).*

LEMMA 3.3 *For a morphism $X \to S$ of log schemes, the following conditions are equivalent.*
*(1) The map $X \to S$ is log syntomic.*
*(2) Let*

$$\begin{array}{ccc} X' & \longrightarrow & S' \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & S \end{array}$$

*be a commutative diagram of log schemes such that $X' \to X \times_S^{\log} S'$ is log étale and $X' \to S'$ is strict. Then the underlying map $X' \to S'$ is flat and locally of complete intersection.*

To deduce Lemma 3.3 from Lemma 3.2, we introduce some basic constructions on log schemes.

LEMMA 3.4 *Let $f : X \to S$ be a morphism of log schemes and $x \in X$. Then there exist charts $P$ and $N$ on open neighborhoods $U$ of $x$ and $V \supset f(U)$ of $s = f(x)$ and a morphism $(U, P) \to (V, N)$ of charted log schemes such that the map* Spec $\mathbb{Z}[P] \to$ Spec $\mathbb{Z}[N]$ *is log smooth.*

*Proof.* We put $\bar{M}_S = M_S/O_S^\times$, $\bar{M}_X = M_X/O_X^\times$, $N = \bar{M}_{S,s}$, $P_0 = \bar{M}_{X,x}$ and let $N \to P_0$ be the canonical map. We take charts $N \to \Gamma(V, M_V)$, $P_0 \to \Gamma(U, M_U)$ on open neighborhoods lifting the identities. We define an fs-monoid $P$ to be the inverse image of $P_0$ by the map $P_0^{\rm gp} \oplus N^{\rm gp} \to P_0^{\rm gp}$ sending $(m,n)$ to $m+f(n)$. Then, shrinking $U$ if necessary, we find a unique map $P \to \Gamma(U, M_X)$ extending the composition $P_0 + N \to \Gamma(X, M_X) + \Gamma(S, M_S) \to \Gamma(X, M_X)$. Thus, we obtain a morphism $(U,P) \to (V,N)$ of charted log schemes. Since the map $N^{\rm gp} \to P^{\rm gp}$ is an isomorphism to a direct summand, the map Spec $\mathbb{Z}[P] \to$ Spec $\mathbb{Z}[N]$ is log smooth. □

For a morphism $f : N \to P$ of fs-monoids, we define an fs-monoid $(P +_N P)^\sim$ to be the inverse image of $P$ by the map $P^{\rm gp} \oplus_{N^{\rm gp}} P^{\rm gp} \to P^{\rm gp}$ sending $(m, m')$ to $m + m'$.

Lemma 3.5 *Let $N \to P$ be a map of fs-monoids and let $(P+_N P)^\sim \subset P^{\rm gp} \oplus_{N^{\rm gp}} P^{\rm gp}$ be as above. Then,*

*1. The map $P \times (P^{\rm gp}/N^{\rm gp}) \to (P +_N P)^\sim$ sending $(m, m')$ to $(m + m', -m')$ is an isomorphism.*

*2. The ring homomorphism $\mathbb{Z}[P] \to \mathbb{Z}[(P +_N P)^\sim]$ induced by the map $P \to (P +_N P)^\sim$ of monoids sending $m$ to $(m, 0)$ is faithfully flat.*

*3. The map $P + P + (P^{\rm gp}/N^{\rm gp}) \to (P +_N P)^\sim$ sending $(m, m', m'') \to (m+m'', m'-m'')$ induces an isomorphism $\mathbb{Z}[P \times P \times (P^{\rm gp}/N^{\rm gp})]/((m, 0, 0) - (0, m, m); m \in P) \to \mathbb{Z}[(P +_N P)^\sim]$ of rings.*

*Proof.* 1. The inverse $(P +_N P)^\sim \to P \times (P^{\rm gp}/N^{\rm gp})$ is given by $(m, m') \to (m + m', -m')$.

2 and 3. Clear from 1. □

Corollary 3.6 *Let $(X, P) \to (S, N)$ be a morphism of charted log schemes and put $S' = S \otimes_{\mathbb{Z}[N]} \mathbb{Z}[P]$ and $X' = X \otimes_{\mathbb{Z}[P]} \mathbb{Z}[(P +_N P)^\sim]$. Then the map $X' \to S'$ is strict, the map $X' \to X \times_S^{\log} S'$ is log étale and $X' \to X$ is faithfully flat.*

*Proof.* The map $X' \to X \times_S^{\log} S'$ is log étale by the definition of $(P +_N P)^\sim$. The map $X' \to S'$ is strict by Lemma 3.5.1. The map $X' \to X$ is faithfully flat by Lemma 3.5.2. □

*Proof of Lemma 3.3.* Since the assertion is local on $X$, we may assume there exist a log smooth scheme $Y$ over $S$, an exact closed immersion $X \to Y$ over $S$ and a morphism $(Y, P) \to (S, N)$ of charted log schemes as in Lemma 3.4. We put $S_1 = S \otimes_{\mathbb{Z}[N]}^{\log} \mathbb{Z}[P], Y_1 = Y \otimes_{\mathbb{Z}[P]}^{\log} \mathbb{Z}[(P +_N P)^\sim]$ and $X_1 = X \times_Y^{\log} Y_1$.

We show (1)⇒(2). We assume $X \to S$ is log syntomic. We consider the diagram in (2). Since the question is local on $X'$, we may assume there exist a log étale scheme $Y'$ over $Y \times_S S'$ and an isomorphism $X' \to X \times_Y^{\log} Y'$. Shrinking $Y'$, we may assume that the map $Y' \to S'$ is strict. Hence by Lemma 3.2, the underlying map $Y' \to S'$ is smooth. It is sufficient to show that the closed immersion $X' \to Y'$ is a regular immersion. We consider a commutative diagram

$$\begin{tikzcd}
 & & & X_1' \arrow[r] \arrow[llld] \arrow[d] & Y_1' \arrow[llld] \arrow[d] \arrow[rd] & \\
X' \arrow[r] \arrow[d] & Y' \arrow[d] \arrow[rd] & & X_1 \times_S^{\log} S' \arrow[r] \arrow[llld] \arrow[d] & Y_1 \times_S^{\log} S' \arrow[r] \arrow[llld] \arrow[d] & S_1' \arrow[llld] \arrow[d] \\
X \times_S^{\log} S' \arrow[r] \arrow[d] & Y \times_S^{\log} S' \arrow[r] \arrow[d] & S' \arrow[d] & X_1 \arrow[r] \arrow[llld] & Y_1 \arrow[r] \arrow[llld] & S_1 \arrow[llld] \\
X \arrow[r] & Y \arrow[r] & S & & &
\end{tikzcd}$$

by putting $S_1' = S_1 \times_S^{\log} S', Y_1' = Y_1 \times_Y^{\log} Y'$ and $X_1' = X_1 \times_X^{\log} X'$.
Since $Y \to S$ is log smooth, $Y_1 \to S_1$ is strict and $Y_1 \to Y \times_S^{\log} S_1$ is log étale, the underlying map $Y_1 \to S_1$ is smooth by Lemma 3.2. Similarly, since $X \to S$ is log flat, $X_1 \to S_1$ is strict and $X_1 \to X \times_S^{\log} S_1$ is log étale, the underlying map $X_1 \to S_1$ is flat by Lemma 3.2. Since $Y_1 \to Y$ is flat by Lemma 3.5.2 and $X \to Y$ is a regular immersion, the immersion $X_1 \to Y_1$ is a regular immersion. Thus $X_1 \to S_1$ is flat and locally of complete intersection. Since the maps $X_1 \to Y_1 \to S_1$ are strict, the underlying map $X_1 \times_S^{\log} S' \to S_1'$ is flat and locally of complete intersection and the immersion $X_1 \times_S^{\log} S' \to Y_1 \times_S^{\log} S'$ is a regular immersion by EGA IV Propositions (19.3.9)(ii) and (19.3.7). Since $Y_1' \to Y_1 \times_S^{\log} S'$ is a base change of $Y' \to Y \times_S^{\log} S'$, the map $Y_1' \to Y_1 \times_S^{\log} S'$ is log étale. Since it is strict, the underlying map $Y_1' \to Y_1 \times_S^{\log} S'$ is étale by Lemma 3.2. Since $X_1' \to Y_1'$ is the base change of the regular immersion $X_1 \times_S^{\log} S' \to Y_1 \times_S^{\log} S'$ by the étale map $Y_1' \to Y_1 \times_S^{\log} S'$, it is also a regular immersion. Since the regular immersion $X_1' \to Y_1'$ is also the base change of the immersion $X' \to Y'$ by the faithfully flat and strict map $Y_1 \to Y$, the immersion $X' \to Y'$ is a regular immersion as required.
We show (2)$\Rightarrow$(1). We assume the condition (2) is satisfied. It is sufficient to show that the exact closed immersion $X \to Y$ is a regular immersion. By (2), the underlying map $X_1 \to S_1$ is flat and locally of complete intersection and the underlying map $Y_1 \to S_1$ is smooth. Hence the immersion $X_1 \to Y_1$ is a regular immersion by EGA IV Proposition (19.3.7). Since the regular immersion $X_1 \to Y_1$ is the base change of the immersion $X \to Y$ by the strict and faithfully flat map $Y_1 \to Y$, the immersion $X \to Y$ is a regular immersion as required. □

COROLLARY 3.7 (cf. [11] Corollary 4.12) *Let $f : X \to S$ and $S' \to S$ be morphisms of log schemes and let $f' : X' = X \times_S^{\log} S' \to S'$ be the log base change. Then, if $f : X \to S$ is log flat (resp. log syntomic), the base change $f' : X' \to S'$ is also log flat (resp. log syntomic).*

*Proof.* Clear from Lemmas 3.2 and 3.3. □

Lemma 3.8 *Let $X \to S$ be a log scheme over $S$ log locally of complete intersection, $Y \to S$ be a log smooth log scheme over $S$ and $X \to Y$ be an exact closed immersion over $S$. Then,*
*1. The immersion $X \to Y$ is a regular immersion.*
*2. Let $Y' \to S$ be another log smooth log scheme over $S$ and $X \to Y'$ be an exact closed regular immersion over $S$. Let $n$ and $n'$ be the relative dimensions of $Y$ and of $Y'$ over $S$ and $r$ and $r'$ be the codimensions of the regular immersions $X \to Y$ and of $X \to Y'$ respectively. Then we have $n - r = n' - r'$.*

*Proof.* 1. Since the assertion is local, we may assume there is an exact regular closed immersion $X \to Y'$ into a log smooth scheme $Y'$ over $S$. By the same argument as in the proof of Lemma 3.4, we may assume that there exist a commutative diagram

$$\begin{array}{ccc} (X, P) & \longrightarrow & (Y, P) \\ \downarrow & & \downarrow \\ (Y', P) & \longrightarrow & (S, N) \end{array}$$

of charted log schemes. We define an fs-monoid $(P +_N P)^\sim \subset P^{\rm gp} \oplus_{N^{\rm gp}} P^{\rm gp}$ as above and put $Y'' = (Y \times_S^{\log} Y') \otimes^{\log}_{\mathbb{Z}[P+P]} \mathbb{Z}[(P +_N P)^\sim]$. Then the projections $Y'' \to Y$ and $Y'' \to Y'$ are log smooth and strict and hence are smooth. Since the immersion $X \to Y'$ is a regular immersion, the immersion $X \to Y''$ is a regular immersion. Since the map $Y'' \to Y$ is also smooth, the immersion $X \to Y$ is also a regular immersion by EGA IV Proposition (19.1.5)(iv)b)⇒a) applied to the immersions $X \times_Y^{\log} Y'' \to Y''$ and $X \to X \times_Y^{\log} Y''$ and by loc.cit (ii). Hence the assertion follows.
2. In the notation above, the relative dimensions of $Y''$ over $Y$ and $Y'$ are $n'$ and $n$ respectively. Hence the assertion follows. □
If $X \to Y$ is an exact regular immersion of codimension $r$, and $Y$ is log smooth over $S$ of relative dimension $n$, we say that $X \to S$ is of relative dimension $n - r$.

Lemma 3.9 *Let $X$ and $S$ be log regular schemes and $f : X \to S$ be a morphism of finite type. Then $f : X \to S$ is log locally of complete intersection.*

*Proof.* Since the assertion is local, we may assume there is a morphism $(X, P) \to (S, N)$ of charted log schemes as in Lemma 3.4. The map $S' = S \otimes^{\log}_{\mathbb{Z}[N]} \mathbb{Z}[P] \to S$ is log smooth and the map $X \to S$ is factorized as $X \to S' \to S$ where $X \to S'$ is strict. Hence by replacing $S$ by $S'$, we may assume $X \to S$ is strict. Further replacing $S$ by a smooth scheme over $S$, we may assume $X \to S$ is an exact immersion. It is sufficient to show that the immersion $X \to S$ is a regular immersion.
Since the question is local, we may assume $S = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$ are local. We put $P = \bar{M}_{S,s}$ and take a chart $\alpha : P = \bar{M}_{S,s} \to A$. We

put $\bar{A} = A/\alpha(P - \{1\})$ and $\bar{B} = B \otimes_A \bar{A}$. Since $\bar{A} \to \bar{B}$ is a surjection of regular local rings, the kernel is generated by a regular sequence $(\bar{t}_1, \ldots, \bar{t}_r)$ of $\bar{A}$. We take a lifting $(t_1, \ldots, t_r)$ in the maximal ideal $\mathfrak{m}_A$. We show that $A_i = A/(t_1, \ldots, t_i)$ is log regular of dimension $\dim A - i$ and that $(t_1, \ldots, t_i)$ is a regular sequence. By induction on $i = 1, \ldots, r$, it is sufficient to show the case $i = 1$. Since $t_1 \neq 0$ and $A$ is normal, we have $\dim A_1 = \dim A - 1$. On the other hand, we have $\dim \bar{A}_1 + \text{rank } P^{\text{gp}} = \dim \bar{A} - 1 + \text{rank } P^{\text{gp}}$. Hence, we have $\dim A_1 = \dim \bar{A}_1 + \text{rank } P^{\text{gp}}$ and $A_1$ is log regular. Thus by induction, $A_r$ is log regular of dimension $\dim A - r$ and $(t_1, \ldots, t_r)$ is a regular sequence. Since $\dim B = \dim \bar{B} + \text{rank } P^{\text{gp}} = \dim \bar{A} - r + \text{rank } P^{\text{gp}} = \dim A_r$ and $A_r$ is normal, the surjection $A_r \to B$ is an isomorphism. Hence the immersion $X \to S$ is a regular immersion of codimension $r$. □

Let $f : X \to S$ be a map of log schemes such that the map of underlying schemes is locally of finite presentation and $x \in X$. We put $s = f(x)$, $X_s = X \otimes_{\kappa(s)} \kappa(x)$ and define

$$\begin{aligned}\dim_x^{\log} f^{-1}(f(x)) &= \\ &= \dim O_{X_s,x}/(\alpha(M_{X,x} - O_{X,x}^\times)) + \text{tr.deg } \kappa(x)/\kappa(s) + \text{rank } \bar{M}_{X,x}^{\text{gp}}/\bar{M}_{S,s}^{\text{gp}}.\end{aligned}$$

Lemma 3.10 *Let $f : X \to S$ be a morphism of log schemes such that the map of underlying schemes is of finite presentation.*

*1. Let $(X, P) \to (S, N)$ be a morphism of charted log schemes and let $x \in X$. Regard $x$ as a log scheme with the log structure defined by the chart $P$. We put $X'_x = (X \times_S x) \otimes_{\mathbb{Z}[P+P]} \mathbb{Z}[(P +_N P)^\sim]$ and let $x \to X'_x$ be the section defined by $x \to X$ and the map $(P +_N P)^\sim \to P \to \kappa(x)$. Then, we have an equality*

$$\dim_x^{\log} f^{-1}(f(x)) = \dim O_{X'_x,x}.$$

*2. If $X \to S$ is log flat, the function $\dim_x^{\log} f^{-1}(f(x))$ is a locally constant function of $x \in X$.*

*3. Assume $X \to S$ is log locally of complete intersection of relative dimension $d$. If we have an equality $\dim_x^{\log} f^{-1}(f(x)) = d$ for all $x \in X$, the map $X \to S$ is log flat and hence log syntomic.*

*Proof.* 1. By Lemma 3.5.3, $X'_x$ is the closed subscheme of $(X_s \otimes_{\kappa(s)} \kappa(x)) \otimes_{\mathbb{Z}} \mathbb{Z}[P'^{\text{gp}}/N^{\text{gp}}]$ defined by the ideal $I$ generated by $(\alpha(m) \otimes 1) - (1 \otimes \alpha_x(m)) \cdot (m)$ for $m \in P$. The ideal $I$ is generated by $\alpha(m) \otimes 1$ for $m \in P \backslash \text{Ker}(P \to \bar{M}_{X,x})$ and $(m) - (1 \otimes \alpha_x(m))^{-1}(\alpha(m) \otimes 1)$ for $m \in \text{Ker}(P \to \bar{M}_{X,x})$. Hence $X'_x$ is the closed subscheme of $(X_s \otimes_{\kappa(s)} \kappa(x)) \otimes_{\mathbb{Z}} \mathbb{Z}[\bar{M}_{X,x}^{\text{gp}}/\bar{M}_{S,s}^{\text{gp}}]$ defined by the ideal $J$ generated by $\alpha(m) \otimes 1$ for $m \in P \backslash \text{Ker}(P \to \bar{M}_{X,x})$. Thus the assertion follows.

2. Let $S' = S \otimes_{\mathbb{Z}[N]}^{\log} \mathbb{Z}[P], X' = X \otimes_{\mathbb{Z}[P]}^{\log} \mathbb{Z}[(P +_N P)^\sim]$ and $f' : X' \to S'$ be the map. Since the map $X' \to X \times_S^{\log} S'$ is log étale, and the composition $X' \to S'$ is strict, the underlying map $X' \to S'$ is flat. Hence the function $\dim_{x'} f'^{-1}(f'(x')) = \dim O_{X'_{f'(x')},x'}$ is locally constant on $x' \in X'$. The function $\dim_x^{\log} f^{-1}(f(x))$ is the pull-back of the locally constant function

$\dim_{x'} f'^{-1}(f'(x'))$ by the section $X \to X'$ induced by the map $(P +_N P)^{\sim} \to P$. Thus the assertion is proved.
3. Since the question is local, we may further assume that there is an exact regular immersion $X \to Y$ to a log scheme $Y$ log smooth over $S$. Let $n$ be the relative dimension of $Y$ over $S$ and $r = n - d$ be the codimension of the regular immersion $X \to Y$. We put $Y' = Y \otimes^{\log}_{\mathbb{Z}[P']} \mathbb{Z}[(P' +_N P')^{\sim}]$. Then we have $X' = X \times^{\log}_Y Y'$. Since $X' \to X$ is faithfully flat by Lemma 3.5.2, it is sufficient to show that the map $X' \to S'$ is flat. Since $Y' \to Y$ is flat, the immersion $X' \to Y'$ is regular of codimension $r$. The map $Y' \to S'$ is smooth of relative dimension $n$. Hence the strict map $X' \to S'$ is locally of complete intersection of relative dimension $d$. By the assumption and the computation above, each fiber of $X' \to S'$ has dimension $d$. Hence by EGA IV Théorème (11.3.8) d)⇒a), $X' \to S'$ is flat.

Corollary 3.11 *Let $f : X \to S$ be a finite morphism of log regular schemes. Assume $\dim X = \dim S$ and $f^*\bar{M}^{\rm gp}_S \otimes \mathbb{Q} \to \bar{M}^{\rm gp}_X \otimes \mathbb{Q}$ is surjective. Then $X$ is log flat and hence log syntomic over $S$.*

*Proof.* By Lemma 3.9, the map $f : X \to S$ is log locally of complete intersection. Further, by the assumption that $X \to S$ is finite and $\dim X = \dim S$, the map $X \to S$ has relative dimension 0. Since $\dim^{\log}_x f^{-1}(f(x)) = 0$ for all $x \in X$, it is sufficient to apply Lemma 3.10 □

For a ring $A$, we call a Zariski fs-log structure on $X = \operatorname{Spec} A$ a log structure on $A$. We call a ring with a log structure a log ring. If $A$ is a local ring, a log structure on $A$ is defined by a chart $P \to A$. We say that a map $A \to B$ of log rings is a surjection if the underlying ring homomorphism $A \to B$ is surjective and the map $f^*M_Y \to M_X$ is surjective where $f : X = \operatorname{Spec} B \to Y = \operatorname{Spec} A$ denotes the corresponding map of log schemes and $M_X$ and $M_Y$ denote the log structures. We say that a surjection $A \to B$ of log rings is an exact surjection if the log structure $M_X$ is the pull-back log structure of $M_Y$. We say that a surjection $A \to B$ is regular if the immersion $\operatorname{Spec} B \to \operatorname{Spec} A$ of the underlying schemes is a regular immersion. For a map $A \to B$ of log rings, let $\Omega_{B/A}(\log/\log)$ denote the module of logarithmic differential forms, denoted by $\omega_{B/A}$ in [6]. If $A$ and $B$ are local and $N$ and $P$ denote the stalks of the log structures at the closed points, we have

$$\Omega_{B/A}(\log/\log) = \left(\Omega_{B/A} \oplus (B \otimes_{\mathbb{Z}} (P^{\rm gp}/N^{\rm gp}))\right)/(dm - m \otimes m : m \in P).$$

We study formally log smooth maps of complete local Noetherian log rings.

Definition 3.12 (cf. [11] Definition 4.4) *Let $A$ and $B$ be complete local Noetherian rings with log structures and $f : A \to B$ a morphism of log rings such that the underlying ring homomorphism is local.*
*1. We say $f : A \to B$ is formally log smooth (resp. formally log étale) if, for a nilpotent exact surjection $R \to R'$ of discrete log $A$-algebras and a continuous*

*homomorphism $B \to R'$ of log $A$-algebras, there exists a (resp. a unique) continuous homomorphism $B \to R$ of log $A$-algebras lifting $B \to R'$.*
*2. We put* $\hat{\Omega}_{B/A}(\log/\log) = \varprojlim_n \Omega_{(B/\mathfrak{m}_B^n)/A}(\log/\log)$.

Lemma 3.13 *Let $A$ and $B$ be complete local Noetherian rings with log structures and $f : A \to B$ a morphism of log rings such that the underlying ring homomorphism is local. Assume that the residue field of $B$ is finitely generated over the residue field of $A$. Then, the following conditions are equivalent.*
*(1) $B$ is formally log smooth over $A$.*
*(2) There exist a log smooth scheme $X$ over $A$, a point $x$ of $X$ over the closed point of* Spec $A$ *and an étale local homomorphism $B \to \hat{O}_{X,x}$ over $A$.*

*Proof.* It is clear that (2) implies (1). The implication (1)⇒(2) is proved similarly as in the proof of [6] (3.5.1)⇒(3.5.2). □

Corollary 3.14 *Let $A \to B$ be as in Lemma and assume $A \to B$ is log smooth.*
*1. The $B$-module* $\hat{\Omega}_{B/A}(\log/\log)$ *is free of finite rank.*
*2. If $A$ is log regular* (cf. [8] Definition (2.1)), *then $B$ is also log regular.*

*Proof.* 1. It follows from Lemma 3.13 (1)⇒(2) and [6] Proposition (3.10).
2. It follows from Lemma 3.13 (1)⇒(2) and [8] Theorem (8.2). □

# 4 Tubular neighborhoods for finite flat and log flat log algebras

In the rest of the paper, the integer ring $O_K$ is considered as a log ring with its canonical log structure defined by the chart $\mathbb{N} \to O_K$ sending $1 \in \mathbb{N}$ to a prime element. The letter $A$ denotes a finite flat and log flat log $O_K$-algebra such that the log structure on $A_K$ is trivial. For a finite étale algebra $L$ over $K$, its integer ring $O_L$ is considered as a log $O_K$-algebra with its canonical log structure defined by taking the product of the canononical log structures on its factors. The log $O_K$-algebra $O_L$ is log flat by Corollary 3.11. Hence it is finite flat and log flat and the log structure on $L$ is trivial.

## 4.1 Log embeddings

Definition 4.1 *1. Let $A$ be a finite flat and log flat log $O_K$-algebra such that the log structure on $A_K$ is trivial. Let $\mathbf{A}$ be a log $O_K$-algebra formally of finite type and formally log smooth over $O_K$. We say that an exact surjection $\mathbf{A} \to A$ of log $O_K$-algebras is a log embedding if it induces an isomorphism $\mathbf{A}/\mathfrak{m}_{\mathbf{A}} \to A/\mathfrak{m}_A$.*
*2. We define $\mathcal{E}mb_{O_K}^{\log}$ to be the category whose objects and morphisms are as follows. An object of $\mathcal{E}mb_{O_K}^{\log}$ is a triple $(\mathbf{A} \to A)$ where:*

- *$A$ is a finite flat and log flat log $O_K$-algebra such that the log structure on $A_K$ is trivial.*

- $\mathbf{A}$ *is a log* $O_K$*-algebra formally of finite type and formally log smooth over* $O_K$.
- $\mathbf{A} \to A$ *is a log embedding.*

*A morphism* $(f, \mathbf{f}) : (\mathbf{A} \to A) \to (\mathbf{B} \to B)$ *is a pair of homomorphisms* $f : A \to B$ *and* $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ *of log* $O_K$*-algebras such that the diagram*

$$\begin{array}{ccc} \mathbf{A} & \longrightarrow & A \\ {\scriptstyle \mathbf{f}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathbf{B} & \longrightarrow & B \end{array}$$

*of log* $O_K$*-algebra homomorphisms is commutative.*
3. *For a finite flat and log flat log* $O_K$*-algebra* $A$ *such that the log structure on* $A_K$ *is trivial, let* $\mathcal{E}mb^{\log}_{O_K}(A)$ *be the subcategory of* $\mathcal{E}mb^{\log}_{O_K}$ *whose objects are of the form* $(\mathbf{A} \to A)$ *and morphisms are of the form* $(\mathrm{id}_A, \mathbf{f})$.
4. *We say that a morphism* $(f, \mathbf{f}) : (\mathbf{A} \to A) \to (\mathbf{B} \to B)$ *of* $\mathcal{E}mb_{O_K}$ *is finite flat and log flat if* $\mathbf{A} \to \mathbf{B}$ *is finite flat and log flat and the map* $\mathbf{B} \otimes^{\log}_{\mathbf{A}} A \to B$ *is an isomorphism of log* $O_K$*-algebras.*
5. *We say that a log embedding* $\mathbf{A} \to A$ *is strict if the maps* $O_K \to A$ *and* $O_K \to \mathbf{A}$ *of log rings are strict.*

For a complete semi-local Noetherian log $O_K$-algebra $R$ such that $R/\mathfrak{m}_R$ is finite over $F$, we put $\hat{\Omega}_{R/O_K}(\log/\log) = \varprojlim_n \Omega_{(R/\mathfrak{m}_R^n)/O_K}(\log/\log)$. If $(\mathbf{A} \to A)$ is a log embedding, the $\mathbf{A}$-module $\hat{\Omega}_{\mathbf{A}/O_K}(\log/\log)$ is locally free of finite rank. If $(\mathbf{A} \to A)$ is a strict object of $\mathcal{E}mb^{\log}_{O_K}$, by forgetting the log structures, we obtain an object $(\mathbf{A} \to A)^\circ$ of $\mathcal{E}mb_{O_K}$. For an object $(\mathbf{A} \to A)$ of $\mathcal{E}mb_{O_K}$, by putting the pull-back log structures on $\mathbf{A}$ and $A$ from that on $O_K$, we obtain an object $(\mathbf{A} \to A)^{\log}$ of $\mathcal{E}mb^{\log}_{O_K}$. Thus, we obtain an equivalence of categories between $\mathcal{E}mb_{O_K}$ and the full subcategory of $\mathcal{E}mb^{\log}_{O_K}$ consisting of strict objects.

Lemma 4.2 *Let* $A$ *be a finite flat and log flat log* $O_K$*-algebra such that the log structure on* $A_K$ *is trivial. We put* $X = \mathrm{Spec}\ A$ *and* $S = \mathrm{Spec}\ O_K$.
1. *For a closed point* $x$ *of* $X = \mathrm{Spec}\ A$, *the stalk* $\bar{M}_{X,x}$ *of the sheaf* $\bar{M}_X = M_X/O_X^\times$ *is isomorphic to* $\mathbb{N}$ *and the map* $\bar{M}_{S,s} = \mathbb{N} \to \bar{M}_{X,x} = \mathbb{N}$ *is the multiplication by an integer* $e_x \geq 1$.
2. *Let* $(\mathbf{A} \to A)$ *be a log embedding. Then, the ring* $\mathbf{A}$ *is regular and the reduced closed fiber* $(\mathbf{A} \otimes_{O_K} F)_{\mathrm{red}}$ *is a regular divisor. The log ring* $\mathbf{A}$ *is log regular and the log structure is defined by the reduced closed fiber* $(\mathbf{A} \otimes_{O_K} F)_{\mathrm{red}}$.
3. *A log embedding* $(\mathbf{A} \to A)$ *is strict if and only if the map* $O_K \to A$ *is strict.*

*Proof.* 1. Clear from Lemma 3.10.1.
2. We may assume $\mathbf{A}$ is local and the log structure is defined by a chart $\mathbb{N} \to \mathbf{A}$. Since $\mathbf{A}$ is formally log smooth over $O_K$, it is log regular by Corollary 3.14.2. Since the stalks of $\bar{M}$ are either $\mathbb{N}$ or 0, the ring $\mathbf{A}$ is regular and the image

$t \in \mathbf{A}$ of $1 \in \mathbb{N}$ defines a regular divisor. Since $\pi/t^{e_x} \in \mathbf{A}^\times$, the assertion follows.
3. We may assume $\mathbf{A}$ is local. Assume the map $O_K \to A$ is strict. Then, in the notation of the proof of 2, we have $e_x = 1$ and $\pi/t \in \mathbf{A}^\times$. Hence the map $O_K \to \mathbf{A}$ is strict. The only if part is obvious. $\square$
To prove the logarithmic version Lemma 4.5 below of Lemma 1.2, we make another definition.

Definition 4.3 *1. Let $A$ be a finite flat and log flat log $O_K$-algebra such that the log structure on $A_K$ is trivial. Let $\mathbf{A}$ be a log $O_K$-algebra formally of finite type, formally smooth and formally log smooth over $O_K$. We say that a surjection $\mathbf{A} \to A$ of log $O_K$-algebra is a log pre-embedding if it induces an isomorphism $\mathbf{A}/\mathfrak{m}_{\mathbf{A}} \to A/\mathfrak{m}_A$ of underlying $F$-algebras.*
*2. We define $pre\mathcal{E}mb^{\log}_{O_K}$ to be the category whose objects and morphisms are as follows. An object of $\mathcal{E}mb^{\log}_{O_K}$ is a triple $(\mathbf{A} \to A)$ where:*

- *$A$ is a finite flat and log flat log $O_K$-algebra such that the log structure on $A_K$ is trivial.*

- *$\mathbf{A}$ is a log $O_K$-algebra formally of finite type, formally smooth and formally log smooth over $O_K$.*

- *$\mathbf{A} \to A$ is a log pre-embedding.*

*A morphism $(f, \mathbf{f}) : (\mathbf{A} \to A) \to (\mathbf{B} \to B)$ is a pair of log $O_K$-homomorphism $f : A \to B$ and $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ such that the diagram*

$$\begin{array}{ccc} \mathbf{A} & \longrightarrow & A \\ {\scriptstyle \mathbf{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{B} & \longrightarrow & B \end{array}$$

*is commutative.*
*3. For a finite flat and log flat log $O_K$-algebra $A$ such that the log structure on $A_K$ is trivial, let $pre\mathcal{E}mb^{\log}_{O_K}(A)$ be the subcategory of $pre\mathcal{E}mb^{\log}_{O_K}$ whose objects are of the form $(\mathbf{A} \to A)$ and morphisms are of the form $(\mathrm{id}_A, \mathbf{f})$.*

A log pre-embedding $(\mathbf{A} \to A)$ is an embedding together with log structures on $\mathbf{A}$ and on $A$ such that the log ring $\mathbf{A}$ is formally log smooth, that the log ring $A$ is log flat and the log structure on $A_K$ is trivial and that the map $\mathbf{A} \to A$ is a surjection of log $O_K$-algebras. Hence, by forgetting the log structures, we obtain a functor $pre\mathcal{E}mb^{\log}_{O_K} \to \mathcal{E}mb_{O_K}$.
We also define a functor $pre\mathcal{E}mb^{\log}_{O_K} \to \mathcal{E}mb^{\log}_{O_K}$. For an object $(\mathbf{A} \to A)$ of $pre\mathcal{E}mb^{\log}_{O_K}$, we attach a log embedding $(\mathbf{A}^\sim \to A)$ as follows. First, we consider the case where $A$ is local. Assume the log structure of $\mathbf{A}$ is defined by a chart $P \to \mathbf{A}$. Let $P \to \mathbb{N}$ be the map $P \to \bar{M}_{X,x} = \mathbb{N}$ where $x$ is the closed point of

$X = \operatorname{Spec} A$ and we identify $\bar{M}_{X,x} = \mathbb{N}$ by the unique isomorphism. Let $P^\sim$ be the inverse image of $\mathbb{N}$ by the induced map $P^{\mathrm{gp}} \to \bar{M}^{\mathrm{gp}}_{X,x} = \mathbb{Z}$. The map $P \to \mathbf{A} \to A$ is extended uniquely to a map $P^\sim \to A$. We define $\mathbf{A}^\sim$ to be the formal completion of the surjection $\mathbf{A} \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P^\sim] \to A$ induced by $P^\sim \to A$. Let $\mathbf{A}^\sim \to A$ be the canonical map. The log ring $\mathbf{A}^\sim$ and the homomorphism $\mathbf{A}^\sim \to A$ are independent of the choice of the chart $P \to \mathbf{A}$ upto a unique isomorphism. In general, we define $\mathbf{A}^\sim$ and $\mathbf{A}^\sim \to A$ by taking the product. By the construction, the canonical map $\mathbf{A} \to \mathbf{A}^\sim$ is formally log étale.

Lemma 4.4 *Let $A$ be a finite flat and log flat log $O_K$-algebra such that the log structure on $A_K$ is trivial.*
*1. The category $pre\mathcal{E}mb^{\log}_{O_K}(A)$ is non-empty.*
*2. Let $(\mathbf{A} \to A)$ be an object of $pre\mathcal{E}mb^{\log}_{O_K}$ and define $\mathbf{A}^\sim$ and $\mathbf{A}^\sim \to A$ as above. Then $(\mathbf{A}^\sim \to A)$ is an object of $\mathcal{E}mb^{\log}_{O_K}$.*

*Proof.* 1. We may assume $A$ is local. Take a system of generators $t_1, \ldots, t_n$ of $A$ over $O_K$ and a chart $\mathbb{N} \to A$. Let $t_0 \in A$ be the image of $1 \in \mathbb{N}$. We define a surjection $O_K[T_0, \ldots, T_n] \to A$ by sending $T_i$ to $t_i$ and a log structure on $O_K[T_0, \ldots, T_n]$ by the chart $\mathbb{N}^2 \to O_K[T_0, \ldots, T_n]$ sending $(1,0)$ and $(0,1) \in \mathbb{N}^2$ to $T_0$ and $\pi$. Then its formal completion $\mathbf{A} \to A$ is a log pre-embedding.
2. By the definition, the $O_K$-algebra $\mathbf{A}^\sim$ is formally of finite type over $O_K$ and the surjection $\mathbf{A}^\sim \to A$ is exact. Since the map $\mathbf{A} \to \mathbf{A}^\sim$ is formally log étale, the log $O_K$-algebra $\mathbf{A}^\sim$ is formally log smooth over $O_K$. Hence the assertion follows. □
By Lemma 4.4.2, we obtain a functor $pre\mathcal{E}mb^{\log}_{O_K} \to \mathcal{E}mb^{\log}_{O_K}$.

Lemma 4.5 *1. For a finite flat and log flat log $O_K$-algebra $A$ such that the log structure on $A_K$ is trivial, the category $\mathcal{E}mb^{\log}_{O_K}(A)$ is non-empty.*
*2. For a morphism $f : A \to B$ of finite flat and log flat log $O_K$-algebras such that the log structures on $A_K$ and $B_K$ are trivial and for objects $(\mathbf{A} \to A)$ and $(\mathbf{B} \to B)$ of $\mathcal{E}mb^{\log}_{O_K}$, there exists a morphism $(f, \mathbf{f}) : (\mathbf{A} \to A) \to (\mathbf{B} \to B)$ lifting $f$.*
*3. For a morphism $f : A \to B$ of finite flat and log flat log $O_K$-algebras such that the log structures on $A_K$ and $B_K$ are trivial, the following conditions are equivalent.*
*(1) The map $f : A \to B$ is log syntomic.*
*(2) Their exists a finite flat and log flat morphism $(f, \mathbf{f}) : (\mathbf{A} \to A) \to (\mathbf{B} \to B)$ of log embeddings.*

*Proof.* 1. Clear from Lemma 4.4.
2. Since $\mathbf{A}$ is formally log smooth, $\mathbf{B} = \varprojlim_n \mathbf{B}/I^n$ where $I = \operatorname{Ker}(\mathbf{B} \to B)$ and the surjection $\mathbf{B}/I^n \to B$ is exact, the assertion follows.
3. (1)⇒(2). We may assume $A$ and $B$ are local. We take log embeddings $\mathbf{A} \to A$ and $\mathbf{B} \to B$. We define a log embedding $\mathbf{B}' \to B$ by applying an argument similar to the proof of Lemma 4.4.2 to $\varprojlim_n (\mathbf{A}/\mathfrak{m}^n_{\mathbf{A}} \otimes^{\log}_{O_K} \mathbf{B}/\mathfrak{m}^n_{\mathbf{B}})^\wedge \to$

$B$. Replacing $\mathbf{B} \to B$ by $\mathbf{B}' \to B$, we may assume that there is a map $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ such that $\mathbf{A} \to \mathbf{B}$ is formally log smooth. Since $A \to B$ is log syntomic, the exact surjection $\mathbf{B} \otimes^{\log}_{\mathbf{A}} A \to B$ is regular by Lemma 3.8.1 and the kernel is generated by a regular sequence $(t_1, \ldots, t_n)$. Take a lifting $(\tilde{t}_1, \ldots \tilde{t}_n)$ in $\mathbf{B}$ and define a map $\mathbf{A}[[T_1, \ldots, T_n]] \to \mathbf{B}$ by sending $T_i$ to $t_i$. We consider $\mathbf{A}[[T_1, \ldots, T_n]]$ as a log ring with the pull-back log structure by the map $\mathbf{A} \to \mathbf{A}[[T_1, \ldots, T_n]]$. Then the composition $\mathbf{A}[[T_1, \ldots, T_n]] \to \mathbf{A} \to A$ sending $T_i$ to 0 defines a log embedding. Replacing $\mathbf{A}$ by $\mathbf{A}[[T_1, \ldots, T_n]]$, we obtain a map $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ such that the map $\mathbf{B} \otimes^{\log}_{\mathbf{A}} A \to B$ is an isomorphism and that $\dim \mathbf{A} = \dim \mathbf{B}$. By Nakayama's lemma, the map $\mathbf{A} \to \mathbf{B}$ is finite. Since $\mathbf{A}$ and $\mathbf{B}$ are regular, the map $\mathbf{A} \to \mathbf{B}$ is flat by EGA Chap $0_{\mathrm{IV}}$ Corollaire (17.3.5) (ii). Further by Corollary 3.11, it is log syntomic.
(2)$\Rightarrow$(1). Since $\mathbf{A}$ and $\mathbf{B}$ are log regular and have the same dimension, $\mathbf{B}$ is log syntomic over $\mathbf{A}$ by Corollary 3.11. Hence $B$ is also log syntomic over $A$ by Lemma 3.7.2. $\square$

The base change of a log embedding by an extension of complete discrete valuation fields is defined as follows.

Lemma 4.6 *Let $K'$ be a complete discrete valuation field and $K \to K'$ be a morphism of fields inducing a local homomorphism $O_K \to O_{K'}$. Let $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb^{\log}_{O_K}$. We define $\mathbf{A}\hat{\otimes}^{\log}_{O_K} O_{K'}$ to be the projective limit $\varprojlim_n (\mathbf{A}/\mathfrak{m}^n_{\mathbf{A}} \otimes^{\log}_{O_K} O_{K'})$. Then the log $O_{K'}$-algebra $\mathbf{A}\hat{\otimes}^{\log}_{O_K} O_{K'}$ is formally of finite type and formally log smooth over $O_{K'}$. The natural surjection $\mathbf{A}\hat{\otimes}^{\log}_{O_K} O_{K'} \to A\hat{\otimes}^{\log}_{O_K} O_{K'}$ defines an object $(\mathbf{A}\hat{\otimes}^{\log}_{O_K} O_{K'} \to A \otimes^{\log}_{O_K} O_{K'})$ of $\mathcal{E}mb^{\log}_{O_{K'}}$.*

*Proof.* Since $\mathbf{A}\hat{\otimes}^{\log}_{O_K} O_{K'}$ is finite over $\mathbf{A}\hat{\otimes}_{O_K} O_{K'}$, it is formally of finite type over $O_{K'}$. The formal log smoothness is clear from the definition. The rest is clear. $\square$.

Thus we obtain a functor $\hat{\otimes}^{\log}_{O_K} O_{K'} : \mathcal{E}mb^{\log}_{O_K} \to \mathcal{E}mb^{\log}_{O_{K'}}$. If $K''$ is an extension of complete discrete valuation fields of $K'$, the composition $\mathcal{E}mb^{\log}_{O_K} \to \mathcal{E}mb^{\log}_{O_{K'}} \to \mathcal{E}mb^{\log}_{O_{K''}}$ is the same as $\hat{\otimes}^{\log}_{O_K} O_{K''} : \mathcal{E}mb^{\log}_{O_K} \to \mathcal{E}mb^{\log}_{O_{K''}}$. If $K'$ is a finite extension, we have $\mathbf{A} \otimes^{\log}_{O_K} O_{K'} = \mathbf{A}\hat{\otimes}^{\log}_{O_K} O_{K'}$. If $(\mathbf{A} \to A)$ is strict, we have $(\mathbf{A} \to A) \otimes^{\log}_{O_K} O_{K'} = ((\mathbf{A} \to A)^\circ \otimes_{O_K} O_{K'})^{\log}$.

Similarly as for $\varinjlim_{K'/K}(\mathrm{Aff}/F')$ defined in Section 1.3, we define a category $\varinjlim_{K'/K} \mathcal{E}mb_{O_{K'}}$. We define a functor $\mathcal{E}mb^{\log}_{O_K} \to \varinjlim_{K'/K} \mathcal{E}mb_{O_{K'}}$ as follows.

Lemma 4.7 *Let $A$ be a finite flat and log flat log $O_K$-algebra. Let $e = e_{A/O_K}$ denote the least common multiple of $e_x$ in Lemma 4.2.1 for the closed points $x$ in $X = \mathrm{Spec}\ A$. Let $K'$ be a finite separable extension of $K$ of ramification index $e_{K'/K}$. If $e_{K'/K}$ is divisible by $e_{A/O_K}$, then the log tensor product $A_{O_{K'}} = A \otimes^{\log}_{O_K} O_{K'}$ is strict over $O_{K'}$.*

*Proof.* We may assume $A$ is local. We put $P = N' = \mathbb{N} \times \mathbb{Z}$ and define maps $\mathbb{N} \to P$ and $\mathbb{N} \to N'$ by sending $1 \in \mathbb{N}$ to $(e_{A/O_K}, 1)$ and to $(e_{O_{K'}/O_K}, 1)$ respectively. There exist morphisms of charts $(\mathbb{N} \to O_K) \to (P \to A)$ and $(\mathbb{N} \to O_K) \to (N' \to O_{K'})$. Since $e_{A/O_K}$ divides $e_{O_{K'}/O_K}$, the saturation $P +^{\mathrm{sat}}_{\mathbb{N}} N'$ is isomorphic to $\mathbb{N} \times (\mathbb{Z}/e_{A/O_K}\mathbb{Z}) \times \mathbb{Z}^2$ and the composition $\mathbb{N} \subset N' \to P +^{\mathrm{sat}}_{\mathbb{N}} N' \to \mathbb{N}$ is the identity. Hence $A \otimes^{\log}_{O_K} O_{K'}$ is strict over $O_{K'}$. □

Let $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb^{\log}_{O_K}$ and define $e = e_{A/K}$ as in Lemma 4.7. Let $\mathcal{C}_e$ be the full subcategory of the category $(\mathrm{Ext}/K)$ of finite separable extensions of $K$ consisting of the extensions with ramification index divisible by $e$. If $K'$ is a finite separable extension in $\mathcal{C}_e$, then by Lemmas 4.7 and 4.2.3, the base change $(\mathbf{A} \otimes^{\log}_{O_K} O_{K'} \to A \otimes^{\log}_{O_K} O_{K'})$ is strict and defines an object $(\mathbf{A} \otimes^{\log}_{O_K} O_{K'} \to A \otimes^{\log}_{O_K} O_{K'})^\circ$ of $\mathcal{E}mb_{O_{K'}}$. We consider a system consisting of $(\mathbf{A} \otimes^{\log}_{O_K} O_{K'} \to A \otimes^{\log}_{O_K} O_{K'})^\circ$ for extensions $K'$ in $\mathcal{C}_e$ and isomorphisms $(\mathbf{A} \otimes^{\log}_{O_K} O_{K'} \to A \otimes^{\log}_{O_K} O_{K'})^\circ \otimes_{O_{K'}} O_{K''} \to (\mathbf{A} \otimes^{\log}_{O_K} O_{K''} \to A \otimes^{\log}_{O_K} O_{K''})^\circ$ for $K$-morphisms $K' \to K''$ of extensions in $\mathcal{C}_e$. Then it defines an object of $\varinjlim_{K'/K} \mathcal{E}mb_{O_{K'}}$. Thus we obtain a functor $\mathcal{E}mb^{\log}_{O_{K'}} \to \varinjlim_{K'/K} \mathcal{E}mb_{O_{K'}}$.

## 4.2 Tubular neighborhoods for log embeddings

For a rational number $j > 0$, a functor $X^j : \varinjlim_{K'/K} \mathcal{E}mb_{O_{K'}} \to \varinjlim_{K'/K}(\text{smooth Affinoid}/K')$ is defined as the limit of the functors $X^{je_{K'/K}} : \mathcal{E}mb_{O_{K'}} \to (\text{smooth Affinoid}/K')$ defined in Section 1.2. We define a functor $\varinjlim_{K'/K}(\text{smooth Affinoid}/K') \to \varinjlim_{K'/K}(\mathrm{Aff}/F')$ as follows. Let $(X_{K'})_{K' \in ob\mathcal{C}}$ be an object of (smooth Affinoid$/K'$). Then the extensions $K'$ in $\mathcal{C}$ such that the stable normalized integral model $\mathcal{A}_{O_{K'}}$ is defined over $K'$ form a cofinal full subcategory $\mathcal{C}'$ by Theorem 1.10. For an extension $K'$ in $\mathcal{C}'$, let $\bar{X}_{F'}$ denote the affine scheme $\mathcal{A}_{O_{K'}} \otimes_{O_{K'}} F'$ over the residue field $F'$ of $K'$. By sending $(X_{K'})_{K' \in ob\mathcal{C}}$ to $(\bar{X}_{F'})_{K' \in ob\mathcal{C}'}$, we obtain a functor $\varinjlim_{K'/K}(\text{smooth Affinoid}/K') \to \varinjlim_{K'/K}(\mathrm{Aff}/F')$. Thus, we have a sequence of functors

$$\begin{aligned}\mathcal{E}mb^{\log}_{O_K} \longrightarrow \varinjlim_{K'/K} \mathcal{E}mb_{O_K} \longrightarrow \varinjlim_{K'/K}(\text{smooth Affinoid}/K') \\ \longrightarrow \varinjlim_{K'/K}(\mathrm{Aff}/F') \longrightarrow G_K\text{-}(\mathrm{Aff}/\bar{F}).\end{aligned}$$

The compositions $X^j_{\log} : \mathcal{E}mb^{\log}_{O_K} \to \varinjlim_{K'/K}(\text{smooth Affinoid}/K')$ and $\bar{X}^j_{\log} : \mathcal{E}mb^{\log}_{O_K} \to G_K\text{-}(\mathrm{Aff}/\bar{F})$ are more concretely described as follows. For an object $(\mathbf{A} \to A)$ of $\mathcal{E}mb^{\log}_{O_K}$ and a finite separable extension $K'$ such that the ramification index $e' = e_{K'/K}$ is divisible by the integer $e_{A/O_K}$ in Lemma 4.7, the base change $(\mathbf{A} \hat{\otimes}^{\log}_{O_K} O_{K'} \to A \otimes^{\log}_{O_K} O_{K'})$ is strict and we define an affinoid variety $X^j_{\log}(\mathbf{A} \to A)_{K'}$ over $K'$ by

$$X^j_{\log}(\mathbf{A} \to A)_{K'} = X^{e'j}((\mathbf{A} \hat{\otimes}^{\log}_{O_K} O_{K'} \to A \otimes^{\log}_{O_K} O_{K'})^\circ).$$

The composite functors $X^j_{\log} : \mathcal{E}mb^{\log}_{O_K} \to \varinjlim_{K'/K}(\text{smooth Affinoid}/K')$ sends an object $(\mathbf{A} \to A)$ of $\mathcal{E}mb^{\log}_{O_K}$ to the system $X^j_{\log}(\mathbf{A} \to A) = (X^j_{\log}(\mathbf{A} \to A)_{K'})_{K'}$ where $K'$ runs over finite separable extensions such that the ramification index $e' = e_{K'/K}$ is divisible by the integer $e_{A/O_K}$.
By Lemma 1.8 and the universality of $\otimes^{\log}$, we obtain a cartesian diagram

$$\begin{array}{ccc} X^j_{\log}(\mathbf{A} \to A)(\bar{K}) & \longrightarrow & Hom_{\text{cont.log}O_K\text{-alg}}(\mathbf{A}, O_{\bar{K}}) \\ \downarrow & & \downarrow \\ Hom_{\log O_K\text{-alg}}(A, O_{\bar{K}}/\mathfrak{m}^j) & \longrightarrow & Hom_{\text{cont.log}O_K\text{-alg}}(\mathbf{A}, O_{\bar{K}}/\mathfrak{m}^j). \end{array}$$

Here $O_{\bar{K}}/\mathfrak{m}^j$ denotes the limit $O_{K'}/\mathfrak{m}^{je_{K'/K}}$ of fs-log rings where $K'$ runs finite extensions in $\bar{K}$ such that $je_{K'/K}$ is an integer. Similarly as in Section 1.2, the surjection $X^j_{\log}(\mathbf{A} \to A)(\bar{K}) \to \pi_0(X^j_{\log}(\mathbf{A} \to A))_{\bar{K}}$ induces a surjection

$$(4.2.1) \qquad Hom_{\text{cont.log}O_K\text{-alg}}(\mathbf{A}, O_{\bar{K}}/\mathfrak{m}^j) \longrightarrow \pi_0(X^j_{\log}(\mathbf{A} \to A))_{\bar{K}}.$$

The map $\mathbf{A} \to A$ also induces a map

$$(4.2.2) \qquad Hom_{\log O_K\text{-alg}}(A, O_{\bar{K}}) \longrightarrow X^j_{\log}(\mathbf{A} \to A)(\bar{K}).$$

Similarly as Lemma 1.9.4, if $(f, \mathbf{f}) : (\mathbf{A} \to O_K) \to (\mathbf{B} \to B)$ is a finite flat and log flat morphism of $\mathcal{E}mb^{\log}_{O_K}$, the map (4.2.2) induces a surjection

$$(4.2.3) \qquad Hom_{\log O_K\text{-alg}}(B, O_{\bar{K}}) \longrightarrow \pi_0(X^j_{\log}(\mathbf{B} \to B)_{\bar{K}}).$$

Let (Finite Flat and Log Flat/$O_K$) denote the category of finite flat and log flat log $O_K$-algebras $A$ such that the log structure on $A \otimes_{O_K} K$ is trivial. We define functors $\Psi_{\log}$ and $\Psi^j_{\log}$ : (Finite Flat and Log Flat/$O_K$) $\to G_K$-(Finite Sets) for a rational number $j > 0$ as in Section 1.2 by sending a finite flat and log flat log $O_K$-algebra $A$ such that the log structure on $A \otimes_{O_K} K$ to the set $\Psi_{\log}(A) = Hom^{\log}_{O_K}(A, O_{\bar{K}})$ and to the set

$$\Psi^j_{\log}(A) = \varprojlim_{(\mathbf{A} \to A) \in \mathcal{E}mb^{\log}_{O_K}(A)} \pi_0(X^j_{\log}(\mathbf{A} \to A)_{\bar{K}})$$

respectively. As in Section 1.2, the surjection (4.2.1) implies that the projective system in the right hand side is constant. Further it induces a map $\Psi_{\log} \to \Psi^j_{\log}$ of functors.
Similarly, for an object $(\mathbf{A} \to A)$ of $\mathcal{E}mb^{\log}_{O_K}$ and a finite separable extension $K'$ such that the ramification index $e' = e_{K'/K}$ is divisible by the integer $e_{A/O_K}$ and that a stable normalized integral model $\mathcal{A}^j_{O_{K'}}$ of $X^j_{\log}(\mathbf{A} \to A)_{K'}$ is defined over $K'$, an affine scheme $\bar{X}^j_{\log}(\mathbf{A} \to A)_{K'}$ over the residue field $F'$ of $K'$ is defined as the closed fiber $\text{Spec}(\mathcal{A}^j_{O_{K'}} \otimes_{O_{K'}} F')$. The system

$\bar{X}^j_{\log}(\mathbf{A} \to A) = (\bar{X}^j_{\log}(\mathbf{A} \to A)_{K'})_{K'}$ defines an object of $\varinjlim_{K'/K}(\mathrm{Aff}/F')$. By identifying the category $\varinjlim_{K'/K}(\mathrm{Aff}/F')$ with $G_K$-$(\mathrm{Aff}/\bar{F})$, we obtain the composite functor $\bar{X}^j_{\log} : \mathcal{E}mb^{\log}_{O_K} \to G_K$-$(\mathrm{Aff}/\bar{F})$. For $j > 0$, the functor $\Psi^j_{\log}$ : (Finite Flat and Log Flat$/O_K$) $\to G_K$-(Finite Sets) is induced by the composition of the functors

$$\mathcal{E}mb^{\log}_{O_K} \xrightarrow{\bar{X}^j_{\log}} G_K\text{-}(\mathrm{Aff}/\bar{F}) \xrightarrow{\pi_0} G_K\text{-(Finite Sets)}.$$

We also have a functor $\bar{C}^j_{\log} : \mathcal{E}mb^{\log}_{O_K} \to G_K$-$(\mathrm{Aff}/\bar{F})$ and a map of functors $\bar{X}^j_{\log} \to \bar{C}^j_{\log}$. Let $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb^{\log}_{O_K}$ and $j > 0$ be a rational number. Let $K'$ be a finite separable extension of $K$ such that the ramification index $e' = e_{K'/K}$ is divisible by $e_{A/O_K}$ and by the denominator of $j$ and that $((A \otimes^{\log}_{O_K} O'_K) \otimes_{O_{K'}} F')_{\mathrm{red}}$ is étale over $F'$. Let $I$ be the kernel of $\mathbf{A} \otimes^{\log}_{O_K} O'_K \to A \otimes^{\log}_{O_K} O'_K$ and we put

$$\bar{C}^j_{\log}(\mathbf{A} \to A)_{K'} = \mathrm{Spec}\left(\bigoplus_{n=0}^{\infty} I^n/I^{n+1} \otimes_{O_{K'}} \mathfrak{m}^{e'jn}_{K'}/m^{e'jn+1}_{K'}\right)_{\mathrm{red}.}$$

Then the system $(\bar{C}^j_{\log}(\mathbf{A} \to A)_{K'})_{K'}$ defines an object $\varinjlim_{K'/K}(\mathrm{Aff}/F')$ and hence an object $\bar{C}^j_{\log}(\mathbf{A} \to A)$ of $G_K$-$(\mathrm{Aff}/\bar{F})$. It is a scheme over $((A \otimes^{\log}_{O_K} O_{K'}) \otimes_{O_{K'}} \bar{F})_{\mathrm{red}}$ for $K'$ as above. In the following, we put $A_{\log \bar{F},\mathrm{red}} = ((A \otimes^{\log}_{O_K} O_{K'}) \otimes_{O_{K'}} \bar{F})_{\mathrm{red}} = (A \otimes^{\log}_{O_K} \bar{F})_{\mathrm{red}}$. In the right hand side, $\bar{F}$ is regarded as the limit of an fs-log ring with the chart $\mathbb{Q}_{\geq 0} \to \bar{F}$ sending positive rational numbers to 0.

We study relations between $X^j$ and $X^j_{\log}$. Let $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb_{O_K}$ and $(\mathbf{B} \to B)$ be an object of $\mathcal{E}mb^{\log}_{O_K}$. Let $(\mathbf{A} \to A)^{\log}$ be the object of $\mathcal{E}mb^{\log}_{O_K}$ defined by the pull-back log structures. An $O_K$-algebra homomorphism $A \to B$ can be lifted to a morphism $(\mathbf{A} \to A)^{\log} \to (\mathbf{B} \to B)$ of $\mathcal{E}mb^{\log}_{O_K}$ by Lemma 4.2. For a rational number $j > 0$, a morphism $(\mathbf{A} \to A)^{\log} \to (\mathbf{B} \to B)$ of $\mathcal{E}mb^{\log}_{O_K}$ induces a morphism $X^j_{\log}(\mathbf{B} \to B) \to X^j_{\log}((\mathbf{A} \to A)^{\log}) = X^j(\mathbf{A} \to A)$ of affinoid varieties.

Let $(\mathbf{A} \to A)$ be a log pre-embedding. We have an embedding $(\mathbf{A} \to A)^\circ$, a log embedding $(\mathbf{A}^\sim \to A)$ and a canonical map $((\mathbf{A} \to A)^\circ)^{\log} \to (\mathbf{A}^\sim \to A)$ of log embeddings by the construction in Lemma 4.4.2. For a rational number $j > 0$, we have affinoid varieties $X^j((\mathbf{A} \to A)^\circ)$ and $X^j_{\log}(\mathbf{A}^\sim \to A)$ and a map of affinoid varieties $X^j_{\log}(\mathbf{A}^\sim \to A) \to X^j((\mathbf{A} \to A)^\circ)$.

Lemma 4.8 *Let $(\mathbf{A} \to A)$ be an object of $pre\mathcal{E}mb^{\log}_{O_K}$ and $j > 0$ be a positive integers.*
*1. The canonical map $X^j_{\log}(\mathbf{A}^\sim \to A) \to X^j((\mathbf{A} \to A)^\circ)$ is an open immersion and $X^j_{\log}(\mathbf{A}^\sim \to A)$ is identified with a rational subdomain.*

*2. Assume $A$ is local and put $S = \mathrm{Spec}\ O_K, X = \mathrm{Spec}\ A$ and $\mathbf{X} = \mathrm{Spec}\ \mathbf{A}$ and let $s$ and $x$ be the closed points of $S$ and of $X$. We put $P = \bar{M}_{\mathbf{X},x}$ and identify $\bar{M}_{X,x}$ and $\bar{M}_{S,s}$ with $\mathbb{N}$. Let $e = e_{A/O_K}$ be the image of $1 \in \bar{M}_{S,s} = \mathbb{N}$ by the composition $\bar{M}_{S,s} \to \bar{M}_{\mathbf{X},x} \to \bar{M}_{X,x} = \mathbb{N}$ as in Lemma 4.7. Let $m_1, \ldots, m_n$ be a system of generators of the monoid $P$ and $e_1, \ldots, e_n$ be their images by $P \to \mathbb{N} = \bar{M}_{X,x}$. Let $j' \geq j + \max_i e_i/e$ be a rational number strictly greater than 1. Then we have an open immersion $X^{j'}((\mathbf{A} \to A)^\circ) \to X^j_{\log}(\mathbf{A}^\sim \to A)$ of rational subdomains $X^j((\mathbf{A} \to A)^\circ)$.*

*Proof.* 1. We may assume $A$ is local. We use the notation in 2. Let $I$ be the kernel of the surjection $\mathbf{A} \to A$ and $J$ be the kernel of the surjection $\mathbf{A}^\sim \to A$. By renumbering the indices if necessary, we may assume $e_1 = 1$. We take a chart $\varphi : P \to \mathbf{A}$ and put $t_i = \varphi(m_i) \in \mathbf{A}$. We define a monoid $P^\sim$ as in Lemma 4.4.2 and $\tilde{\varphi} : P^\sim \to \mathbf{A}^\sim$ be the extension. The monoid $P^\sim$ is generated by $P$ and $(m_i m_1^{-e_i})^{\pm 1}, i = 2, \ldots, n$. Hence the ring $\mathbf{A}^\sim$ is the completion of the subring generated by $\tilde{\varphi}(m_i m_1^{-e_i})^{\pm 1}$ over $\mathbf{A}$. For $i = 2, \ldots, n$, take liftings $u_i \in \mathbf{A}^\times$ of the image of $\tilde{\varphi}(m_i m_1^{-e_i})$ in $A^\times$. Then, the ideal $J$ is generated by the image of $I$ and $\tilde{\varphi}(m_i m_1^{-e_i}) - u_i, i = 2, \ldots, n$. Hence $X^j_{\log}(\mathbf{A}^\sim \to A)$ is the rational subdomain $X^j((\mathbf{A} \to A)^\circ)$ defined by the conditions $\mathrm{ord}(t_i t_1^{-e_i} - u_i) \geq j$ for $i = 2, \ldots, n$.
2. Similarly as in the proof of Lemma 1.17, we have $\mathrm{ord}\ t_1 = 1/e$ on $X^{j'}((\mathbf{A} \to A)^\circ)$ by the assumption $j' > 1$. Since $t_i - u_i t_1^{-e_i} \in I$ for $i = 2, \ldots, n$, we have $\mathrm{ord}(t_i - u_i t_1^{e_i}) \geq j' \geq j + e_i/e$ on $X^{j'}((\mathbf{A} \to A)^\circ)$. Hence the assertion follows. □

Corollary 4.9 *Let $(\mathbf{A} \to A)$ be a log pre-embedding constucted in the proof of Lemma 4.4.1. Then, for a rational number $j > 0$, we have open immersions*

$$X^{j+1}((\mathbf{A} \to A)^\circ) \longrightarrow X^j_{\log}(\mathbf{A}^\sim \to A) \longrightarrow X^j((\mathbf{A} \to A)^\circ)$$

*of rational subdomains.*

*Proof.* The log structure on $\mathbf{A}$ is defined by a chart $\mathbb{N}^2 \to \mathbf{A}$ and we have $e_1 = 1$ and $e_2 = e_{L/K}$ for $m_1 = (1, 0)$ and $m_2 = (0, 1)$ in the notation of Lemma 4.8.2. Hence the assertion follows. □

The affinoid varieties $X^j_{\log}(\mathbf{A} \to A)$ and $\mathcal{Y}^j_{Z,P}$ defined in [1] Section 3.2 are related as follows. Let $L$ be a finite separable extension of $K$ and $A = O_L$ be the integer ring. Let $Z = (z_i)_{i \in I}$ be a finite system of generators of $O_L$ over $O_K$ and $P \subset I$ be a subset such that $z_i$ is a prime element of $O_L$ for some $i \in P$ and $z_i$ is not zero for any $i \in P$. We recall a description of $\mathcal{Y}^j_{Z,P}$ for a rational number $j > 0$. We put $e_i = \mathrm{ord}_L z_i$ and $e = e_{L/K}$ and let $\pi$ be a prime element of $K$. Let $I_Z$ be the kernel of the surjection $O_K[T_i; i \in I] \to A$ sending $T_i$ to $z_i$ and $(f_1, \ldots, f_m)$ be a finite set of generators of $I_Z$. For $i \in P$ and $(i, j) \in P^2$, we take polynomials $g_i, h_{i,j} \in O_K[T_i; i \in I]$ such that the images in $O_L$ are

$u_i = z_i^e/\pi^{e_i}$ and $u_{i,j} = z_j^{e_i}/z_i^{e_j}$. If $z_\iota$ is a prime element for $\iota \in P$, then we have

$$\mathcal{Y}^j_{Z,P}(\bar{K}) = \left\{ (x_i)_{i\in I} \in O_{\bar{K}}^I \left| \begin{array}{cc} \mathrm{ord} f_l(x_i) \geq j & \text{for } 1 \leq l \leq m \\ \mathrm{ord}(x_\iota^e/\pi^{e_\iota} - g_\iota(x_i)) \geq j & \\ \mathrm{ord}(x_k^{e_\iota}/x_\iota^{e_k} - h_{k,\iota}(x_i)) \geq j & \text{for } k \in P \end{array} \right. \right\}$$

by [1] Lemma 3.9 (2). Furthermore, for $(x_i)_{i\in I} \in \mathcal{Y}^j_{Z,P}(\bar{K})$, we have $x_i/x_\iota^{e_i} \in O_{\bar{K}}^\times$ for $i \in P$.

We define a log structure on $O_K[T_i, i \in I]$ by the chart $M = \mathbb{N} \times \mathbb{N}^P \to O_K[T_i, i \in I]$ sending $(1,0)$ to $\pi$ and $(0, f_i)$ to $T_i$ where $f_i \in \mathbb{N}^P$ is the $i$-th standard basis. Let $\mathbf{A}$ be the formal completion of the surjection $O_K[T_i, i \in I] \to A$ sending $T_i$ to $z_i$.

Lemma 4.10 *Let the notation be as above. Then $(\mathbf{A} \to A)$ is a log pre-embedding and the affinoid variety $X^j_{\log}(\mathbf{A}^\sim \to A)_{\bar{K}}$ defined by the log embedding $(\mathbf{A}^\sim \to A)$ is the same as $\mathcal{Y}^j_{Z,P}$ defined in [1] Section 3.2.*

*Proof.* It is clear that $(\mathbf{A} \to A)$ is a log pre-embedding. We describe the log $O_K$-algebra $\mathbf{A}^\sim$. As in Lemma 4.4.2, let $P^\sim \subset P^{\mathrm{gp}} = \mathbb{Z} \times \mathbb{Z}^P$ be the inverse image of $\mathbb{N}$ by the map $\mathbb{Z} \times \mathbb{Z}^P \to \mathbb{Z}$ sending $T_0 = (1,0)$ to $e$ and the standard basis $T_i$ of $\mathbb{Z}^P$ to $e_i$ for $i \in P$. We consider a chart $\mathbb{N} \to O_K$ and a map of monoids $\mathbb{N} \to P^\sim$ sending $1 \in \mathbb{N}$ to a prime element $\pi \in O_K$ and to $T_0 \in P^\sim$. We put $A_{I,P} = O_K \otimes_{\mathbb{Z}[\mathbb{N}]} \mathbb{Z}[P^\sim][T_i, i \in I - P]$ and define a log structure by the chart $P^\sim \to A_{I,P}$. Then, $\mathbf{A}^\sim$ is identified with the formal completion of the natural surjection $A_{I,P} \to A$.

Let $K'$ be a finite separable extension of $K$ containing $L$ as a subfield. We compute the log tensor product $A_{I,P} \otimes^{\log}_{O_K} O_{K'}$. By choosing a numbering, we assume $P = \{1, \ldots, r\} \subset I = \{1, \ldots, m\}$ and $z_r$ is a prime element. Let $T_i, i = 0, \ldots, r$ be the standard basis of $P = \mathbb{N} \times \mathbb{N}^P$ and put $U_i = T_i T_r^{-e_i}$ for $i = 1, \ldots, r-1$ and $U_0 = T_0 T_r^{-e}$. Then the monoid $P^\sim$ is generated by $U_i^{\pm 1}, i = 0, \ldots, r-1$ and $T_r$ and is isomorphic to $\mathbb{Z}^r \times \mathbb{N}$. Let $N'$ be the monoid $\mathbb{N} \times \mathbb{Z}$ with the map $\mathbb{N} \to \mathbb{N} \times \mathbb{Z}$ sending $1 \in \mathbb{N}$ to $(e', 1)$. Let $\pi'$ be a prime element of $K'$ and $e' = e_{K'/K}$ be the ramification index and define a unit $u'$ of $O_{K'}$ by $\pi = u'\pi'^{e'}$. We consider a chart $N' \to O_{K'}$ sending $U' = (0,1)$ to $u'$ and $T' = (1,0)$ to $\pi'$. By the assumption $L \subset K'$, $\bar{e} = e'/e$ is an integer and the saturation $P^\sim +^{\mathrm{sat}}_{\mathbb{N}} N'$ is generated by $U_i^{\pm 1}, i = 1, \ldots, r-1, V^{\pm 1}, U'^{\pm 1}$ and $T'$ where $V = T_r T'^{-\bar{e}}$ and is isomorphic to $\mathbb{Z}^{r+1} \times \mathbb{N}$. Hence $A_{I,P} \otimes^{\log}_{O_K} O_{K'} = O_{K'} \otimes_{\mathbb{Z}[N']} \mathbb{Z}[P^\sim +^{\mathrm{sat}}_{\mathbb{N}} N'][T_{r+1}, \ldots, T_m]$ is isomorphic to $O_{K'}[U_1^{\pm 1}, \ldots, U_{r-1}^{\pm 1}, T_{r+1}, \ldots, T_m, V^{\pm 1}]$. The log structure is the pull-back of that on $O_{K'}$.

The base change $\mathbf{A} \hat{\otimes}^{\log}_{O_K} O_{K'}$ is the formal completion of the surjection $A_{I,P} \otimes^{\log}_{O_K} O_{K'} \to O_L \otimes^{\log}_{O_K} O_{K'}$. We claim that the kernel of the surjection $A_{I,P} \otimes^{\log}_{O_K} O_{K'} \to O_L \otimes^{\log}_{O_K} O_{K'}$ is generated by $I_Z$ and $U_0 - \pi/z_r^e, U_i - z_i/z_r^{e_i}, i = 1, \ldots, r$. The kernel $\mathrm{Ker}(\mathbf{A} \hat{\otimes}^{\log}_{O_K} O_{K'} \to O_L \otimes^{\log}_{O_K} O_{K'})$ is generated by $\mathrm{Ker}(A_{I,P} \to O_L)$

since the surjection $A_{I,P} \to O_L$ is exact. Since $P^{\sim}$ is generated by $U_0 = T_0T_r^{-e}, U_1, \ldots, U_{r-1}$ and $P$, the ring $A_{I,P}$ is also generated by $U_0, U_1, \ldots, U_{r-1}$ over $O_K[T_1, \ldots, T_m]$. Hence, $\mathrm{Ker}(A_{I,P} \to O_L)$ is generated by $I_Z$ and $U_0 - \pi/z_r^e, U_i - z_i/z_r^{e_i}, i = 1, \ldots, r$ and the claim is proved.
For an element $(u_1, \ldots, u_{r-1}, v, x_{r+1}, \ldots, x_m) \in O_{\bar{K}}^{\times r} \times O_{\bar{K}}^{m-r}$, we put $x_r = v\pi'^{\bar{e}}$ and $x_i = u_i x_r^{e_i}$ for $i = 1, \ldots, r-1$. Then, the underlying set of $X^j_{\log}(\mathbf{A} \to A)_{\bar{K}}$ is

$$\left\{ \begin{matrix} (u_1, \ldots, u_{r-1}, v, x_{r+1}, \ldots, x_m) \\ \in O_{\bar{K}}^{\times r} \times O_{\bar{K}}^{m-r} \end{matrix} \,\middle|\, \begin{matrix} \mathrm{ord} f_l(x_i) \geq j \quad \text{for } 1 \leq l \leq m \\ \mathrm{ord}(v^e/u' - g_r(x_i)) \geq j \\ \mathrm{ord}(u_k - h_k(x_i)) \geq j \quad \text{for } k = 1, \ldots, r \end{matrix} \right\}.$$

Hence the map $X^j_{\log}(\mathbf{A} \to A)_{\bar{K}} \to \mathcal{Y}^j_{Z,P}$ sending $(u_1, \ldots, u_{r-1}, v, x_{r+1}, \ldots, x_m)$ to $(x_1, \ldots, x_m)$ is an isomorphism. $\square$

### 4.3 Étale covering of log tubular neighborhoods

Let $A$ and $B$ be the integer rings of finite étale $K$-algebras. For a finite flat and log flat morphism $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ of log embeddings, we study conditions for the induced finite morphism $X^j_{\log}(\mathbf{A} \to A) \to X^j_{\log}(\mathbf{B} \to B)$ to be étale.

Proposition 4.11 *Let $A$ and $B = O_L$ be the integer rings of finite separable extensions of $K$ and $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ be a finite flat and log flat morphism of log embeddings. Let $j > 0$ be a rational number, $\pi_L$ a prime element of $L$ and $e = \mathrm{ord}\pi_L$ be the ramification index.*
*1. Assume $A = O_K$. Suppose that, for each $j' > j$, there exists a finite separable extension $K'$ of $K$ such that $X^{j'}_{\log}(\mathbf{B} \to B)_{K'}$ is isomorphic to the disjoint union of finitely many copies of $X^{j'}_{\log}(\mathbf{A} \to A)_{K'}$ as an affinoid variety over $X^{j'}_{\log}(\mathbf{A} \to A)_{K'}$. Then there is an integer $0 \leq n < ej$ such that $\pi_L^n$ annihilates $\Omega_{B/A}(\log/\log)$.*
*2. If there is an integer $0 \leq n < ej$ such that $\pi_L^n$ annihilates $\Omega_{B/A}(\log/\log)$, then the finite flat map $X^j_{\log}(\mathbf{B} \to B) \to X^j_{\log}(\mathbf{A} \to A)$ is étale.*

Corollary 4.12 *Let $A = O_K$ and let $B$ be the integer ring of a finite étale $K$-algebra and $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ be a finite flat and log flat morphism of log embeddings. Let $j > 0$ be a rational number. Suppose that, for each $j' > j$, there exists a finite separable extension $K'$ of $K$ such that $X^{j'}_{\log}(\mathbf{B} \to B)_{K'}$ is isomorphic to the disjoint union of finitely many copies of $X^{j'}_{\log}(\mathbf{A} \to A)_{K'}$ as in Proposition 4.11.1. Let $I$ be the kernel of the surjection $\mathbf{B} \to B$ and let $N_{B/\mathbf{B}}$ be the $B$-module $I/I^2$. Then we have the following.*
*1. The finite map $X^j_{\log}(\mathbf{B} \to B) \to X^j_{\log}(\mathbf{A} \to A)$ is étale and is extended to a finite étale map of stable normalized integral models.*
*2. The finite map $\bar{X}^j_{\log}(\mathbf{B} \to B) \to \bar{X}^j_{\log}(\mathbf{A} \to A)$ is étale.*

*3. The twisted normal cone* $\bar{C}^j_{\log}(\mathbf{B} \to B)$ *is canonically isomorphic to the covariant vector bundle defined by the* $B_{\bar F,\mathrm{red}}$*-module* $(Hom_B(N_{B/\mathbf{B}}, B) \otimes_{O_K} N^j) \otimes_{B_{\bar F}} B_{\log \bar F,\mathrm{red}}$ *and the finite map* $\bar{X}^j_{\log}(\mathbf{B} \to B) \to \bar{C}^j_{\log}(\mathbf{B} \to B)$ *is étale.*

To prove Proposition 4.11, we use the following.

Lemma 4.13 *Let* $A = O_L$ *be the integer ring of a finite separable extension* $L$*,* $\mathbf{A} \to A$ *be a log embedding and let* $\mathbf{M}$ *be an* $\mathbf{A}$*-module of finite type. Let* $j > 0$ *be a rational number and* $K'$ *be a finite separable extension of* $K$ *such that the map* $O_{K'} \to A \otimes^{\log}_{O_K} O_{K'}$ *is strict and the stable normalized integral model* $\mathcal{A}^j_{O_{K'}}$ *of* $X^j_{\log}(\mathbf{A} \to A)$ *is defined over* $K'$*. Let* $e$ *and* $e'$ *be the ramification indices of* $L$ *and of* $K'$ *over* $K$ *and* $\pi_L$ *and* $\pi'$ *be prime elements of* $L$ *and* $K'$*. Assume that* $e'/e$ *and* $e'j$ *are integers. Then the following conditions are equivalent.*
*(1) There exists an integer* $0 \leq n < ej$ *such that the* $A$*-module* $M = \mathbf{M} \otimes_{\mathbf{A}} A$ *is annihilated by* $\pi_L^n$*.*
*(2) The* $\mathcal{A}^j_{O_{K'}}$*-module* $\mathcal{M}^j = \mathbf{M} \otimes_{\mathbf{A}} \mathcal{A}^j_{O_{K'}}$ *is annihilated by* $\pi'^{e'j-1}$*.*

*Proof of Lemma 4.13.* The proof is similar to that of Lemma 1.17. The image of an element in the kernel $I$ of the surjection $\mathbf{A} \otimes^{\log}_{O_K} O_{K'} \to A \otimes^{\log}_{O_K} O_{K'}$ in $\mathcal{A}^j_{O_{K'}}$ is divisible by $\pi'^{e'j}$. Hence we have a commutative diagram

$$\begin{array}{ccc}
\mathbf{A} & \longrightarrow & \mathcal{A}^j_{O_{K'}} \\
\big\downarrow & & \big\downarrow \\
A & \longrightarrow & \mathcal{A}^j_{O_{K'}}/(\pi'^{e'j})
\end{array}$$

of log rings. The image of $\pi_L \in A$ is a unit times $\pi'^{e'/e}$ in $\mathcal{A}^j_{O_{K'}}/(\pi'^{e'j})$. The rest of the proof is the same as that of Lemma 1.17.

*Proof of Proposition 4.11.* Proof is similar to that of Proposition 1.15.
1. For $j > 0$, the affinoid variety $X^j_{\log}(\mathbf{A} \to A)$ is a polydisk. By the proof of Lemma 1.7, there exist a finite separable extension $K'$ of $K$ of ramification index $e'$, an embedding $(\mathbf{B} \otimes^{\log}_{O_K} O_{K'} \to B')$ in $\mathcal{E}mb_{O_{K'}}$ isomorphic to $(O_{K'}[[T_1, \ldots, T_n]]^N \to O_{K'}^N)$ for some $N > 0$, a positive rational number $\epsilon < j$ and an open immersion $X^j_{\log}(\mathbf{B} \to B)_{K'} \to X^{e'\epsilon}((\mathbf{B} \otimes^{\log}_{O_K} O_{K'} \to B')^\circ)$ as a rational subdomain. The affinoid variety $X^{e'\epsilon}((\mathbf{B} \otimes^{\log}_{O_K} O_{K'} \to B')^\circ)$ is the disjoint union of finitely many copies of polydisks. Enlarging $K'$ if necessary, we may assume that $e'j$ and $e'\epsilon$ are integers. We may further assume that there is a rational number $j < j' < j + \epsilon$ such that $e'j'$ is an integer, that the stable normalized integral models $\mathcal{B}^{j'}_{O_{K'}}$ and $\mathcal{B}'^{e'\epsilon}_{O_{K'}}$ of $X^{j'}_{\log}(\mathbf{B} \to B)_{K'}$ and of $X^{e'\epsilon}((\mathbf{B} \otimes^{\log}_{O_K} O_{K'} \to B')^\circ)_{K'}$ are defined over $K'$ and $X^{j'}_{\log}(\mathbf{B} \to B)_{K'}$ is isomorphic to the disjoint union of copies of $X^{j'}_{\log}(\mathbf{A} \to A)_{K'}$. Since $e'j'$ is an integer, the stable normalized integral model $\mathcal{A}^{j'}_{O_{K'}}$ of $X^{j'}_{\log}(\mathbf{A} \to A)$ is also

defined over $K'$. We have a commutative diagram

$$\begin{array}{ccccc}
\mathbf{A} & & \longrightarrow & & \mathcal{A}^{j'}_{O_{K'}} \\
\big\downarrow & & & & \big\downarrow \\
\mathbf{B} & \longrightarrow & \mathcal{B}'^{e'\epsilon}_{O_{K'}} & \longrightarrow & \mathcal{B}^{j'}_{O_{K'}}.
\end{array}$$

We consider the modules
$\hat{\Omega}_{\mathbf{A}/O_K}(\log/\log) = \varprojlim_n \Omega_{(\mathbf{A}/\mathfrak{m}^n_{\mathbf{A}})/O_K)}(\log/\log), \hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}} = \varprojlim_n \Omega_{(\mathcal{A}^{j'}_{O_{K'}}/\pi'^n\mathcal{A}^{j'}_{O_{K'}})/O_{K'}}$ etc. Since $\mathbf{A}$ is strict over $O_K$ and $\mathbf{B} \otimes^{\log}_{O_K} O_{K'}$ is strict over $O_{K'}$, the canonical maps $\hat{\Omega}_{\mathbf{A}/O_K}(\log/\log) \to \hat{\Omega}_{\mathbf{A}/O_K}$ and $(\mathbf{B} \otimes^{\log}_{O_K} O_{K'}) \otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K}(\log/\log) \to \hat{\Omega}_{(\mathbf{B}\otimes^{\log}_{O_K} O_{K'})/O_{K'}}$ are isomorphisms. Thus, as in the proof of Proposition 1.15, we have a commutative diagram

$$\begin{array}{ccccc}
\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K}(\log/\log) & & \longrightarrow & & \mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathcal{A}^{j'}_{O_{K'}}} \hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}} \\
\big\downarrow & & & & \big\downarrow \\
\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K}(\log/\log) & \longrightarrow & \mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathcal{B}'^{e'\epsilon}_{O_{K'}}} \hat{\Omega}_{\mathcal{B}'^{e'\epsilon}_{O_{K'}}/O_{K'}} & \longrightarrow & \hat{\Omega}_{\mathcal{B}^{j'}_{O_{K'}}/O_{K'}}.
\end{array}$$

We show that the modules are free $\mathcal{B}^{j'}_{O_{K'}}$-modules of rank $n$, the maps are injective and that we have an inclusion $\pi'^{e'j'}\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K}(\log/\log) \subset \pi'^{e'\epsilon}\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K}(\log/\log)$ as submodules of $\hat{\Omega}_{\mathcal{B}^{j'}_{O_{K'}}/O_{K'}}$. By the assumption on the covering $X^{j'}_{\log}(\mathbf{B} \to B)_{K'} \to X^{j'}_{\log}(\mathbf{A} \to A)_{K'}$, the $\mathcal{A}^{j'}_{O_{K'}}$-algebra $\mathcal{B}^{j'}_{O_{K'}}$ is isomorphic to the product of finitely many copies of $\mathcal{A}^{j'}_{O_{K'}}$. Hence the right vertical map $\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathcal{A}^{j'}_{O_{K'}}} \hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}} \to \hat{\Omega}_{\mathcal{B}^{j'}_{O_{K'}}/O_{K'}}$ is an isomorphism. Similarly as in the proof of Proposition 1.15.1, by the canonical map $\mathcal{A}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K}(\log/\log) \to \hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}}$, the module $\mathcal{A}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K}(\log/\log)$ is identified with the submodules $\pi'^{e'j'}\hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}}$ of the free module $\hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}}$. Also by $\mathcal{B}'^{e'\epsilon}_{O_{K'}} \otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K}(\log/\log) \to \hat{\Omega}_{\mathcal{B}'^{e'\epsilon}_{O_{K'}}/O_{K'}}$, the module $\mathcal{B}'^{e'\epsilon}_{O_{K'}} \otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K}(\log/\log)$ is identified with the submodule $\pi'^{e'\epsilon}\hat{\Omega}_{\mathcal{B}'^{e'\epsilon}_{O_{K'}}/O_{K'}}$ of the free module $\hat{\Omega}_{\mathcal{B}'^{e'\epsilon}_{O_{K'}}/O_{K'}}$. Hence we obtain an inclusion $\pi'^{e'j'}\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K}(\log/\log) \subset \pi'^{e'\epsilon}\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K}(\log/\log)$ as submodules of $\hat{\Omega}_{\mathcal{B}^{j'}_{O_{K'}}/O_{K'}}$.

Thus the $\mathcal{B}^{j'}_{O_{K'}}$-module $\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{B}} \Omega_{\mathbf{B}/\mathbf{A}}(\log/\log) = \mathrm{Coker}(\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K}(\log/\log) \to \mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K}(\log/\log))$ is annihilated by $\pi'^{e'(j'-\epsilon)}$.

Since $0 < j - \epsilon < j' - \epsilon < j$, applying Lemma 4.13 (2)$\Rightarrow$(1), the assertion is proved.

2. Let $K'$ be a finite separable extension such that $e'j$ is an integer, that $B \otimes^{\log}_{O_K} O_{K'}$ is strict over $O_{K'}$ and that the stable normalized integral models $\mathcal{A}^j_{O_{K'}}$ and $\mathcal{B}^j_{O_{K'}}$ are defined over $K'$. By Lemma 4.13 (1)$\Rightarrow$(2), the $\mathcal{B}^j_{O_{K'}}$-module $\mathcal{B}^j_{O_{K'}} \otimes_{\mathbf{B}} \Omega_{\mathbf{B}/\mathbf{A}}(\log/\log)$ is annihilated by $\pi'^{n'}$ for an integer $n' < e'j$. The rest of proof is the same as that of Proposition 1.15.2. $\square$

*Proof of Corollary 4.12.* The same as that of Corollary 1.16. $\square$

## 5 Filtration by ramification groups: the logarithmic case

### 5.1 Construction

In this subsection, we rephrase the definition of the logarithmic filtration by ramification groups given in the previous paper [1] by using the preceeding constructions.

Let $\Phi$ : (Finite Étale/$K$) $\to$ $G_K$-(Finite Sets) be the fiber functor as in Section 2.1. For a rational number $j > 0$, we define a functor $\Phi^j_{\log}$ : (Finite Étale/$K$) $\to$ $G_K$-(Finite Sets) as the composition of the functor (Finite Étale/$K$) $\to$ (Finite Flat and log Flat/$O_K$) sending a finite étale $K$-algebra $L$ to the integral closure $O_L$ of $O_K$ in $L$ with the standard log structure and the functor $\Psi^j_{\log}$ : (Finite Flat and log Flat/$O_K$) $\to$ $G_K$-(Finite Sets) defined in Section 4.2. The map (4.2.3) defines a surjection $\Phi \to \Phi^j_{\log}$ of functors. In [1], we define the logarithmic filtration by ramification groups on $G_K$ by using the family of surjections $(\Phi \to \Phi^j_{\log})_{j>0,\in\mathbb{Q}}$ of functors. The filtration by the log ramification groups $G^j_{K,\log} \subset G_K, j > 0, \in \mathbb{Q}$ is characterized by the condition that the canonical map $\Phi(L) \to \Phi^j_{\log}(L)$ induces a bijection $\Phi(L)/G^j_{K,\log} \to \Phi^j_{\log}(L)$ for each finite étale algebra $L$ over $K$.

The functor $\Phi^j_{\log}$ is defined by the commutativity of the diagram

$$\begin{tikzcd}
(\text{Finite Étale}/K) \arrow[r, "\Phi^j_{\log}"] \arrow[d] & G_K\text{-(Finite Sets)} \\
(\text{Finite Flat and Log Flat}/O_K) \arrow[ur, "\Psi^j_{\log}"'] & G_K\text{-}(\mathrm{Aff}/\bar{F}) \arrow[u, "\pi_0"'] \\
\mathcal{E}mb^{\log}_{O_K} \arrow[u] \arrow[d, "(\ \otimes^{\log} O_{K'})_{K'}"'] & \varinjlim_{K'/K} (\mathrm{Aff}/F') \arrow[u] \\
\varinjlim_{K'/K} \mathcal{E}mb_{O_{K'}} \arrow[r, "(X^{e_{K'/K}j})_{K'}"'] & \varinjlim_{K'/K} (\text{smooth Affinoid}/K') \arrow[u, "(X_{K'})_{K'} \mapsto (\bar{X}_{F'})_{K'}"']
\end{tikzcd}$$

We briefly recall how the other arrows in the diagram are defined. The forgetful functor $\mathcal{E}mb^{\log}_{O_K} \to$ (Finite Flat and Log Flat$/O_K$) sends $(\mathbf{A} \to A)$ to $A$. The functor $\mathcal{E}mb^{\log}_{O_K} \to \varinjlim_{K'/K} \mathcal{E}mb_{O_{K'}}$ sends a log embedding to the system of strict base changes. The functor $\varinjlim_{K'/K} \mathcal{E}mb_{O_K} \to \varinjlim_{K'/K}$(smooth Affinoid$/K'$) is defined by the system of tubular neighborhoods. The functor $\varinjlim_{K'/K}$(smooth Affinoid$/K'$) $\to \varinjlim_{K'/K}(\mathrm{Aff}/F')$ is defined by the closed fiber of the stable normalized integral models. The functor $\varinjlim_{K'/K}(\mathrm{Aff}/F') \to G_K$-$(\mathrm{Aff}/\bar{F})$ is the equivalence of category defined in Section 1.3. The functor $\pi_0$ is defined by the set of connecteds components. They induce a functor $\Psi^j_{\log}$ : (Finite Flat and log flat$/O_K$) $\to G_K$-(Finite Sets). The functor $\Phi^j_{\log}$ is defined as the composition of $\Psi^j_{\log}$ with the functor sending a finite étale algebra $L$ to the integral closure $O_L$ in $L$ of $O_K$ with the canonical log structure. More concretely, we have

$$\Phi^j_{\log}(L) = \varprojlim_{(\mathbf{A}\to O_L)\in \mathcal{E}mb^{\log}_{O_K}(O_L)} \pi_0(\varinjlim_{K'/K} \bar{X}^{e_{K'/K}j}((\mathbf{A} \otimes^{\log}_{O_K} O_{K'} \to O_L \otimes^{\log}_{O_K} O_{K'})^\circ))$$

for a finite étale $K$-algebra $L$. This definition agrees with that given in [1] by Lemma 4.10.

For a rational number $j \geq 0$, we define a functor $\Phi^{j+}_{\log}$ : (Finite Étale$/K$) $\to\to G_K$-(Finite Sets) by $\Phi^{j+}_{\log}(L) = \varinjlim_{j'>j} \Phi^{j'}_{\log}(L)$ for a finite étale $K$-algebra $L$. We define a closed normal subgroup $G^{j+}_{K,\log}$ to be $\overline{\cup_{j'>j} G^{j'}_K}$. Then we have $\Phi^{j+}_{\log}(L) = \Phi(L)/G^{j+}_{K,\log}$. Similarly as Lemma 2.1, the finite set $\Phi^{j+}_{\log}(L)$ has the following geometric description.

Lemma 5.1 *Let $B$ be the integer ring with the standard log structure of a finite étale algebra $L$ over $K$ and $j > 0$ be a rational number. Let $(f, \mathbf{f}) : (\mathbf{A} \to O_K) \to (\mathbf{B} \to B)$ be a finite flat and log flat morphism of embeddings. Let $f^j : X^j_{\log}(\mathbf{B} \to B) \to X^j_{\log}(\mathbf{A} \to O_K)$ and $\bar{f}^j : \bar{X}^j_{\log}(\mathbf{B} \to B) \to \bar{X}^j_{\log}(\mathbf{A} \to O_K)$ be the canonical maps. Let $0 \in X^j_{\log}(\mathbf{A} \to O_K)$ be the point corresponding to the map $\mathbf{A} \to O_K$ and $\bar{0} \in \bar{X}^j_{\log}(\mathbf{A} \to O_K)$ be its specialization. Then the maps* (1.8.0), (1.12.1) *and the specialization map form a commutative diagram*

$$(5.1.1) \qquad \begin{array}{ccccc} \Phi(L) & \longrightarrow & \Phi^{j+}_{\log}(L) & \longrightarrow & \Phi^j_{\log}(L) \\ \downarrow & & \downarrow & & \downarrow \\ (f^j)^{-1}(0) & \longrightarrow & (\bar{f}^j)^{-1}(\bar{0}) & \longrightarrow & \pi_0(\bar{X}^j_{\log}(\mathbf{B} \to B)) \end{array}$$

*and the vertical arrows are bijections.*

For a finite étale algebra $L$ over $K$ and a rational number $j > 0$, we say that the log ramification of $L$ is bounded by $j$ if the canonical map $\Phi(L) \to \Phi^j_{\log}(L)$

is a bijection. Let $A = O_K$ and let $B = O_L$ be the integer ring of a finite étale $K$-algebra $L$ and $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ be a finite flat and log flat morphism of log embeddings. Then, since the map $X^j_{\log}(\mathbf{B} \to B) \to X^j_{\log}(\mathbf{A} \to A)$ is finite flat of degree $[L : K]$, the ramification of $L$ is bounded by $j$ if and only if there exists a finite separable extension $K'$ of $K$ such that the affinoid variety $X^j_{\log}(\mathbf{B} \to B)_{K'}$ is isomorphic to the disjoint union of finitely many copies of $X^j_{\log}(\mathbf{A} \to A)_{K'}$ over $X^j_{\log}(\mathbf{A} \to A)_{K'}$. We say that the log ramification of $L$ is bounded by $j+$ if the log ramification of $L$ is bounded by every rational number $j' > j$. The log ramification of $L$ is bounded by $j+$ if and only if the canonical map $\Phi(L) \to \Phi^{j+}_{\log}(L)$ is a bijection.

Lemma 5.2 *Let $K \to K'$ be a map of complete discrete valuation fields inducing a local homomorphism $O_K \to O_{K'}$ of integer rings. Assume that the ramification index $e = e_{K'/K}$ is prime to $p$ and that the residue field $F'$ of $K'$ is a separable extension of the residue field $F$ of $K$. Then, for a rational number $j > 0$, the map $G_{K'} \to G_K$ induces a surjection $G^{ej}_{\log,K'} \to G^j_{\log,K}$.*

*Proof.* Let $A$ be the integer ring of a finite étale $K$-algebra $L$ and $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb_{O_K}$. By the assumption, the log tensor product $A \otimes^{\log}_{O_K} O_{K'}$ is the integer ring of $L \otimes_K K'$. The rest is the same as the proof of Lemma 2.2.
□

The two filtrations by ramification groups are related as follows.

Lemma 5.3 *Let $K$ be a complete discrete valuation field and $j > 0$ be a rational number. Then, we have inclusions $G^j_K \supset G^j_{K,\log} \supset G^{j+1}_K$.*

*Proof.* By Corollary 4.9, there are natural morphisms $\Phi^{j+1} \to \Phi^j_{\log} \to \Phi^j$ of functors. Hence the assertion follows. □

## 5.2 Functoriality of the closed fibers of log tubular neighborhoods

For a positive rational number $j > 0$, let $(\text{Finite Étale}/K)^{\le j+}_{\log}$ denote the full subcategory of (Finite Étale/$K$) consisting of étale $K$-algebras whose log ramification is bounded by $j+$. At the end of the section, we prove Theorem 5.12. As in the proof of Theorem 2.15, we reduce it to the case where the condition

(F) There exists a perfect subfield $F_0$ of $F$ such that $F$ is finitely generated over $F_0$.

is satisfied. Assuming the condition (F), we define a twisted tangent space $\Theta^j_{\log}$ and show that the functor $\bar{X}^j_{\log} : \mathcal{E}mb^{\log}_{O_K} \to G_K\text{-}(\text{Aff}/\bar{F})$ induces a functor

$$\bar{X}^j_{\log} : (\text{Finite Étale}/K)^{\le j+}_{\log} \to G_K\text{-}(\text{Finite Étale}/\Theta^j_{\log}).$$

In this subsection, $L$ denotes a finite étale $K$-algebra and $A = O_L$ denotes the integer ring with the canonical log structure.
We assume that the condition (F) is satisfied. Let $K_0$ be a subfield of $K$ such that $O_{K_0} = O_K \cap K_0$ is a complete discrete valuation ring with perfect residue field $F_0$ and $F$ is finitely generated over $F_0$ as in Section 2.3. Let $\pi_0$ denote a prime element of $O_{K_0}$. We consider $O_{K_0}$ as a log ring with the *trivial* log structure. We introduce a new category $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ and a functor $\mathcal{E}mb^{\log}_{K,O_{K_0}} \to \mathcal{E}mb^{\log}_{O_K}$ similarly as in Section 2.3.

DEFINITION 5.4 *Let $K$ be a complete discrete valuation field and $K_0$ be a subfield of $K$ such that $O_{K_0} = O_K \cap K_0$ is a complete discrete valuation ring with perfect residue field $F_0$ and that $F$ is finitely generated over $F_0$. We put $m = \mathrm{tr.deg}(F/F_0)$. We consider $O_{K_0}$ as a log ring with the* trivial *log structure.*
*1. We define $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ to be the category whose objects and morphisms are as follows. An object of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ is a triple $(\mathbf{A}_0 \to A)$ where:*

- *$A$ is the integer ring of a finite étale $K$-algebra with the canonical log structure.*
- *$\mathbf{A}_0$ is a complete semi-local Noetherian log $O_{K_0}$-algebras formally smooth and formally log smooth of relative dimension $m+1 = \mathrm{tr.deg}(F/F_0)+1$ over $O_{K_0}$.*
- *$\mathbf{A}_0 \to A$ is an exact and regular surjection of codimension 1 of log $O_{K_0}$-algebras and induces an isomorphism $\mathbf{A}_0/\mathfrak{m}_{\mathbf{A}_0} \to A/\mathfrak{m}_A$ of underlying $F_0$-algebras.*

*A morphism $(f, \mathbf{f}) : (\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ is a pair of a log $O_K$-homomorphism $f : A \to B$ and a log $O_{K_0}$-homomorphism $\mathbf{f} : \mathbf{A}_0 \to \mathbf{B}_0$ such that the diagram*

$$\begin{array}{ccc} \mathbf{A}_0 & \longrightarrow & A \\ {\scriptstyle \mathbf{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{B}_0 & \longrightarrow & B \end{array}$$

*is commutative.*
*2. For the integer ring $A$ of a finite étale $K$-algebra, we define $\mathcal{E}mb^{\log}_{K,O_{K_0}}(A)$ to be the subcategory of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ whose objects are of the form $(\mathbf{A}_0 \to A)$ and morphisms are of the form $(\mathrm{id}_A, \mathbf{f})$.*
*3. We say that a morphism $(\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ is finite flat and log flat if $\mathbf{A}_0 \to \mathbf{B}_0$ is finite flat and log flat and the canonical map $\mathbf{B}_0 \otimes^{\log}_{\mathbf{A}_0} A \to B$ is an isomorphism.*

An object $(\mathbf{A}_0 \to A)$ of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ is an object $(\mathbf{A}_0 \to A)$ of $\mathcal{E}mb_{K,O_{K_0}}$ together with a log strucure on $\mathbf{A}_0$ such that the log ring $\mathbf{A}_0$ is formally log smooth over $O_{K_0}$ and that the surjection $\mathbf{A}_0 \to A$ is exact.

Lemma 5.5 *1. Let $A$ be the integer ring of a finite étale $K$-algebra with the canonical log structure. Then, the category $\mathcal{E}mb^{\log}_{K,O_{K_0}}(A)$ is non-empty.*

*2. Let $(\mathbf{A}_0 \to A)$ and $(\mathbf{B}_0 \to B)$ be objects of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ and $A \to B$ be an $O_K$-homomorphism. Then there exists a homomorphism $(\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ in $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ extending $A \to B$.*

*3. Every morphism in $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ is finite flat and log flat.*

*Proof.* 1. We may assume $A$ is local. Take a transcendental basis $(\bar{t}_1, \ldots, \bar{t}_m)$ of the residue field $E$ of $A$ over $F_0$ such that $E$ is a finite separable extension of $F_0(\bar{t}_1, \ldots, \bar{t}_m)$. Take a lifting $(t_1, \ldots, t_m)$ in $A$ of $(\bar{t}_1, \ldots, \bar{t}_m)$ and prime elements $t_0$ of $A$ and $\pi_0$ of $O_{K_0}$. Then $A$ is unramified over the completion of the local ring of $O_{K_0}[T_0, \ldots, T_m]$ at the prime ideal $(\pi_0, T_0)$ by the map defined by sending $T_i$ to $t_i$. Hence there are an étale scheme $X$ over $\mathbb{A}^{m+1}_{O_{K_0}}$, a point $\xi$ of $X$ above $(\pi_0, T_0)$ and a regular immersion $\varphi : \hat{O}_{X,\xi} \to A$ of codimension 1. Let $\mathbf{A}_0$ be the $O_{K_0}$-algebra $\hat{O}_{X,\xi}$ with the log structure defined by the chart $\mathbb{N} \to \mathbf{A}_0$ sending $1 \in \mathbb{N}$ to $T_0$. Then $(\mathbf{A}_0 \to A)$ is an object of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$.
2. Since $\mathbf{A}_0$ is formally log smooth over $O_{K_0}$, it follows from that $\mathbf{B}_0$ is the formal completion of itself with respect to the surjection $\mathbf{B}_0 \to B$.
3. We may assume $A$ and $B$ are local. We show that the map $\mathbf{B}_0 \otimes^{\log}_{\mathbf{A}_0} A \to B$ is an isomorphism. Let $f$ be a generator of the kernel of $\mathbf{A}_0 \to A$. It is sufficient to show that the image of $f$ in $\mathfrak{m}_{\mathbf{B}_0}/\mathfrak{m}^2_{\mathbf{B}_0}$ is not 0. We take charts $\mathbb{N} \to \mathbf{A}_0$ and $\mathbb{N} \to \mathbf{B}_0$ and let $t_0 \in \mathbf{A}_0$ and $t'_0 \in \mathbf{B}_0$ be the images of $1 \in \mathbb{N}$. The charts $\mathbb{N} \to \mathbf{A}_0$ and $\mathbb{N} \to \mathbf{B}_0$ induces isomorphisms $\mathbb{N} \to M_{\mathbf{Y},y}$ and $\mathbb{N} \to M_{\mathbf{X},x}$ where $y$ and $x$ are the closed points of the log schemes $\mathbf{Y} = \mathrm{Spec}\,\mathbf{A}_0$ and $\mathbf{X} = \mathrm{Spec}\,\mathbf{X}_0$. The map $\mathbb{N} = M_{\mathbf{Y},y} \to \mathbb{N} = M_{\mathbf{X},x}$ is the multiplication by the ramification index $e$ of $B \otimes_{O_K} K$ over $A \otimes_{O_K} K$.
Since $dt_0$ is in the kernel of the surjection $\hat{\Omega}_{\mathbf{A}_0/O_{K_0}} \otimes_{\mathbf{A}_0} A/\mathfrak{m}_A \to \Omega_{(A/\mathfrak{m}_A)/F_0}$ and is non-zero, $(\pi_0, t_0)$ is a basis of $\mathfrak{m}_{\mathbf{A}_0}/\mathfrak{m}^2_{\mathbf{A}_0}$. We put $f = a\pi_0 + bt_0$ in $\mathfrak{m}_{\mathbf{A}_0}/\mathfrak{m}^2_{\mathbf{A}_0}$ for some element $a, b$ in the residue field $E$ of $A$. Since the surjection $\mathbf{A}_0 \to A$ is regular of codimension 1, either of $a$ and $b$ is not 0. Since the image of $t_0$ is a basis of $\mathfrak{m}_A/\mathfrak{m}^2_A$ and the image of $f$ is 0, we have $a \neq 0$. Similarly $(\pi_0, t'_0)$ is a basis of $\mathfrak{m}_{\mathbf{B}_0}/\mathfrak{m}^2_{\mathbf{B}_0}$. Since the map $\mathbb{N} = M_{\mathbf{Y},y} \to \mathbb{N} = M_{\mathbf{X},x}$ is the multiplication by the ramification index $e$, the image of $t_0$ is a unit times $t'^e_0$. Hence the image of $f$ in $\mathfrak{m}_{\mathbf{B}_0}/\mathfrak{m}^2_{\mathbf{B}_0}$ is not zero. Thus the map $\mathbf{B}_0 \otimes^{\log}_{\mathbf{A}_0} A \to B$ is an isomorphism. Since $B$ is finite over $A$, $\mathbf{B}_0$ is also finite over $\mathbf{A}_0$ by Nakayama's lemma. Since $\dim \mathbf{A}_0 = \dim \mathbf{B}_0 = 2$ the assertion follows by Corollary 3.11. □

Corollary 5.6 *Every morphism in $\mathcal{E}mb^{\log}_{K,O_{K_0}}(A)$ is an isomorphism.*

*Proof.* If $(\mathbf{A}_0 \to A) \to (\mathbf{A}'_0 \to A)$ is a map, the map $\mathbf{A}_0 \to \mathbf{A}'_0$ is finite flat of degree 1 and is an isomorphism. □

We define a functor $\mathcal{E}mb^{\log}_{K,O_{K_0}} \to \mathcal{E}mb^{\log}_{O_K}$ as follows. Let $(\mathbf{A}_0 \to A)$ be an object of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$. We define an embedding $((\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge \to A)$ by regarding $(\mathbf{A}_0 \to A)$ as an object of $\mathcal{E}mb_{K,O_{K_0}}$. Since the underlying ring of $\mathbf{A}/\mathfrak{m}_{\mathbf{A}}^n \otimes_{O_{K_0}}^{\log} O_K/\mathfrak{m}_K^n$ is $\mathbf{A}/\mathfrak{m}_{\mathbf{A}}^n \otimes_{O_{K_0}} O_K/\mathfrak{m}_K^n$, we define a log structure on $(\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge$ as the limit of those on $\mathbf{A}/\mathfrak{m}_{\mathbf{A}}^n \otimes_{O_{K_0}}^{\log} O_K/\mathfrak{m}_K^n$. We let $(\mathbf{A}_0 \hat{\otimes}_{O_{K_0}}^{\log} O_K)^\wedge$ denote the log ring $(\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge$ with this log structure.

LEMMA 5.7 *Let $(\mathbf{A}_0 \to A)$ be an object of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$. Then, $((\mathbf{A}_0 \hat{\otimes}_{O_{K_0}}^{\log} O_K)^\wedge \to A)$ is a log pre-embedding and hence $((\mathbf{A}_0 \hat{\otimes}_{O_{K_0}}^{\log} O_K)^{\wedge\sim} \to A)$ is a log embedding.*

*Proof.* By the construction, the log $O_K$-algebra $(\mathbf{A}_0 \hat{\otimes}_{O_{K_0}}^{\log} O_K)^\wedge$ is formally log smooth and $((\mathbf{A}_0 \hat{\otimes}_{O_{K_0}}^{\log} O_K)^\wedge \to A)$ is a log pre-embedding. The rest follows from Lemma 4.4.2. □

In the following, we put $\mathbf{A} = (\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^{\wedge\sim}$. We obtain a functor $\mathcal{E}mb^{\log}_{K,O_{K_0}} \to \mathcal{E}mb^{\log}_{O_K}$ sending $(\mathbf{A}_0 \to A)$ to $(\mathbf{A} \to A) = ((\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^{\wedge\sim} \to A)$ by Lemma 5.7. For a rational number $j > 0$, we have a sequence of functors

$$\mathcal{E}mb^{\log}_{K,O_{K_0}} \longrightarrow \mathcal{E}mb^{\log}_{O_K} \xrightarrow{X^j_{\log}} \varinjlim_{K'/K}(\text{smooth Affinoid}/K') \longrightarrow G_K\text{-}(\mathrm{Aff}/\bar{F}).$$

We also let $\bar{X}^j_{\log}$ denote the composite functor $\mathcal{E}mb^{\log}_{K,O_{K_0}} \to G_K\text{-}(\mathrm{Aff}/\bar{F})$. Thus, for an object $(\mathbf{A}_0 \to A)$ of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$, we have $\bar{X}^j_{\log}(\mathbf{A}_0 \to A) = \bar{X}^j_{\log}((\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^{\wedge\sim} \to A)$.

For a rational number $j > 0$, the composition

$$\mathcal{E}mb^{\log}_{K,O_{K_0}} \longrightarrow \mathcal{E}mb^{\log}_{O_K} \xrightarrow{\bar{C}^j_{\log}} G_K\text{-}(\mathrm{Aff}/\bar{F}).$$

defines a functor $\bar{C}^j_{\log} : \mathcal{E}mb^{\log}_{K,O_{K_0}} \to G_K\text{-}(\mathrm{Aff}/\bar{F})$. We compute the twisted normal cone $\bar{C}^j_{\log}(\mathbf{A} \to A)$ for an object $(\mathbf{A}_0 \to A)$ of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ and $\mathbf{A} = (\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^{\wedge\sim}$. It is a scheme over $(A_{\log \bar{F}})_{\mathrm{red}} = (A \otimes^{\log}_{O_K} \bar{F})_{\mathrm{red}}$. Let $N_{A/\mathbf{A}} = I/I^2$ be the conormal module where $I$ is the kernel of the surjection $\mathbf{A} \to A$. We put $\hat{\Omega}_{O_K/O_{K_0}}(\log) = \varprojlim_n \Omega_{(O_K/\mathfrak{m}_K^n)/O_{K_0}}(\log)$ with respect to the canonical log structure on $O_K$ and the trivial log structure on $O_{K_0}$. Similarly, we put $\hat{\Omega}_{\mathbf{A}/\mathbf{A}_0}(\log/\log) = \varprojlim_n \Omega_{(\mathbf{A}/\mathfrak{m}_{\mathbf{A}}^n)/\mathbf{A}_0}(\log/\log)$. Since the map $\mathbf{A} \to \mathbf{A}_0$ is strict, we have $\hat{\Omega}_{\mathbf{A}/\mathbf{A}_0}(\log/\log) = \hat{\Omega}_{\mathbf{A}/\mathbf{A}_0}$. Let $\Omega_F(\log)$ be the $F$-vector space $\Omega_{F/F_0}(\log)$ with respect to the trivial log structure on $F_0$ and the log structure on $F$ defined by the chart $\mathbb{N} \to F$ sending $1 \in \mathbb{N}$ to 0. The canonical map $\hat{\Omega}_{O_K/O_{K_0}}(\log) \otimes_{O_K} F \to \Omega_F(\log)$ is an isomorphism. We have an exact sequence $0 \to \Omega_{F/F_0} \to \Omega_{F/F_0}(\log) \xrightarrow{\mathrm{res}} F \to 0$. We have canonical maps $N_{A/\mathbf{A}} \to \hat{\Omega}_{\mathbf{A}/\mathbf{A}_0} \otimes_{\mathbf{A}} A$ and $\hat{\Omega}_{O_K/O_{K_0}}(\log) \otimes_{O_K} A \to \hat{\Omega}_{\mathbf{A}/\mathbf{A}_0}(\log/\log) \otimes_{\mathbf{A}} A$. Similarly as Lemma 2.11, we have the following.

Lemma 5.8 *Let* $(\mathbf{A}_0 \to A)$ *be an object of* $\mathcal{E}mb^{\log}_{K,O_{K_0}}$.
*1. If* $m$ *is the transcendental dimension of* $F$ *over* $F_0$*, the dimension of the* $F$*-vector space* $\Omega_F(\log)$ *is* $m+1$.
*2. The map* $N_{A/\mathbf{A}} \to \Omega_{\mathbf{A}/\mathbf{A}_0} \otimes_{\mathbf{A}} A$ *is a surjection and the map* $\Omega_{O_K/O_{K_0}}(\log) \otimes_{O_K} A \to \Omega_{\mathbf{A}/\mathbf{A}_0} \otimes_{\mathbf{A}} A$ *is an isomorphism. They induce an isomorphism* $N_{A/\mathbf{A}} \otimes_A A/\mathfrak{m}_A \to \Omega_F(\log) \otimes_F A/\mathfrak{m}_A$.
*3. Let* $(\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ *be a morphism of* $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ *and put* $\mathbf{B} = (\mathbf{B}_0 \hat{\otimes}_{O_{K_0}} O_K)^{\wedge\sim}$. *Then, the diagram*

$$\begin{array}{ccc} N_{A/\mathbf{A}} \otimes_A A/\mathfrak{m}_A & \longrightarrow & \Omega_F(\log) \otimes_F A/\mathfrak{m}_A \\ \downarrow & & \downarrow \\ N_{B/\mathbf{B}} \otimes_B B/\mathfrak{m}_B & \longrightarrow & \Omega_F(\log) \otimes_F B/\mathfrak{m}_B \end{array}$$

*is commutative.*

For a rational number $j > 0$, let $\Theta^j_{\log}$ be the $\bar{F}$-vector space $Hom_F(\Omega_F(\log), N^j)$ regarded as an affine scheme over $\bar{F}$. Similarly as Corollary 2.12, we have the following.

Corollary 5.9 *Let* $(\mathbf{A}_0 \to A)$ *be an object of* $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ *and let* $(\mathbf{A} \to A)$ *be its image in* $\mathcal{E}mb^{\log}_{O_{K_0}}$. *Let* $j > 0$ *be a rational number.*
*1. Let* $\bar{C}^j_{\log}(\mathbf{A} \to A)$ *be the twisted normal cone. The isomorphism in Lemma 5.8.2 induces an isomorphism* $\bar{C}^j_{\log}(\mathbf{A} \to A) \to \Theta^j_{\log} \otimes_{\bar{F}} (A_{\log \bar{F}})_{\rm red}$.
*2. Let* $(\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ *be a morphism of* $\mathcal{E}mb^{\log}_{K,O_{K_0}}$. *Then the diagram*

$$\begin{array}{ccccc} \bar{X}^j_{\log}(\mathbf{B} \to B) & \longrightarrow & \bar{C}^j_{\log}(\mathbf{B} \to B) & \longrightarrow & \Theta^j_{\log} \otimes_{\bar{F}} (B_{\log \bar{F}})_{\rm red} \\ \downarrow & & \downarrow & & \downarrow \\ \bar{X}^j_{\log}(\mathbf{A} \to A) & \longrightarrow & \bar{C}^j_{\log}(\mathbf{A} \to A) & \longrightarrow & \Theta^j_{\log} \otimes_{\bar{F}} (A_{\log \bar{F}})_{\rm red} \end{array}$$

*is commutative.*
*3. If the ramification of* $A \otimes_{O_K} K$ *is bounded by* $j+$*, then the composition* $\bar{X}^j_{\log}(\mathbf{A} \to A) \to \bar{C}^j_{\log}(\mathbf{A} \to A) \to \Theta^j_{\log}$ *is finite and étale.*

For a rational number $j > 0$, we regard $\Theta^j_{\log}$ as an object of $G_K$-(Aff$/\bar{F}$) with the natural $G_K$-action. Let $G_K$-(Finite Étale$/\Theta^j_{\log}$) denote the subcategory of $G_K$-(Aff$/\bar{F}$) whose objects are finite étale schemes over $\Theta^j_{\log}$ and morphisms are over $\Theta^j_{\log}$. Let $\mathcal{E}mb^{\log,\le j+}_{K,O_{K_0}}$ denote the full subcategory of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ consisting of the objects $(\mathbf{A}_0 \to A)$ such that the log ramifications of $A \otimes_{O_K} K$ are bounded by $j+$. By Corollary 5.9, the functor $\bar{X}^j_{\log} : \mathcal{E}mb^{\log}_{K,O_{K_0}} \to G_K$-(Aff$/\bar{F}$) induces a functor $\bar{X}^j_{\log} : \mathcal{E}mb^{\log,\le j+}_{K,O_{K_0}} \to G_K$-(Finite Étale$/\Theta^j_{\log}$).

The functor $\bar{X}^j_{\log} : \mathcal{E}mb^{\log,\le j+}_{K,O_{K_0}} \to G_K$-(Finite Étale/$\Theta^j_{\log}$) further induces a functor $\bar{X}^j_{\log}$ : (Finite Étale/$K)^{\le j+}_{\log} \to G_K$-(Finite Étale/$T^j_{\log}$). In fact, similarly as Lemma 2.13 and Corollary 2.14, we have the following.

Lemma 5.10 *Let $f : A \to B$ be a map over $O_K$ and let $(f,\mathbf{f}),(g,\mathbf{g}) : (\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ be maps in $\mathcal{E}mb^{\log}_{K,O_{K_0}}$. If $f = g$, then the induced maps*

$$(f,\mathbf{f})_*,(g,\mathbf{g})_* : \bar{X}^j_{\log}(\mathbf{A}_0 \to A) \longrightarrow \bar{X}^j_{\log}(\mathbf{B}_0 \to B)$$

*are equal.*

Corollary 5.11 *Let $j > 0$ be a rational number.*
*1. Let $L$ be a finite étale $K$-algebra $L$ such that the log ramification is bounded by $j+$. Then the system $\bar{X}^j_{\log}(\mathbf{A}_0 \to O_L)$ parametrized by the objects $(\mathbf{A}_0 \to O_L)$ of $\mathcal{E}mb^{\log}_{K,O_{K_0}}(O_L)$ is constant and the limit*

$$\bar{X}^j_{\log}(L) = \varprojlim_{(\mathbf{A}_0 \to O_L)\in \mathcal{E}mb^{\log}_{K,O_{K_0}}(O_L)} \bar{X}^j_{\log}(\mathbf{A}_0 \to O_L)$$

*is a finite étale scheme over $\Theta^j_{\log}$.*
*2. The functor $\bar{X}^j_{\log} : \mathcal{E}mb^{\log,\le j+}_{K,O_{K_0}} \to G_K$-(Finite Étale/$\Theta^j_{\log}$) induces a functor*

$$\bar{X}^j_{\log} : (\text{Finite Étale}/K)^{\le j+}_{\log} \to G_K\text{-(Finite Étale}/\Theta^j_{\log}).$$

Using the functor $\bar{X}^j_{\log}$ : (Finite Étale/$K)^{\le j+} \to G_K$-(Finite Étale/$\Theta^j$) defined under the condition (F), we obtain the following theorem by the same argument as the proof of Theorem 2.15.

Theorem 5.12 *Let $K$ be a complete discrete valuation field and let $j > 0$ be a rational number. Let $m$ be the prime-to-$p$ part of the denominator of $j$ and $I_m$ be the subgroup of the inertia group $I \subset G_K$ of index $m$. Then we have the following.*
*1. The graded piece $Gr^j G_K = G^j_{K,\log}/G^{j+}_{K,\log}$ is abelian.*
*2. The commutator $[I_m, G^j_{K,\log}]$ is a subgroup of $G^{j+}_{K,\log}$. In particular, $Gr^j_{\log} G_K$ is a subgroup of the center of the pro-$p$-group $G^{0+}_{K,\log}/G^{j+}_{K,\log}$.*

Similarly as in the proof of Theorem 2.15, assuming the condition (F), we obtain a canonical surjection

$$\pi_1^{\rm ab}(\Theta^j_{\log}) \longrightarrow Gr^j_{\log} G_K. \tag{5.12.1}$$

The canonical surjections $\pi_1^{\rm ab}(\Theta^j_{\log}) \to Gr^j_{\log} G_K$ and $\pi_1^{\rm ab}(\Theta^j) \to Gr^j G_K$ are related as follows. The exact sequences $0 \to N \to \tilde{\Omega}_F \to \Omega_F \to 0$ and $0 \to \Omega_F \to \Omega_F(\log) \to F \to 0$ induces canonical maps $\Theta^j_{\log} \to \Theta^j$ and $\Theta^{j+1} \to \Theta^j_{\log}$.

Lemma 5.13 *Assume that the condition (F) is satisfied and that $p$ is not a prime element of $K$. Then, for a rational number $j > 0$, we have a commutative diagram*

$$\begin{array}{ccccc}
\pi_1^{\rm ab}(\Theta^{j+1}) & \longrightarrow & \pi_1^{\rm ab}(\Theta^j_{\log}) & \longrightarrow & \pi_1^{\rm ab}(\Theta^j) \\
\downarrow & & \downarrow & & \downarrow \\
Gr^{j+1}G_K & \longrightarrow & Gr^j_{\log}G_K & \longrightarrow & Gr^jG_K.
\end{array}$$

*Proof.* We show the commutativity of the left square. Let $L$ be a finite separable extension of $K$ such that the log ramification is bounded by $j+$ and $A$ be the integer ring of $L$. By Lemma 5.3, the ramification of $L$ is bounded by $(j+1)+$. Let $(\mathbf{A}_0 \to A)$ be an object of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$. By Lemma 5.7, the surjection $\mathbf{A} = \mathbf{A}_0 \otimes^{\log}_{O_{K_0}} O_K \to A$ defines a log pre-embedding $(\mathbf{A}_0 \otimes^{\log}_{O_{K_0}} O_K \to A)$. By forgetting the log structure, we obtain an embedding $(\mathbf{A} \to A)^\circ$. By applying Lemma 4.4, we obtain a log embedding $(\mathbf{A}^\sim \to A)$. Then, by Lemma 4.8 we have an open immersion $X^{j+1}((\mathbf{A} \to A)^\circ) \to X^j_{\log}(\mathbf{A}^\sim \to A)$ of affinoid subdomains of $X^j((\mathbf{A} \to A)^\circ)$. It induces a map $\bar{X}^{j+1}(L) \to \bar{X}^j_{\log}(L)$. By the functoriality, we obtain a commutative diagram

$$\begin{array}{ccc}
\bar{X}^{j+1}(L) & \longrightarrow & \bar{X}^j_{\log}(L) \\
\downarrow & & \downarrow \\
\Theta^{j+1} = \bar{X}^{j+1}(K) & \longrightarrow & \Theta^j_{\log} = \bar{X}^j_{\log}(K).
\end{array}$$

From this diagram, we deduce the commutativity of the left square. The proof for the right square is similar and omitted. $\square$

# 6 The perfect residue field case

## 6.1 The Newton polygon of a polynomial

We recall the notion of Newton polygons and establish some properties. We say that a function $l : [0, n] \to \mathbb{R} \cup \{\infty\}$ is convex if for every $0 \le x \le y \le n$, the graph of $l$ is below the line segment connecting $(x, l(x))$ and $(y, l(y))$. If at least one of $l(x)$ and $l(y)$ is $\infty$, we define the line segment connecting $(x, l(x))$ and $(y, l(y))$ to be the union $\{(z, \infty) | x < z < y\} \cup \{(x, l(x)), (y, l(y))\}$. For a polynomial $h(T) = \sum_{i=0}^n b_i T^{n-i} \in \bar{K}[T]$ of degree $\le n$, we define its Newton polygon to be the graph of the maximum convex function $l_h : [0, n] \to \mathbb{R} \cup \{\infty\}$ satisfying $l_h(i) \le \operatorname{ord} b_i$.

If $b_0 = 1$, the Newton polygon of $h$ and the solutions of the equation $h(T) = 0$ are related as follows. Let $z_1, \ldots, z_n$ be the solution of $h(T) = \prod_{i=1}^n (T - z_i) = 0$ and assume $\operatorname{ord} z_i$ is increasing in $i$. Then, since $b_i = (-1)^i \sum_{1 \le k_1 < \ldots < k_i \le n} z_{k_1} \cdots z_{k_i}$, the slope of $l_h$ on the interval $(i-1, i)$ is equal to $\operatorname{ord} z_i$. If $l(x) = \infty$, we define the slope of $l$ at $x$ to be $\infty$.

Lemma 6.1 *Let $f(T)=\sum_{i=0}^n a_iT^{n-i}\in O_K[T]$ be a polynomial of degree $n$ and $z$ be an element of $\bar K^\times$ such that* $\mathrm{ord} z=\frac{1}{n}$. *We assume $a_0=1$ and* $\mathrm{ord} a_i\geq 1$ *for $1\leq i<n$. We put*

$$h(T)=\frac{f(z(T+1))-f(z)}{z^n}=\sum_{i=0}^{n-1}b_iT^{n-i}\in\bar K[T]$$

*and let $l_h:[0,n]\to\mathbb{R}\cup\{\infty\}$ be the function defining the Newton polygon of $h(T)$. Then, for an integer $0<i<n$, the equality $l_h(i)=\mathrm{ord} b_i$ implies $i=n-p^k$ for some integer $k\geq 0$.*

*Proof.* For an integer $0\leq r\leq n$, we put $f_r(T)=a_{n-r}T^r$, $h_r(T)=(f_r(z(T+1))-f_r(z))/z^n=a_{n-r}z^{-(n-r)}((T+1)^r-1)$ and let $l_r:[0,n]\to\mathbb{R}\cup\{\infty\}$ denote the function defining the Newton polygon of $h_r(T)$. We have

$$h(T)=\sum_{r=1}^n h_r(T)=\sum_{i=1}^n\left(\sum_{r=i}^n a_{n-r}z^{-(n-r)}\binom{r}{i}\right)T^i.$$

Since $\mathrm{ord} z=\frac{1}{n}$, we have $\mathrm{ord}\ b_{n-i}=\min_{i\leq r\leq n}(\mathrm{ord}\ a_{n-r}z^{-(n-r)}\binom{r}{i})$. Hence $l_h$ is the maximum convex function satisfying $l_h\leq l_r$ for $1\leq r\leq n$.
We compute the function $l_r$ for $1\leq r\leq n$. We have $h_r(T)=a_{n-r}z^{-(n-r)}\sum_{i=1}^r\binom{r}{i}T^i$. For an integer $0<i\leq p^k|r$, we have

$$\mathrm{ord}\binom{r}{i}=\mathrm{ord}\ \frac{r}{i}+\sum_{j=1}^{i-1}\mathrm{ord}\ \frac{r-j}{j}=\mathrm{ord}\ \frac{r}{i}\geq\mathrm{ord}\ \frac{r}{p^k}.$$

The equality holds only for $i=p^k$. Hence, $l_r$ is the maximum convex function satisfying

$$l_r(i)=(\mathrm{ord}\ a_{n-r}-1)+\frac{r}{n}+\begin{cases}0 & \text{if } i=n-r\\ \mathrm{ord}\ \dfrac{r}{p^k} & \text{if } i=n-p^k \text{ for an integer } 0\leq k\leq\mathrm{ord}_p r.\end{cases}$$

Thus, for an integer $i$ satisfying $n-p^{\mathrm{ord}_p r}\leq i\leq n$, the equality $l_h(i)=l_r(i)$ implies $i=n-p^k$ for an integer $1\leq k\leq\mathrm{ord}_p r$. It also follows that we have $0=l_h(0)<l_r(n-r)=l_r(n-p^{\mathrm{ord}_p r})$ for $1\leq r<n$. Hence the equality $l_h(i)=l_r(i)$ implies $i\geq n-p^{\mathrm{ord}_p r}$. Thus the assertion is proved. $\square$

For a polynomial $h(T)\in\bar K[T],\neq 0$, let ord $h(T)$ denote the minimum of the valuations of the coefficients. For a rational number $u$, let $\pi^u$ denote an element of $\bar K^\times$ satisfying $\mathrm{ord}\pi^u=u$. We define a function $\varphi_h:[0,\infty)\to[0,\infty)$ by $\varphi_h(u)=\mathrm{ord}\ h(\pi^uT)$. The function $\varphi_h$ is continuous, convex and piecewise linear.

Lemma 6.2 *Let $h(T)=\sum_{i=0}^n b_iT^{n-i}=\prod_{i=1}^n(T-z_i)\in\bar K[T]$ be a monic polynomial of degree $n$. Let $l_h:[0,n]\to\mathbb{R}\cup\{\infty\}$ be the function defining the*

*Newton polygon of* $h(T)$ *and* $\varphi_h : [0,\infty) \to [0,\infty)$ *be the function* $\varphi_h(u) = \mathrm{ord}\ h(\pi^u T)$ *defined above. Then,*
*1. The minimum value of the function* $l_h(t) + (n-t)u$ *on* $t \in [0,n]$ *is equal to* $\varphi_h(u)$.
*2. We have an equality*

$$\varphi_h(u) = \sum_{i=1}^{n} \min(u, \mathrm{ord}\ z_i).$$

*3. If the coefficient of* $T^r$ *in* $\overline{\left(\dfrac{h(\pi^u T)}{\pi^{\varphi_h(u)}}\right)} \in \bar{F}[T]$ *is not zero, then the function* $l_h(t) + (n-t)u$ *attains the minimum value at* $t = r$ *and we have* $l_h(r) = \mathrm{ord} b_r$.

*Proof.* 1. Since the function $l_h(t) + (n-t)u$ defines the Newton polygon of $h(\pi^u T)$, the assertion follows.
2. We put $s_i = \mathrm{ord}\ z_i$. Let $t_0 \in [0,n]$ be the minimum where the function $l_h(t) + (n-t)u$ takes the minimum value. Then $t_0$ is the maximum such that the function $l_h(t) + (n-t)u$ is strictly decreasing on $[0, t_0]$. Hence $t_0$ is the cardinality of the set $\{i | s_i < u\}$ and the minimum value of $l_h(t) + (n-t)u$ is given by

$$l_h(t_0) + (n-t_0)u = \sum_{s_i < u} s_i + \sum_{s_i \geq u} u = \sum_{i=1}^{n} \min(s_i, u).$$

Thus the assertion follows from 1.
3. The coefficient of $T^r$ in $\overline{h(\pi^u T)/\pi^{\varphi_h(u)}} \in \bar{F}[T]$ is not zero if and only if the value of the function defining the Newton polygon of $h(\pi^u T)/\pi^{\varphi_h(u)}$ at $r$ is zero and $l_h(r) = \mathrm{ord} b_r$. Hence the assertion follows from 1. □

### 6.2 The structure of graded pieces

In this subsection, we assume that the residue field $F$ is perfect. Since the residue map $\Omega_F(\log) \to F$ is an isomorphism in this case, we have an isomorphism $\Theta^j_{\log} \to N^j$ of $\bar{F}$-vector spaces of dimension 1. Let $\pi_1^{\mathrm{ab,gp}}(N^j)$ denote the quotient of $\pi_1^{\mathrm{ab}}(N^j)$ classifying the étale isogenies to the algebraic group $N^j$.

Proposition 6.3 *Let* $K$ *be a complete discrete valuation field with perfect residue field and* $j > 0$ *be a positive rational number. Then,*
*1.* ([1] Propositions 3.7 (3) and 3.15 (4)) *We have* $G^j_{\log,K} = G^{j+1}_K$. *If* $p$ *is not a prime element of* $K$, *the horizontal arrows in the diagram of Lemma 5.13 are isomorphism.*
*2. The canonical surjection* $\pi_1^{\mathrm{ab}}(N^j) \to Gr^j_{\log} G_K$ (5.12.1) *induces an isomorphism* $\pi_1^{\mathrm{ab,gp}}(N^j) \to Gr^j_{\log} G_K$.

Contrary to the proof given in [12], we give a proof without using the "lower numbering" filtration or local class field theory.

Before starting proof, we introduce some notations. Let $L$ be a finite separable extension of $K$ and $\pi_L$ be a prime element of $L$. Let $K_1$ be the maximum unramified extension of $K$ in $L$ and let $f(T) \in O_{K_1}[T]$ be the minimal polynomial of $\pi_L$ over $K_1$. Since, $L$ is totally ramified over $K_1$, the polynomial $f(T)$ is an Eisenstein polynomial. We put $n = [L : K_1] = \deg f$.

We put $A = O_L$ and $K_0 = K$ and define an object $(\mathbf{A} \to A)$ of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ as follows. We define a log structure on $O_{K_1}[T]$ by the chart $\mathbb{N} \to O_{K_1}[T]$ sending 1 to $T$. We define a log $O_{K_0}$-algebra $\mathbf{A} = O_{K_1}[[T]]$ to be the formal completion of the surjection $O_{K_1}[T] \to O_L$ sending $T$ to $\pi_L$ with the induced log structure. Then the surjection $\mathbf{A} \to A$ defines an object $(\mathbf{A} \to A)$ of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$. By Lemma 5.7, it defines a log pre-embedding $(\mathbf{A} \otimes^{\log}_{O_{K_0}} O_K \to A)$. The log ring $\mathbf{A} \otimes^{\log}_{O_{K_0}} O_K$ is the ring $\mathbf{A}$ itself with the log structure defined by the chart $\mathbb{N}^2 \to \mathbf{A}$ sending $(1, 0)$ to $T$ and $(0, 1)$ to a prime element $\pi$ of $O_K$. By forgetting the log structure, we obtain an embedding $(\mathbf{A} \to A)^\circ$. By applying Lemma 4.4, we obtain a log embedding $(\mathbf{A}^\sim \to A)$. The log ring $\mathbf{A}^\sim$ is identified with the formal completion of the surjection $O_K[T, U^{\pm 1}]/(T^n - U\pi) \to A$ of log $O_K$-algebras sending $T$ to $\pi_L$ and $U$ to $\pi_L^n/\pi \in A^\times$ with log structure defined by the chart $\mathbb{N} \to O_{K_1}[T, U^{\pm 1}]/(T^n - U\pi)$ sending 1 to $T$. Let $K'$ be a finite separable extension of $K$ containing the conjugates of $K_1$ over $K$ and an element $z$ of ord $z = 1/n$. Then, the log tensor product $\mathbf{A}^\sim \otimes^{\log}_{O_K} O_{K'}$ is further identified with the formal completion of the surjection $O_{K_1} \otimes_{O_K} O_{K'}[W^{\pm 1}] = \prod_{\sigma : K_1 \to K'} O_{K'}[W^{\pm 1}] \to A \otimes^{\log}_{O_K} O_{K'}$ of strict log $O_{K'}$-algebras sending $W$ to $(\pi_L \otimes 1)/(1 \otimes z)$. With this identification, the canonical map $\mathbf{A}^\sim \to \mathbf{A}^\sim \otimes^{\log}_{O_K} O_{K'}$ sends $T$ to $(1 \otimes z)W$ and $U$ to $((1 \otimes z)^n/\pi) \cdot W^n$. Further, we identify the affinoid variety $X^j_{\log}(\mathbf{A}^\sim \to A)_{K'}$ as an affinoid subdomain of $\coprod_{\sigma : K_1 \to K'} \mathrm{Sp} K' \langle W^{\pm 1} \rangle$.

Similarly as for $(\mathbf{A} \to A)$, by taking a prime element $\pi$ of $O_K$, we define an object $(\mathbf{B} \to O_K)$ of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ as the formal completion of the surjection $O_{K_0}[S] \to O_K$ sending $S$ to $\pi$. By Lemma 4.4, the log ring $\mathbf{B}^\sim$ is identified with the formal completion of the surjection $O_K[V^{\pm 1}] \to A$ of strict log $O_K$-algebras sending $V$ to 1. With this identification, the canonical map $\mathbf{B} \to \mathbf{B}^\sim$ sends $S$ to $\pi V$. Further, we identify the affinoid variety $X^j_{\log}(\mathbf{B}^\sim \to O_K)_K$ with the subdisk $D(1, \pi^j) \subset \mathrm{Sp} K \langle V^{\pm 1} \rangle$.

We define a map $(\mathbf{B} \to O_K) \to (\mathbf{A} \to A)$ of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ as follows. Since $f(T)$ is an Eisenstein polynomial of degree $n$, $g(T) = (T^n - f(T))/\pi$ is in $O_{K_1}[T]$ and its image is invertible in $\mathbf{A}$. By sending $S$ to $T^n g(T)^{-1}$, we obtain a map $(\mathbf{B} \to O_K) \to (\mathbf{A} \to A)$ of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$.

The Herbrand functions $\varphi$ and $\psi : [0, \infty) \to [0, \infty)$ are defined as follows (cf. [4] Appendix). We put $h(T) = f(\pi_L(T+1))/\pi_L^n$ and define $\varphi$ to be the function $\varphi_h$ in Lemma 6.2. The function $\varphi$ is strictly increasing, continuous and piecewise linear. We define $\psi : [0, \infty) \to [0, \infty)$ to be the inverse $\varphi^{-1}$. The function $\psi$ is

also strictly increasing, continuous and piecewise linear.
For an embedding $\sigma : K_1 \to \bar{K}$ over $K$, let $f^\sigma(T) \in O_{\bar{K}}[T]$ denote the image of $f(T)$ by $\sigma$. For $w \in \bar{K}$ and a rational number $u > 0$, let $D(w, \pi^u)$ denote the disk with center $w$ and radius $\pi^u$.

Lemma 6.4 *Let the notation be as above.*
*1. The open immersion* $X^{j+1}((\mathbf{A} \to A)^\circ) \subset X^j_{\log}(\mathbf{A}^\sim \to A)$ *in Corollary 4.9 is an isomorphism.*
*2. As affinoid subdomains of* $\coprod_{\sigma : K_1 \to K'} \mathrm{Sp} K'\langle W^{\pm 1}\rangle$*, we have an equality*

$$X^j_{\log}(\mathbf{A}^\sim \to A) = \coprod_{\sigma : K_1 \to K'} \bigcup_{f^\sigma(z_i^\sigma) = 0} D(\frac{z_i^\sigma}{z}, \pi^{\psi(j)}). \tag{2}$$

*The log ramification of $L$ is bounded by $j$ if and only if $\psi(j)$ is larger than the slope $s_{n-1}$ of the Newton polygon of $h$ on the interval $(n-2, n-1)$.*
*3. Let $\sigma : K_1 \to \bar{K}$ be an embedding and $z_i^\sigma \in O_{\bar{K}}$ be a solution of $f^\sigma(T) = 0$. We put*

$$h_i^\sigma(T) = -\frac{f^\sigma(z(\pi^{\psi(j)} T + \frac{z_i^\sigma}{z}))}{\pi^j f^\sigma(0)}.$$

*Then we have $h_i^\sigma \in O_{\bar{K}}[T]$. Let $\bar{h}_i^\sigma \in \bar{F}[T]$ be the reduction and let $\bar{h}_i^\sigma : \mathbb{A}^1 \to \mathbb{A}^1$ be the map defined by the polynomial $\bar{h}_i^\sigma$. Then the isomorphisms $\times \pi^{\psi(j)} + \frac{z_i^\sigma}{z} : D(0,1) \to D(\frac{z_i^\sigma}{z}, \pi^{\psi(j)}) \subset X^j_{\log}(\mathbf{A}^\sim \to A)$ and $\times \pi^j + 1 : D(0,1) \to D(1, \pi^j)$ induce a commutative diagrams*

$$\begin{array}{ccc}
\mathbb{A}^1 & \longrightarrow & \overline{D(\frac{z_i^\sigma}{z}, \pi^{\psi(j)})} \subset \bar{X}^j_{\log}(\mathbf{A}^\sim \to A) \\
{\scriptstyle \bar{h}_i^\sigma}\downarrow & & \downarrow \\
\mathbb{A}^1 & \longrightarrow & \overline{D(1, \pi^j)} = \bar{X}^j_{\log}(\mathbf{B}^\sim \to O_K) = N^j.
\end{array}$$

*Proof.* 1. As in Lemma 4.8, we identify $X^j_{\log}(\mathbf{A}^\sim \to A)$ and $X^{j+1}((\mathbf{A} \to A)^\circ)$ as affinoid subdomains of $X^j((\mathbf{A} \to A)^\circ)$. The kernels of the surjections $\mathbf{A} \to A$ and $\mathbf{A}^\sim \to A$ are generated by $f(T)$ and $U^{-1} - g(T) = f(T)/\pi$ respectively. Hence, the affinoid subdomains $X^j_{\log}(\mathbf{A}^\sim \to A)$ and $X^{j+1}((\mathbf{A} \to A)^\circ)$ of $X^j((\mathbf{A} \to A)^\circ)$ are defined by the conditions ord $f(x)/\pi \geq j$ and by ord $f(x) \geq j+1$ respectively. Hence the assertion follows.
2. Since the kernel of surjection $\mathbf{A}^\sim \to A$ is generated by $f(T)/\pi$, the kernel of surjection $\mathbf{A}^\sim \otimes^{\log}_{O_K} O_{K'} \to A \otimes^{\log}_{O_K} O_{K'}$ is generated by $(z^n/\pi) \cdot (f(zW)/z)$. Hence we have

$$X^j_{\log}(\mathbf{A}^\sim \to A)(\bar{K}) = \coprod_{\sigma : K_1 \to K'} \{w \in O_{\bar{K}} | \mathrm{ord}\ f^\sigma(zw)/z^n \geq j\}$$

We fix an embedding $\sigma : K_1 \to \bar{K}$ and drop $\sigma$ in the notation. For $i = 1, \ldots, n$, we put $U_i = \{w | \mathrm{ord}(w - z_i/z) \geq \mathrm{ord}(w - z_k/z)$ for $k = 1, \ldots, n\}$. By the equality above, to prove (2), it is sufficient to show

$$\{w \in O_{\bar K} | \text{ord } f(zw)/z^n \geq j\} \cap U_i \subset \\ D(z_i/z, \pi^{\psi(j)}) \subset \{w \in O_{\bar K} | \text{ord } f(zw)/z^n \geq j\}$$

for each $i$. Let $w \in O_{\bar K}$. We put $I_1 = \{k : \text{ord}(w - z_i/z) > \text{ord}(w - z_k/z)\}$ and $I_2 = \{k : \text{ord}(w - z_i/z) \leq \text{ord}(w - z_k/z)\}$. For $i \in I_1$, we have $\text{ord}(w - z_k/z) = \text{ord}(z_k - z_i)/z < \text{ord}(w - z_i/z)$ and, for $i \in I_2$, we have $\text{ord}(w - z_k/z) \geq \text{ord}(w - z_i/z)$ with the equality if $x \in U_i$. Since $f(zW)/z^n = \prod_{k=1}^n (W - z_k/z)$, we have an inequality

$$\text{ord}\frac{f(zw)}{z^n} = \sum_{k=1}^n \text{ord}(w - \frac{z_k}{z}) \geq \\ \geq \sum_{k \in I_1} \text{ord}(\frac{z_k}{z} - \frac{z_i}{z}) + \sum_{k \in I_2} \text{ord}(w - \frac{z_i}{z}) = \varphi(\text{ord}(w - \frac{z_i}{z})).$$

We have an equality if $x \in U_i$. Thus the equality (2) is proved. The last assertion follows from the equality (2) and $s_{n-1} = \max_{i \neq k} \text{ord } (z_i/z - z_k/z)$.
3. We show $h_i^\sigma(T) \in O_{\bar K}[T]$. We extend $\sigma : K_1 \to \bar K$ to $\sigma_i : L \to \bar K$ by sending $\pi_L$ to $z_i^\sigma$ and put $u = \psi(j)$. Then we have $h_i^\sigma(T) = -h^{\sigma_i}(\pi^u \cdot (z/z_i) \cdot T)/\pi^{\varphi(u)} f(0)$. Since $z/z_i$ and $f(0)/z^n$ are units, we have $h_i^\sigma(T) \in O_{\bar K}[T]$ by the definition of $\varphi(u)$.
We show the commutativity of the diagram. Since $\mathbf{B} \to \mathbf{A}$ sends $S$ to $T^n g(T)^{-1}$, the induced map $\mathbf{B}^\sim \to \mathbf{A}^\sim \otimes_{O_K}^{\log} O_{K'}$ sends $V$ to

$$\frac{T^n}{\pi \cdot g(T)} = \frac{f(T)}{\pi \cdot g(T)} + 1 = \frac{f((1 \otimes z)W)}{\pi \cdot g((1 \otimes z)W)} + 1.$$

We fix $\sigma : K_1 \to K$ and we drop $\sigma$ in the notation. We define a map $D(z_i/z, \pi^{\psi(j)}) \to D(1, \pi^j)$ by sending $w$ to $(f(zw)/(\pi g(zw))) + 1$. Then, we have a commutative diagram

$$\begin{array}{ccc} D(\frac{z_i}{z}, \pi^{\psi(j)}) & \xrightarrow{\subset} & X^j_{\log}(\mathbf{A}^\sim \to A) \\ \downarrow & & \downarrow \\ D(1, \pi^j) & = & X^j_{\log}(\mathbf{B}^\sim \to O_K). \end{array}$$

The polynomial $g(zW)$ is congruent to the constant $-f(0)/\pi$ modulo the maximal ideal. Hence, by substituting $W = \pi^{\psi(j)}T + z_i/z$, we get the assertion. $\square$

*Proof of Proposition 6.3.* 1. The equality $G^j_{K,\log} = G^{j+1}_K$ follows from Lemma 6.4.1. The rest is clear.
2. First we show that the map $\pi_1^{\text{ab}}(N^j) \to Gr^j_{\log} G_K$ factors the quotient $\pi_1^{\text{ab,gp}}(N^j)$. By Lemma 6.4.3, it is sufficient to show that the map $\bar h_i^\sigma : \mathbb{A}^1 \to \mathbb{A}^1$ is an isogeny. In other words, it is enough to show that if the coefficient of $T^r$

in $\bar h_i^\sigma$ is non-zero, then $r$ is a power of $p$. Since $h_i^\sigma(T) = -h^{\sigma_i}(\pi^u\cdot(z/z_i)\cdot T)/\pi^{\varphi(u)}f(0)$ and $z/z_i$ and $f(0)/z^n$ are units, the coefficient of $T^r$ in $\bar h_i^\sigma$ is non-zero if and only if the coefficient of $T^r$ in $\overline{h(\pi^uT)/\pi^{\varphi(u)}}$ is non-zero. Let $l_h$ be the function defining the Newton polygon of $h$. We apply Lemma 6.2 to $h(T) = f(\pi_L(T+1))/\pi_L^n = \sum_{i=0}^{n-1} b_iT^{n-i}$. Then, if the coefficient of $T^r$ in $\overline{h(\pi^uT)/\pi^{\varphi(u)}}$ is non-zero, we have $l_h(r) = \mathrm{ord}\ b_r$. Since $\mathrm{ord}\ z = 1/n$, we may apply Lemma 6.1 to the polynomial $h(T)$. Thus the equality $l_h(r) = \mathrm{ord}\ b_r$ implies that $r$ is a power of $p$ as required.

We show that the surjection $\pi_1^{\mathrm{ab,gp}}(N^j) \to Gr^j_{\log}G_K$ is an isomorphism. By Lemma 5.2, we may replace $K$ by the completion of a maximum unramified extension and assume the residue field $F$ is algebraically closed. To show the isomorphism, it is sufficient to construct every étale isogeny of degree $p$ to $N^j$ from a finite separable extension of $K$. Recall that every étale isogeny of degree $p$ to $N^j$ is obtained by pulling-back the isogeny $\mathbb{A}^1 \to \mathbb{A}^1$ defined by the polynomial $T^p - T$ by an isomorphism $N^j \to \mathbb{A}^1$.

We show the following Lemma.

Lemma 6.5 *Let $n, m, l \geq 1$ be integers such that $m \leq n$ and $pl \leq n$, $m$ and $l$ are prime to $p$ and that $p^2|n$. Let $\pi$ be a prime element of $O_K$ and $a, b$ be element of $O_K$. We put $m' = n\cdot \mathrm{ord}\ a + m$ and $l' = n\cdot\mathrm{ord}\ b + pl$ and assume $pl' < m' < pl' + n\cdot\mathrm{ord}\ p$ and $pl' < n\cdot\mathrm{ord}\ p\cdot\mathrm{ord}_p(n/p^2)$. Let $f(T)$ be the Eisenstein polynomial*

$$f(T) = T^n - \pi(aT^m - bT^{pl} + 1)$$

*and let $z = \pi_L$ be the image of $T$ in $L = K[T]/f(T)$. We put*

$$j = \frac{p}{p-1}\cdot\frac{m'-l'}{n} \qquad \text{and} \qquad \pi^j = maz^m\left(\frac{maz^m}{bz^{pl}}\right)^{\frac{1}{p-1}}.$$

*Then,*

*1. The log ramification of the extension $L = K[T]/(f(T))$ is bounded by $j+$.*

*2. We define a map $(\mathbf{B}^\sim \to O_K) \to (\mathbf{A}^\sim \to O_L)$ as above and consider $X^j_{\log}(\mathbf{A}^\sim \to A)_L$ as an affinoid subdomain of $\mathrm{Sp}O_L\langle W\rangle$ by taking $K' = L$ and $z$ to be the image of $T$. Let $\mathbb{A}^1 \to \mathbb{A}^1$ be the map defined by the polynomial $T^p - T$. Then, $D(1, \pi^{\psi(j)})$ is a connected component of $X^j_{\log}(\mathbf{A}^\sim \to A)$. Further, the isomorphism $\times\pi^j + 1 : D(0,1) \to D(1,\pi^j)$ induce a commutative diagram*

$$\begin{array}{ccc}
\mathbb{A}^1 & \longrightarrow & \overline{D(1,\pi^{\psi(j)})} \subset \bar X^j_{\log}(\mathbf{A}^\sim \to A) \\
\downarrow & & \downarrow \\
\mathbb{A}^1 & \longrightarrow & \overline{D(1,\pi^j)} = \bar X^j_{\log}(\mathbf{B}^\sim \to O_K) = N^j.
\end{array}$$

*Proof.* 1. We put $h(T) = f(z(T+1))/z^n$ and let $l_h : [0,n] \to \mathbb{R}\cup\{\infty\}$ be the function defining the Newton polygon of $h(T)$. Let $l_1 : [0,n] \to \mathbb{R}\cup$

$\{\infty\}$ be the linear function characterized by $l_1(n-1) = m'/n$ and $l_1(n-p) = pl'/n$. We claim that we have an equality $l_h = l_1$ on and only on the interval $(n-p, n-1)$. By Lemma 6.1, it is sufficient to show $l_h(n-1) = m'/n, l_h(n-p) = pl'/n$ and $l_h(n-p^2) > l_1(n-p^2)$. By the proof of Lemma 6.1, we have $l_h(n-1) = \min(\text{ord } n, \text{ord } maz^m, \text{ord } plbz^{pl}) = \min(\text{ord } p \cdot \text{ord}_p n, m'/n, \text{ord } p + pl'/n)$. By the assumptions, we have $m' < n \cdot \text{ord } p + pl' < n \cdot \text{ord } p \cdot \text{ord}_p n$ and $l_h(n-1) = m'/n$. Similarly, we have $l_h(n-p) = \min(\text{ord } \binom{n}{p}, m'/n, \text{ord } \binom{pl}{p} bz^{pl}) = \min(\text{ord } p \cdot \text{ord}_p(n/p), m'/n, pl'/n) = pl'/n$ and $l_h(n-p^2) \geq \min(\text{ord } \binom{n}{p^2}, m'/n, pl'/n) = pl'/n \geq l_1(n-p) > l_1(n-p^2)$. Thus the claim is proved.

By Lemma 6.4.2, it is sufficient to show that the slope $s_{n-1}$ of $l_h$ on the interval $(n-2, n-1)$ is $\psi(j)$. By the claim above, we have $s_{n-1} = (l_h(n-1) - l_h(n-p))/(p-1)$ and $\varphi(s_{n-1}) = l_h(n-1) + s_{n-1} = (p \cdot l_h(n-1) - l_h(n-p))/(p-1) = p(m'-l')/(p-1)n = j$. Thus the assertion follows.

2. In Lemma 6.4.3, we put $\pi^{\psi(j)} = (maz^m/bz^{pl})^{1/(p-1)}$ and $\pi^j = maz^m \pi^{\psi(j)}$. Then we have

$$-\frac{f(z(\pi^{\psi(j)}T+1))}{\pi^j f(0)} \equiv -\frac{-\binom{pl}{p} bz^{pl} \pi^{p\psi(j)} T^p + maz^m \pi^{\psi(j)} T}{\pi^j} \equiv T^p - T.$$

Hence the assertion follows. □

We complete the proof of Proposition 6.3.2. By Lemma 6.5, it is sufficient to show the following: For every rational number $j > 0$, there exist integers $n, m', l' > 0$ satisfying the conditions in Lemma 6.5 and, for every non-zero element $x$ of $N^j$, there exist $a, b \in O_K$ such that ord $a$ is the integral part of $m'/n$, ord $b$ is the integral part of $pl'/n$ and $x \equiv maz^m(maz^m/bz^{pl})^{1/(p-1)}$. First, we prove the claim for $j$. Assume $p$ is odd (resp. even). Let $n > 0$ be an integer such that $n(p-1)j/p$ (resp. $n(p-1)j/2p$) and $n/p^2$ are integers and $(p-1)j/p \in [(p+1)/n, (p-1)n/p^2 \cdot \text{ord } p \cdot \text{ord}_p(n/p^2) - (p+1)/n]$. Then there exist integers $l', m'$ such that $(p-1)j/p = (m'-l')/n$, $l'$ and $m'$ are prime to $p$, $pl' < m' < pl' + n \cdot \text{ord } p$ and $pl' < n \cdot \text{ord } p \cdot \text{ord}_p(n/p^2)$. Thus the claim is proved for $j$. Since we may multiply $a$ an arbitrary unit, the claim for $x$ is clear. Hence the assertion is proved. □

## References


[1] A. Abbes, T. Saito, *Ramification of local fields with imperfect residue fields*, American J. of Math. 124.5 (2002), 879-920.

[2] S. Bosch, U. Güntzer, and R. Remmert, Non-Archimedean Analysis, Fundamental Principles of Math. Sciences, vol. 261, Springer-Verlag, Berlin, 1984.

[3] A. J. de Jong, *Crystalline Dieudonné module theory via formal and rigid geometry*, Publ. Math. IHES 82 (1995), 5-96.

[4] P. Deligne, *Les corps locaux de characteristic p, limites de corps locaux de characteristic 0*, in J.-N. Bernstein, P. Deligne, D. Kazhdan, M.-F. Vign-

eras, Représentations des groupes réductifs sur un corps local, Travaux en cours, Hermann, Paris (1984), 119-157.

[5] L. Illusie, *Champs toriques et log lissité*, preprint, 2000.

[6] K. Kato, *Logarithmic structure of Fontaine-Illusie*, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp.191-224.

[7] —–, *Logarithmic structure of Fontaine-Illusie II*, unpublished preprint.

[8] —–, *Toric singularities*, American J. of Math., 116 (1994), pp. 1073–1099.

[9] —–, *Semi-stable reduction and $p$-adic etale cohomology*, in J.-M. Fontaine (ed.), Périodes $p$-adiques, Astérisque 223 (1994), pp. 269-293.

[10] K. Kato, T. Saito, *Conductor formula of Bloch*, preprint (revised version), 2003.

[11] M. Olsson, *Logarithmic geometry via algebraic stacks*, to appear in Ann. Sci. Ecole Norm. Sup.

[12] T. Saito, *Ramification groups and local constants*, preprint, Univ. of Tokyo, 1996.

[13] J.-P. Serre, Corps Locaux, Hermann, Paris, 1968.

Ahmed Abbes
LAGA, Institut Galilée
Université Paris-Nord
93430 Villetaneuse, France
abbes@math.univ-paris13.fr

Takeshi Saito
Department of Mathematical Sciences
University of Tokyo
Tokyo 153-8914 Japan
t-saito@ms.u-tokyo.ac.jp