On Base Change Theorem and Coherence in Rigid Cohomology

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Abstract. We prove that the base change theorem in rigid cohomology holds when the rigid cohomology sheaves both for the given morphism and for its base extension morphism are coherent. Applying this result, we give a condition under which the rigid cohomology of families becomes an overconvergent isocrystal. Finally, we establish generic coherence of rigid cohomology of proper smooth families under the assumption of existence of a smooth lift of the generic fiber. Then the rigid cohomology becomes an overconvergent isocrystal generically. The assumption is satisfied in the case of families of curves. This example relates to P. Berthelot's conjecture of the overconvergence of rigid cohomology for proper smooth families.

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1 Introduction

Let p be a prime number and let $\mathcal{V}$ (resp. k, resp. K) be a complete discrete valuation ring (resp. the residue field of $\mathcal{V}$ with characteristic p, resp. the quotient field of $\mathcal{V}$ with characteristic 0). Let $f : X \to \operatorname{Spec} k$ be a separated morphism of schemes of finite type. The finiteness of rigid cohomology $H^*_{\mathrm{rig}}(X/K, E)$ for an overconvergent F-isocrystal E on X/K are proved by recent developments [2] [6] [8] [9] [11] [18] [19] [20] [21]. However, if one takes another embedding $\operatorname{Spec} k \to \mathcal{S}$ for a smooth $\mathcal{V}$-formal scheme $\mathcal{S}$, we do not know whether the "same" rigid cohomology, $\mathbb{R}^* f_{\mathrm{rig}\mathfrak{S}*}E$ in our notation, with respect to the base $\mathfrak{S} = (\operatorname{Spec} k, \operatorname{Spec} k, \mathcal{S})$ becomes a sheaf of coherent $\mathcal{O}_{]\operatorname{Spec} k[_{\mathcal{S}}}$-modules or not, and whether the base change homomorphism

$$H^*_{\mathrm{rig}}(X/K, E) \otimes_K \mathcal{O}_{]\operatorname{Spec} k[_{\mathcal{S}}} \to \mathbb{R}^* f_{\mathrm{rig}\mathfrak{S}*}E$$

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is an isomorphism or not. In this case, if one knows the coherence of $\mathbb{R}^* f_{\mathrm{rig}\mathfrak{S}*}E$, then the homomorphism above is an isomorphism. Moreover, if the coherence holds for any $\mathcal{S}$, then there exists a rigid cohomology isocrystal $\mathbb{R}^* f_{\mathrm{rig}*}E$ on $\mathrm{Spec}\, k/K$ and $\mathbb{R}^* f_{\mathrm{rig}\mathfrak{S}*}E$ is a realization with respect to the base $\mathfrak{S}$.
In this paper we discuss the coherence, base change theorems, and the overconvergence of the Gauss-Manin connections, for rigid cohomology of families. Up to now, only few results are known. One of the difficulities to see the coherence of rigid cohomology comes from the reason that there is no global lifting. If a proper smooth family over $\mathrm{Spec}\, k$ admits a proper smooth formal lift over $\mathrm{Spf}\, \mathcal{V}$, then the rigid cohomology of the family is coherent by R. Kiehl's finiteness theorem for proper morphisms in rigid analytic geometry. Hence it becomes an overconvergent isocrystal. This was proved by P. Berthelot [4, Théorème 5]. (See 4.1.)
In general it is too optimistic to believe the existence of a proper smooth lift for a proper smooth family. So we present a problem on the existence of a projective smooth lift of the generic fiber up to "alteration" (Problem 4.2.1). Assuming a positive solution of this problem, we have generic coherence of rigid cohomology. This means that the rigid cohomology becomes an overconvergent isocrystal on a dense open subscheme. In the case of families of curves this problem is solved [12, Exposé III, Corollaire 7.4], so the rigid cohomology sheaves become overconvergent isocrystals generically.
In [1] Y. André and F. Baldassarri had a result on the generic overconvergence of Gauss-Manin connections of de Rham cohomologies for overconvergent isocrystals on families of smooth varieties (not necessary proper) which come from algebraic connections of characteristic 0.

Now let us explain the contents. See the notation in the convention.
Let

$$\begin{array}{ccc} \mathfrak{X} & \stackrel{v}{\longleftarrow} & \mathfrak{Y} \\ f\downarrow & & \downarrow g \\ \mathfrak{S} & \underset{u}{\longleftarrow} & \mathfrak{T} \end{array}$$

be a cartesian square of $\mathcal{V}$-triples separated of finite type such that $\widehat{f} : \mathcal{X} \to \mathcal{S}$ is smooth around X. In section 2 we discuss base change homomorphisms

$$\widetilde{u}^* \mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E \to \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*} v^* E$$

such that $\mathcal{T} \to \mathcal{S}$ is flat. In a rigid analytic space one can not compare sheaves by stalks because of G-topology. Only coherent sheaves can be compared by stalks. The base change homomorphism is an isomorphism if both the source and the target are coherent (Proposition 2.3.1). By the hypothesis we can use the stalk argument.
In section 3 we review the Gauss-Manin connection on the rigid cohomology sheaf and give a condition under which the Gauss-Manin connection becomes overconvergent. Let $f : \mathfrak{X} \to \mathfrak{T}$ and $u : \mathfrak{T} \to \mathfrak{S}$ be morphisms of $\mathcal{V}$-triples such that $\widehat{f} : \mathcal{X} \to \mathcal{T}$ and $\widehat{u} : \mathcal{T} \to \mathcal{S}$ are smooth around X and T, respectively. Then

the Gauss-Manin connection ∇^{GM} on the rigid cohomology sheaf $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E$ for an overconvergent isocrystal E on $(X, \overline{X})/\mathcal{S}_K$ is overconvergent if $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}'*}E$ is coherent for any triple $\mathfrak{T}' = (T, \overline{T}, \mathcal{T}')$ over $\mathfrak{T}$ such that $\mathcal{T}' \to \mathcal{T}$ is smooth around T (Theorem 3.3.1). If the Gauss-Manin connection is overconvergent, then there exists an overconvergent isocrystal $\mathbb{R}^q f_{\mathrm{rig}*}E$ on $(T, \overline{T})/\mathcal{S}_K$ such that the rigid cohomology sheaf $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}'*}E$ is the realization of $\mathbb{R}^q f_{\mathrm{rig}*}E$ on $\mathfrak{T}'$ for any embedding $\overline{T} \to \mathcal{T}'$ such that $\mathcal{T}' \to \mathcal{S}$ is smooth around T. We also prove the existence of the Leray spectral sequence (Theorem 3.4.1).
In section 4 we discuss Berthelot's conjecture [4, Sect. 4.3]. Let $f : (X, \overline{X}) \to (T, \overline{T})$ be a proper smooth family of k-pairs of finite type over a triple $\mathfrak{S}$. We give a proof of Berthelot's theorem using the result in the previous sections (Theorems 4.1.1, 4.1.4). Finally, we discuss the generic coherence of rigid cohomology of proper smooth families as mentioned above.

Convention. The notation follows [5] and [9].
Throughout this paper, k is a field of characteristic $p > 0$, K is a complete discrete valuation field of characteristic 0 with residue field k and $\mathcal{V}$ is the ring of integers of K. $|\ |$ is denoted an p-adic absolute value on K.
A k-pair $(X, \overline{X})$ consists of an open immersion $X \to \overline{X}$ over $\operatorname{Spec} k$. A $\mathcal{V}$-triple $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ separated of finite type consists of a k-pair $(X, \overline{X})$ and a formal $\mathcal{V}$-scheme $\mathcal{X}$ separated of finite type with a closed immersion $\overline{X} \to \mathcal{X}$ over $\operatorname{Spf} \mathcal{V}$. Let $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ and $\mathfrak{Y} = (Y, \overline{Y}, \mathcal{Y})$ be $\mathcal{V}$-triples separated of finite type. A morphism $f : \mathfrak{Y} \to \mathfrak{X}$ of $\mathcal{V}$-triples is a commutative diagram

$$\begin{array}{ccccc} Y & \to & \overline{Y} & \to & \mathcal{Y} \\ \mathring{f} \downarrow & & \overline{f} \downarrow & & \downarrow \widehat{f} \\ X & \to & \overline{X} & \to & \mathcal{X}. \end{array}$$

over $\operatorname{Spf} \mathcal{V}$. The associated morphism between tubes denotes $\widetilde{f} :]\overline{Y}[_{\mathcal{Y}} \to]\overline{X}[_{\mathcal{X}}$.
A Frobenius endomorphism over a formal $\mathcal{V}$-scheme is a continuous lift of p-power endomorphisms.

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2 Base change theorems

2.1 Base change homomorphisms

We recall the definition of rigid cohomology in [9, Sect. 10] and introduce base change homomorphisms. Let $\mathcal{V} \to \mathcal{W}$ be a ring homomorphism of complete discrete valuation rings whose valuations are extensions of that of the ring $\mathbb{Z}_p$ of p-adic integers and let k and K (resp. l and L) be the residue field and the quotient field of $\mathcal{V}$ (resp. $\mathcal{W}$), respectively. Let $\mathfrak{S} = (S, \overline{S}, \mathcal{S})$ (resp. $\mathfrak{T} = (T, \overline{T}, \mathcal{T})$) be a $\mathcal{V}$-triple (resp. a $\mathcal{W}$-triple) separated of finite type and let $u : \mathfrak{T} \to \mathfrak{S}$ be a morphism of triples. Let

$$\begin{array}{ccc} (X, \overline{X}) & \overset{v}{\longleftarrow} & (Y, \overline{Y}) \\ f \downarrow & & \downarrow g \\ (S, \overline{S}) & \underset{u}{\longleftarrow} & (T, \overline{T}) \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, k & \longleftarrow & \mathrm{Spec}\, l \end{array}$$

be a commutative diagram of pairs such that the vertical arrows are separated of finite type and the upper square is cartesian. Then there always exists a Zariski covering $\mathfrak{X}'$ of $(X, \overline{X})$ over $\mathfrak{S}$ (resp. $\mathfrak{Y}'$ of $(Y, \overline{Y})$ over $\mathfrak{T}$) with a commutative diagram

$$\begin{array}{ccc} \mathfrak{X}' & \overset{v'}{\longleftarrow} & \mathfrak{Y}' \\ f' \downarrow & & \downarrow g' \\ \mathfrak{S} & \underset{u}{\longleftarrow} & \mathfrak{T} \end{array}$$

as triples such that the induced morphism $\mathcal{Y}' \to \mathcal{X}' \times_{\mathcal{S}} \mathcal{T}'$ is smooth around Y. Let $\mathfrak{X}'_{\bullet}$ be the Čech diagram as $(X, \overline{X})$-triples over $\mathfrak{S}$ and let $\mathrm{DR}^{\dagger}(\mathfrak{X}'_{\bullet}/\mathfrak{S}, (E_{\mathfrak{X}'_{\bullet}}, \nabla_{\mathfrak{X}'_{\bullet}}))$ be the de Rham complex

$$E_{\mathfrak{X}'_{\bullet}} \xrightarrow{\nabla_{\mathfrak{X}'}} E_{\mathfrak{X}'_{\bullet}} \otimes_{j^{\dagger}\mathcal{O}_{]\overline{X}'_{\bullet}[_{\mathcal{X}'_{\bullet}}}} j^{\dagger}\Omega^1_{]\overline{X}'_{\bullet}[_{\mathcal{X}'_{\bullet}}/]\overline{S}[_{\mathcal{S}}} \xrightarrow{\nabla_{\mathfrak{X}'}} E_{\mathfrak{X}'_{\bullet}} \otimes_{j^{\dagger}\mathcal{O}_{]\overline{X}'_{\bullet}[_{\mathcal{X}'_{\bullet}}}} j^{\dagger}\Omega^2_{]\overline{X}'_{\bullet}[_{\mathcal{X}'_{\bullet}}/]\overline{S}[_{\mathcal{S}}} \to \cdots$$

on $]\overline{X}[_{\mathcal{X}'_{\bullet}}$ associated to the realization $(E_{\mathfrak{X}'_{\bullet}}, \nabla_{\mathfrak{X}'_{\bullet}})$ of E with respect to $\mathfrak{X}'_{\bullet}$. Since $\mathfrak{X}'_{\bullet}$ is a universally de Rham descendable hypercovering of $(X, \overline{X})$ over $\mathfrak{S}$, one can calculate the q-th rigid cohomology $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E$ with respect to $\mathfrak{S}$ as the q-th hypercohomology of the total complex of $\mathrm{DR}^{\dagger}(\mathfrak{X}'_{\bullet}/\mathfrak{S}, (E_{\mathfrak{X}'_{\bullet}}, \nabla_{\mathfrak{X}'_{\bullet}}))$. From our choice of $\mathfrak{X}'$ and $\mathfrak{Y}'$, there is a canonical homomorphism

$$\mathbb{L}\widetilde{u}^*\mathbb{R}f_{\mathrm{rig}\mathfrak{S}*}E \to \mathbb{R}g_{\mathrm{rig}\mathfrak{T}*}v^*E$$

in the derived category of complexes of sheaves of abelian groups on $]\overline{T}[_{\mathcal{T}}$. The canonical homomorphism does not depend on the choices of $\mathfrak{X}'$ and $\mathfrak{Y}'$. If $\widehat{u} : \mathcal{T} \to \mathcal{S}$ is flat around T, we have a base change homomorphism

$$\widetilde{u}^*\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E \to \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*}v^*E$$

of sheaves of $j^{\dagger}\mathcal{O}_{]\overline{T}[_{\mathcal{T}}}$-modules for any q.
The following is the finite flat base change theorem in rigid cohomology.

2.1.1 Theorem [9, Theorem 11.8.1]

With notation as above, we assume furthermore that $\widehat{u} : \mathcal{T} \to S$ is finite flat, $\widehat{u}^{-1}(\overline{S}) = \overline{T}$ and $\overline{u}^{-1}(S) = T$. Then the base change homomorphism

$$\widetilde{u}^*\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E \to \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*}v^*E$$

is an isomorphism for any q.

2.2 The condition (F)

Let $\mathfrak{T} = \mathfrak{S} \times_{(\operatorname{Spec} k, \operatorname{Spec} k, \operatorname{Spf} \mathcal{V})} (\operatorname{Spec} l, \operatorname{Spec} l, \operatorname{Spf} \mathcal{W})$ and let q be an integer. For an overconvergent isocrystal E on $(X, \overline{X})/\mathcal{S}_K$, we say that the condition $(\mathrm{F})^q_{f,\mathcal{W}/\mathcal{V},E}$ holds if and only if the base change homomorphism

$$\widetilde{u}^* \mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*} E \to \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*} v^* E$$

is an isomorphism. By Theorem 2.1.1 we have

2.2.1 Proposition

If l is finite over k, then the condition $(\mathrm{F})^q_{f,\mathcal{W}/\mathcal{V},E}$ holds for any q and any E.

2.2.2 Example

Let $\mathfrak{S} = (\operatorname{Spec} k, \operatorname{Spec} k, \operatorname{Spf} \mathcal{V})$ and let $j^\dagger \mathcal{O}_{]\overline{X}[}$ be the overconvergent isocrystal on $(X, \overline{X})/K$ associated to the structure sheaf with the natural connection. If $\overline{X}$ is proper over $\operatorname{Spec} k$, then the condition $(\mathrm{F})^q_{f,\mathcal{W}/\mathcal{V},j^\dagger\mathcal{O}_{]\overline{X}[}}$ holds for any q and any extension $\mathcal{W}/\mathcal{V}$ of complete discrete valuation rings.

Proof. Using an alteration [14, Theorem 4.1] and the spectral sequence for proper hypercoverings [21, Theorem 4.5.1], we may assume that $\overline{X}$ is smooth. Note that the Gysin isomorphism [20, Theorem 4.1.1] commutes with any base extension. The assertion follows from induction on the dimension of X by a similar method of Berthelot's proof of finiteness of the rigid cohomology [6, Théorème 3.1] since the crystalline cohomology satisfies the base change theorem [5, Chap. 5, Théorème 3.5.1]. □

2.3 A base change theorem

We give a sufficient condition for a base change homomorphism to be an isomorphism.

2.3.1 Proposition

With notation in 2.1, assume furthermore that $\mathcal{W} = \mathcal{V}$ and $\widehat{u} : \mathcal{T} \to \mathcal{S}$ is smooth around T. Let q be an integer and suppose that $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*} E$ (resp. $\mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*} v^* E$) is a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{S}[_\mathcal{S}}$-modules (resp. a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{T}[_\mathcal{T}}$-modules) for an overconvergent isocrystal E on $(X, \overline{X})/\mathcal{S}_K$. Then the base change homomorphism

$$\widetilde{u}^* \mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*} E \to \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*} v^* E$$

is an isomorphism.

Proof. Since both sheaves are coherent, we may assume $T = \overline{T}$ by the faithfulness of the forgetful functor from the category of sheaves of coherent $j^\dagger\mathcal{O}_{]\overline{T}[_\mathcal{T}}$-modules to the category of sheaves of coherent $\mathcal{O}_{]T[_\mathcal{T}}$-modules [5, Corollaire 2.1.11]. Then we have only to compare stalks of both sides at each closed point of $]T[_\mathcal{T}$ by [7, Corollary 9.4.7] since both sides are coherent. Hence we may assume that $T = \overline{T}$ consists of a k-rational point by Proposition 2.2.1. Then the assertion follows from the following lemma. □

2.3.2 Lemma

Under the assumption of Proposition 2.3.1, assume furthermore that $T = \overline{T} = \operatorname{Spec} k$. Then the base change homomorphism

$$\widetilde{u}^*\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E \to \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*}v^*E$$

is an isomorphism.

Proof. We may assume that $S = \overline{S} = \operatorname{Spec} k$. By the fibration theorem [5, Théorème 1.3.7] we may assume that $\mathcal{T} = \widehat{\mathbb{A}}^d_\mathcal{S}$ is a formal affine space over $\mathcal{S}$ with coordinates $x_1, \cdots, x_d$ such that $T = S$ is included in the zero section of $\mathcal{T}$ over $\mathcal{S}$. Applying Proposition 2.2.1, we have only to compare stalks of both sides at a K-rational point $t \in]T[_\mathcal{T}$ with $x_i(t) = 0$ for all i after a suitable change of coordinates.
Let $\mathcal{T}_n = \operatorname{Spf} \mathcal{V}[x_1, \cdots, x_d]/(x_1^{n_1}, \cdots, x_d^{n_d}) \times_{\operatorname{Spf}\mathcal{V}} \mathcal{S}$ for $n = (n_1, \cdots, n_d)$ with $n_i > 0$ for all i and denote by $u_n : \mathfrak{T}_n = (T, \overline{T}, \mathcal{T}_n) \to \mathfrak{S}$ (resp. $w_n : \mathfrak{T}_n \to \mathfrak{T}$) the natural structure morphism. Observe a sequence of base change homomorphisms:

$$\widetilde{u}^*\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E \to \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*}v^*E \to \widetilde{w}_{n*}\mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}_n*}v_n^*E.$$

By the finite flat base change theorem (Theorem 2.1.1) the induced homomorphism

$$\widetilde{u}_n^*\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E \to \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}_n*}v_n^*E$$

is an isomorphism since the rigid cohomology is determined by the reduced subscheme. Hence, the base change homomorphism $\widetilde{u}^*\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E \to \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*}v^*E$ is injective.
Let us define an overconvergent isocrystal $F = v^*E/(x_1, \cdots, x_d)v^*E$ on $(Y, \overline{Y})/\mathcal{T}_K$ and observe a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\oplus_i \widetilde{u}^*\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E & \xrightarrow{\oplus_i x_i} & \widetilde{u}^*\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E & \longrightarrow & \widetilde{w}_{1*}\widetilde{u}_1^*\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\oplus_i \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*}v^*E & \xrightarrow{\oplus_i x_i} & \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*}v^*E & \longrightarrow & \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*}F, & &
\end{array}$$

where the subscript 1 of u_1 and w_1 means the multi-index $(1, \cdots, 1)$. Indeed, one can prove $\mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*}((x_1, \cdots, x_d)v^*E) \cong (x_1, \cdots, x_d)\mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*}v^*E$ inductively since $x_i v^*E \cong v^*E$ as overconvergent isocrystals. Hence the bottom row is exact. By the finite flat base change theorem we have $\widetilde{u}_1^*\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E \cong$

$\mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}_1 *} v^* E$. Since $\widetilde{w}_1 :]\overline{T}[_{\mathcal{T}_1} \to]\overline{T}[_{\mathcal{T}}$ is a closed immersion of rigid analytic spaces, $R^q \widetilde{w}_{1*} \mathcal{F} = 0 \, (q > 0)$ for any sheaf $\mathcal{F}$ of coherent $\mathcal{O}_{]\overline{T}[_{\mathcal{T}_1}}$-modules. Hence the right vertical arrow is an isomorphism.

Let $\mathcal{G}$ be a cokernel of the base change homomorphism $\widetilde{u}^* \mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*} E \to \mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*} v^* E$. By the snake lemme we have

$$\mathcal{G} = (x_1, \cdots, x_d) \mathcal{G}.$$

Since $\mathcal{G}$ is coherent and the ideal $(x_1, \cdots, x_d) \mathcal{O}_{]\overline{T}[_{\mathcal{T}}, t}$ is included in the unique maximal ideal of the stalk $\mathcal{O}_{]\overline{T}[_{\mathcal{T}}, t}$ of $\mathcal{O}_{]\overline{T}[_{\mathcal{T}}}$ at t, the stalk $\mathcal{G}_t$ vanishes by Nakayama's lemma. Hence the homomorphism between stalks at t which is induced by the base change homomorphism is an isomorphism. This completes the proof. $\square$

2.3.3 Corollary

With notation in 2.1, assume furthermore that the induced morphism $\mathcal{T} \to \mathcal{S} \times_{\mathrm{Spf}\, \mathcal{V}} \mathrm{Spf}\, \mathcal{W}$ is smooth around T. Suppose that, for an integer q and an overconvergent isocrystal E on $(X, \overline{X})/\mathcal{S}_K$, the condition $(\mathrm{F})^q_{f, \mathcal{W}/\mathcal{V}, E}$ holds and $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*} E$ (resp. $\mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*} v^* E$) is a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{S}[_{\mathcal{S}}}$-modules (resp. a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{T}[_{\mathcal{T}}}$-modules). Then the base change homomorphism

$$\widetilde{u}^* \mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*} E \to \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*} v^* E$$

is an isomorphism.

3 A condition for the overconvergence of Gauss-Manin connections

3.1 The condition (C)

Let $\mathfrak{S} = (S, \overline{S}, \mathcal{S})$ be a $\mathcal{V}$-triple separated of finite type and let $(X, \overline{X})$ be a pair separated of finite type over $(S, \overline{S})$ with structure morphism $f : (X, \overline{X}) \to (S, \overline{S})$.

Let E be an overconvergent isocrystal on $(X, \overline{X})/\mathcal{S}_K$ and let q be an integer. We say that the condition $(\mathrm{C})^q_{f, \mathfrak{S}, E}$ holds if and only if, for any $\mathcal{V}$-morphism $\widehat{u} : \mathcal{T} \to \mathcal{S}$ separated of finite type with a closed immersion $\overline{S} \to \mathcal{T}$ over $\mathcal{S}$ such that $\widehat{u}$ is smooth around S, the rigid cohomology $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*} v^* E$ with respect to $\mathfrak{T} = (S, \overline{S}, \mathcal{T})$ is a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{S}[_{\mathcal{T}}}$-modules.

Since an open covering (resp. a finite closed covering) of $\overline{S}$ induces an admissible covering of $]\overline{S}[_{\mathcal{S}}$ [5, Proposition 1.1.14], we have the proposition below by the gluing lemma.

3.1.1 Proposition

Let $u : \mathfrak{S}' \to \mathfrak{S}$ be a separated morphism of $\mathcal{V}$-triples locally of finite type such that $S' = \overline{u}^{-1}(S)$, and let

$$\begin{array}{ccc} (X, \overline{X}) & \xleftarrow{v} & (X', \overline{X}') \\ f \downarrow & & \downarrow f' \\ (S, \overline{S}) & \xleftarrow[u]{} & (S', \overline{S}') \end{array}$$

be a cartesian diagram of pairs. Let E be an overconvergent isocrystal on $(X, \overline{X})/\mathcal{S}_K$ and let $E' = v^*E$ be the inverse image on $(X', \overline{X}')/\mathcal{S}'_K$.

(1) *Suppose one of the situations* (i) *and* (ii).

  (i) $\widehat{u} : \mathcal{S}' \to \mathcal{S}$ *is an open immersion and* $\overline{S}' = \widehat{u}^{-1}(\overline{S})$.

  (ii) $\overline{u} : \overline{S}' \to \overline{S}$ *is a closed immersion and* $\overline{S}' \to \mathcal{S}' = \mathcal{S}$ *is the natural closed immersion.*

  Then, the condition $(\mathrm{C})^q_{f,\mathfrak{S},E}$ *implies the condition* $(\mathrm{C})^q_{f',\mathfrak{S}',E'}$.

(2) *Suppose one of the situations* (iii) *and* (iv).

  (iii) $\widehat{u} : \mathcal{S}' \to \mathcal{S}$ *is an open covering and* $\overline{S}' = \widehat{u}^{-1}(\overline{S})$.

  (iv) $\overline{u} : \overline{S}' \to \overline{S}$ *is a finite closed covering and the closed immersion* $\overline{S}' \to \mathcal{S}'$ *is a disjoint sum of the natural closed immersion into* $\mathcal{S}$ *for each component of* $\overline{S}'$.

  Then, the condition $(\mathrm{C})^q_{f,\mathfrak{S},E}$ *holds if and only if the condition* $(\mathrm{C})^q_{f',\mathfrak{S}',E'}$ *holds.*

3.2 The overconvergence of Gauss-Manin connections

Let $\mathfrak{S} = (S, \overline{S}, \mathcal{S})$ be a $\mathcal{V}$-triple separated of finite type and let $\mathfrak{T} = (T, \overline{T}, \mathcal{T})$ be a $\mathfrak{S}$-triple separated of finite type such that $\mathcal{T} \to \mathcal{S}$ is smooth around T. Let $f : (X, \overline{X}) \to (T, \overline{T})$ be a morphism of pairs separated of finite type. Then, for an overconvergent isocrystal E on $(X, \overline{X})/\mathcal{S}_K$, we have an integrable connection

$$\nabla^{\mathrm{GM}} : \mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E \to \mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E \otimes_{j^\dagger \mathcal{O}_{]\overline{T}[_\mathcal{T}}} j^\dagger \Omega^1_{]\overline{T}[_\mathcal{T}/]\overline{S}[_\mathcal{S}}$$

of sheaves of $j^\dagger \mathcal{O}_{]\overline{T}[_\mathcal{T}}$-modules over $j^\dagger \mathcal{O}_{]\overline{S}[_\mathcal{S}}$, which is called the Gauss-Manin connection and constructed as follows (cf. [16]). Here $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E$ needs not be coherent and the integrable connection means a $j^\dagger \mathcal{O}_{]\overline{S}[_\mathcal{S}}$-homomorphism ∇ such that $\nabla(ae) = a\nabla(e) + e \otimes da$ for $e \in E, a \in j^\dagger \mathcal{O}_{]\overline{T}[_\mathcal{T}}$ and such that $\nabla^2 = 0$. Let us take a formal $\mathcal{V}$-scheme $\mathcal{X}$ separated of finite type over $\mathcal{T}$ with a $\mathcal{T}$-closed immersion $\overline{X} \to \mathcal{X}$ such that the structure morphism $\widehat{f} : \mathcal{X} \to \mathcal{T}$ is smooth around X. In general, one can not take such a global $\mathcal{X}$ and one needs to take a Zariski covering $\mathfrak{U}$ of $(X, \overline{X})$ over $\mathfrak{S}$ in order to define the rigid cohomology. For simplicity, we assume here that there exists a global $\mathcal{X}$. The following construction also works if one replaces the triple $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ by the Čech diagram $\mathfrak{U}_.$ of $\mathfrak{U}$ as $(X, \overline{X})$-triples over $\mathfrak{S}$. (See [9, Sect. 10].)

Let $\mathrm{DR}^\dagger(\mathfrak{X}/\mathfrak{S}, (E_\mathfrak{X}, \nabla_\mathfrak{X}))$ be the de Rham complex associated to the realization $(E_\mathfrak{X}, \nabla_\mathfrak{X})$ of E with respect to $\mathfrak{X}$ and let us define a decreasing filtration $\{\mathrm{Fil}^q\}_q$ of $\mathrm{DR}^\dagger(\mathfrak{X}/\mathfrak{S}, (E_\mathfrak{X}, \nabla_\mathfrak{X}))$ by

$$\begin{aligned}\mathrm{Fil}^q = \mathrm{Image}(\mathrm{DR}^\dagger(\mathfrak{X}/\mathfrak{S}, (E_\mathfrak{X}, \nabla_\mathfrak{X}))[-q] \otimes_{\widetilde{f}^{-1} j^\dagger \mathcal{O}_{]\overline{T}[_\mathcal{T}}} \widetilde{f}^{-1} j^\dagger \Omega^q_{]\overline{T}[_\mathcal{T}/]\overline{S}[_\mathcal{S}} \\ \to \mathrm{DR}^\dagger(\mathfrak{X}/\mathfrak{S}, (E_\mathfrak{X}, \nabla_\mathfrak{X})))\end{aligned}$$

for any q, where $[-q]$ means the $-q$-th shift of the complex. Since

$$0 \to \widetilde{f}^* j^\dagger \Omega^1_{]\overline{T}[_\mathcal{T}/]\overline{S}[_\mathcal{S}} \to j^\dagger \Omega^1_{]\overline{X}[_\mathcal{X}/]\overline{S}[_\mathcal{S}} \to j^\dagger \Omega^1_{]\overline{X}[_\mathcal{X}/]\overline{T}[_\mathcal{T}} \to 0$$

is an exact sequence of sheaves of locally free $j^\dagger \mathcal{O}_{]\overline{X}[_\mathcal{X}/]\overline{S}[_\mathcal{S}}$-modules of finite type, the filtration $\{\mathrm{Fil}^q\}_q$ is well-defined and we have

$$\mathrm{gr}^q_{\mathrm{Fil}} = \mathrm{Fil}^q/\mathrm{Fil}^{q+1} = \mathrm{DR}^\dagger(\mathfrak{X}/\mathfrak{T}, (E_\mathfrak{X}, \overline{\nabla}_\mathfrak{X}))[-q] \otimes_{\widetilde{f}^{-1} j^\dagger \mathcal{O}_{]\overline{T}[_\mathcal{T}}} \widetilde{f}^{-1} j^\dagger \Omega^q_{]\overline{T}[_\mathcal{T}/]\overline{S}[_\mathcal{S}},$$

where $\overline{\nabla}_\mathfrak{X}$ is the connection induced by the composition

$$E_\mathfrak{X} \xrightarrow{\nabla_\mathfrak{X}} E_\mathfrak{X} \otimes_{j^\dagger \mathcal{O}_{]\overline{X}[_\mathcal{X}}} j^\dagger \Omega^1_{]\overline{X}[_\mathcal{X}/]\overline{S}[_\mathcal{S}} \longrightarrow E_\mathfrak{X} \otimes_{j^\dagger \mathcal{O}_{]\overline{X}[_\mathcal{X}}} j^\dagger \Omega^1_{]\overline{X}[_\mathcal{X}/]\overline{T}[_\mathcal{T}}.$$

From this decreasing filtration we have a spectral sequence

$$\underline{E}_1^{qr} = \mathbb{R}^{q+r} \widetilde{f}_* \mathrm{gr}^q_{\mathrm{Fil}} \Rightarrow \mathbb{R}^{q+r} \widetilde{f}_* \mathrm{DR}^\dagger(\mathfrak{X}/\mathfrak{S}, (E_\mathfrak{X}, \nabla_\mathfrak{X})),$$

where

$$\begin{aligned}\underline{E}_1^{qr} &= \mathbb{R}^r \widetilde{f}_* (\mathrm{DR}^\dagger(\mathfrak{X}/\mathfrak{T}, (E_\mathfrak{X}, \overline{\nabla}_\mathfrak{X})) \otimes_{\widetilde{f}^{-1} j^\dagger \mathcal{O}_{]\overline{T}[_\mathcal{T}}} \widetilde{f}^{-1} j^\dagger \Omega^q_{]\overline{T}[_\mathcal{T}/]\overline{S}[_\mathcal{S}}) \\ &\cong \mathbb{R}^r f_{\mathrm{rig}\mathfrak{T}*} E \otimes_{j^\dagger \mathcal{O}_{]\overline{T}[_\mathcal{T}}} j^\dagger \Omega^q_{]\overline{T}[_\mathcal{T}/]\overline{S}[_\mathcal{S}}.\end{aligned}$$

Then the Gauss-Manin connection $\nabla^{\mathrm{GM}} : \mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*} E \to \mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*} E \otimes_{j^\dagger \mathcal{O}_{]\overline{T}[_\mathcal{T}}} j^\dagger \Omega^1_{]\overline{T}[_\mathcal{T}/]\overline{S}[_\mathcal{S}}$ is defined by the differential

$$d_1^{0r} : \underline{E}_1^{0r} \to \underline{E}_1^{1r}.$$

Indeed, one can check that d_1^{0r} is an integrable connection by an explicit calculation (see [16, Sect. 3]).

3.2.1 Theorem

Let E be an overconvergent isocrystal on $(X, \overline{X})/\mathcal{S}_K$ and let q be an integer. If the condition $(\mathrm{C})^q_{f,\mathfrak{T},E}$ holds, then the Gauss-Manin connection $\nabla^{\mathrm{GM}} : \mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E \to \mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E \otimes_{j^\dagger \mathcal{O}_{]\overline{T}[_{\mathcal{T}}}} j^\dagger \Omega^1_{]\overline{T}[_{\mathcal{T}}/]\overline{S}[_{\mathcal{S}}}$ is overconvergent along $\partial T = \overline{T} \setminus T$.

Proof. Let us put $\mathfrak{X}^2 = (X, \overline{X}, \mathcal{X} \times_{\mathcal{S}} \mathcal{X})$, denote by $p_{\mathfrak{X}i} : \mathfrak{X}^2 \to \mathfrak{X}$ the i-th projection for $i = 1, 2$, and the same for $\mathfrak{T}$. By definition, the overconvergent connection $\nabla_{\mathfrak{X}}$ is induced from an isomorphism

$$\epsilon_{\mathfrak{X}} : \widetilde{p}^*_{\mathfrak{X}1}E \to \widetilde{p}^*_{\mathfrak{X}2}E$$

of sheaves of $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}^2}}$-modules which satisfies the cocycle condition [5, Definition 2.2.5]. Consider the commutative diagram

$$\begin{array}{ccccccc}
\mathfrak{X}_{\mathfrak{T}1} & = & (X, \overline{X}, \mathcal{X} \times_{\mathcal{S}} \mathcal{T}) & \to & \mathfrak{X}^2 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathfrak{T}^2 & \to & (X, \overline{X}, \mathcal{T} \times_{\mathcal{S}} \mathcal{X}) & = & \mathfrak{X}_{\mathfrak{T}2}
\end{array}$$

of triples. Then the rigid cohomology $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}^2*}E$ can be calculated as the hypercohomology of the de Rham complex by using any of $\mathfrak{X}^2$, $\mathfrak{X}_{\mathfrak{T}1}$ and $\mathfrak{X}_{\mathfrak{T}2}$. Hence, we have an isomorphism

$$\epsilon_{\mathfrak{T}} : \widetilde{p}^*_{\mathfrak{T}1}\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E \cong \mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}^2*}E \xrightarrow[\cong]{\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}^2*}(\epsilon_{\mathfrak{X}})} \mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}^2*}E \cong \widetilde{p}^*_{\mathfrak{T}2}\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E$$

of sheaves of $j^\dagger \mathcal{O}_{]\overline{T}[_{\mathcal{T}^2}}$-modules which satisfies the cocycle condition by $(\mathrm{C})^q_{f,\mathfrak{T},E}$ (Proposition 2.3.1). By an explicit calculation, the Gauss-Manin connection ∇^{GM} is induced from the isomorphism $\epsilon_{\mathfrak{T}}$ (see [3, Capt.4, Proposition 3.6.4]). Therefore, ∇^{GM} is an overconvergent connection along $\partial T = \overline{T} \setminus T$. □

3.2.2 Proposition

Let $w : \mathfrak{T}' \to \mathfrak{T}$ be a morphism separated of finite type over $\mathfrak{S}$ which satisfies the conditions

(i) $\overset{\circ}{w} : T' \to T$ *is an isomorphism;*

(ii) $\overline{w} : \overline{T}' \to \overline{T}$ *is proper;*

(iii) $\widehat{w} : \mathcal{T}' \to \mathcal{T}$ *is smooth around T',*

and let $f' : (X', \overline{X}') \to (T', \overline{T}')$ be the base extension of $f : (X, \overline{X}) \to (T, \overline{T})$ by $w : (T, \overline{T}) \to (T', \overline{T}')$. Let q be an integer and let E (resp. E') be an overconvergent isocrystal on $(X, \overline{X})/\mathcal{S}_K$ (resp. the inverse image of E on $(X', \overline{X}')/\mathcal{S}_K$).

(1) *If the condition $(\mathrm{C})^q_{f,\mathfrak{T},E}$ holds, then the condition $(\mathrm{C})^q_{f',\mathfrak{T}',E'}$ holds.*

(2) *If the condition* $(\mathrm{C})^{q'}_{f',\mathfrak{T}',E'}$ *holds for all* $q' \leq q$*, then the condition* $(\mathrm{C})^{q}_{f,\mathfrak{T},E}$ *holds.*

In both cases, the base change homomorphism

$$\widetilde{w}^*\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E \to \mathbb{R}^q f'_{\mathrm{rig}\mathfrak{T}'*}E'$$

is an isomorphism with respect to connections.

Proof. If $\widehat{w} : \mathcal{T}' \to \mathcal{T}$ is etale around T', then there are strict neighborhoods of $]T[_{\mathcal{T}}$ and $]T'[_{\mathcal{T}'}$ (resp. $]X[_{\mathcal{X}}$ and $]X'[_{\mathcal{X}'}$, where $\mathcal{X}' = \mathcal{X} \times_{\mathcal{T}} \mathcal{T}'$) which are isomorphic [5, Théorème 1.3.5]. Using the argument of the proof of [5, Théorème 2.3.5], we may assume that $\mathcal{T}'$ is a formal affine space over $\mathcal{T}$ and $\overline{w} : \overline{T}' \to \overline{T}$ is an isomorphism by Proposition 3.1.1. Then the assertion (1) follows from Proposition 2.3.1.

Now we prove the assertion (2). We may assume that $\mathcal{T}'$ is a formal affine line over $\mathcal{T}$ by induction. Since the equivalence between categories of realizations of overconvergent isocrystals with respect to $\mathcal{T}$ and $\mathcal{T}'$ is given by the functors w^* and $\mathbb{R}^0 w_{\mathrm{rig}\mathfrak{T}*}$ [5, Théorème 2.3.5] [9, Proposition 8.3.5], $\mathbb{R}^0 w_{\mathrm{rig}\mathfrak{T}*}\mathbb{R}^{q'} f'_{\mathrm{rig}\mathfrak{T}'*}E'$ is a sheaf of coherent $j^\dagger\mathcal{O}_{]\overline{T}[_{\mathcal{T}}}$-modules with an overconvergent connection for $q' \leq q$ by Theorem 3.2.1. Moreover, the canonical homomorphism

$$\mathbb{R}^0 w_{\mathrm{rig}\mathfrak{T}*}\mathbb{R}^{q'} f'_{\mathrm{rig}\mathfrak{T}'*}E' \to \mathbb{R}\widetilde{w}_*\mathrm{DR}^\dagger(\mathfrak{T}'/\mathfrak{T}, (\mathbb{R}^{q'} f'_{\mathrm{rig}\mathfrak{T}'*}E', \nabla^{\mathrm{GM}}))$$

is an isomorphism for $q' \leq q$.

Let us put $C^\bullet = \mathbb{R}\widetilde{f'}_*\mathrm{DR}^\dagger(\mathfrak{X}'/\mathfrak{T}', (E'_{\mathfrak{X}'}, \nabla_{\mathfrak{X}'}))$ and $D^\bullet = C^\bullet \otimes_{j^\dagger\mathcal{O}_{]\overline{T}'[_{\mathcal{T}'}}} j^\dagger\Omega^1_{]\overline{T}'[_{\mathcal{T}'}/]\overline{T}[_{\mathcal{T}}}$. Observe the filtration of $\mathrm{DR}^\dagger(\mathfrak{X}'/\mathfrak{T}, (E'_{\mathfrak{X}'}, \nabla_{\mathfrak{X}'}))$ with respect to w and f' in 3.2. Then

$$\mathbb{R}\widetilde{f'}_*\mathrm{DR}^\dagger(\mathfrak{X}'/\mathfrak{T}, (E'_{\mathfrak{X}'}, \nabla_{\mathfrak{X}'})) \cong \mathrm{Cone}(C^\bullet \to D^\bullet)[-1].$$

since $\mathcal{T}'$ is an affine line over $\mathcal{T}$. Let us denote by $C^{\bullet > i}$ (resp. $C^{\bullet \geq i}$) a subcomplex of $(C^\bullet, d^\bullet)$ defined by $(C^{\bullet>i})^j = 0\,(j < i-1), (C^{\bullet>i})^i = C^i/\mathrm{Ker}\, d^i$ and $(C^{\bullet>i})^j = C^j\ (j > i)$ (resp. $(C^{\bullet\geq i})^j = 0\,(j < i-1), (C^{\bullet\geq i})^i = C^i/\mathrm{Im}\, d^{i-1}$ and $(C^{\bullet\geq i})^j = C^j\ (j > i)$) and the same for $D^\bullet$. Then

$$\begin{aligned}&\mathrm{DR}^\dagger(\mathfrak{T}'/\mathfrak{T}, (\mathbb{R}^t f'_{\mathrm{rig}\mathfrak{T}'*}E', \nabla^{\mathrm{GM}}))[-t]\\ &\quad\cong \mathrm{Cone}(\mathrm{Cone}(C^{\bullet\geq t} \to D^{\bullet\geq t})[-1] \to \mathrm{Cone}(C^{\bullet>t} \to D^{\bullet>t})[-1])[-1]\end{aligned}$$

for any t. Hence we have an isomorphism

$$\mathbb{R}^{q'}(wf')_{\mathrm{rig}\mathfrak{T}*}E' \cong \mathbb{R}^0 w_{\mathrm{rig}\mathfrak{T}*}\mathbb{R}^{q'} f'_{\mathrm{rig}\mathfrak{T}'*}E'$$

for any $q' \leq q$ inductively.

On the contrary, if we denote by $v : (X', \overline{X}') \to (X, \overline{X})$ the structure morphism, then the spectral sequence

$$\underline{E}_1^{st} = \mathbb{R}^t v_{\mathrm{rig}\mathfrak{X}*}E' \otimes_{j^\dagger\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}} j^\dagger\Omega^s_{]\overline{X}[_{\mathcal{X}}/]\overline{T}[_{\mathcal{T}}} \Rightarrow \mathbb{R}^{s+t}\widetilde{v}_*\mathrm{DR}^\dagger(\mathfrak{X}'/\mathfrak{T}, (E'_{\mathfrak{X}'}, \nabla_{\mathfrak{X}'}))$$

with respect to f and v in 3.2 induces an isomorphism

$$\mathbb{R}^q(wf')_{\mathrm{rig}\mathfrak{T}*}E' \cong \mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E$$

since $\mathbb{R}^0 v_{\mathrm{rig}\mathfrak{X}*}E' = E$ and $\mathbb{R}^t v_{\mathrm{rig}\mathfrak{X}*}E' = 0\,(t > 0)$. Hence,

$$\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E \cong \mathbb{R}^0 w_{\mathrm{rig}\mathfrak{T}*}\mathbb{R}^q f'_{\mathrm{rig}\mathfrak{T}'*}E'$$

and $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E$ is a sheaf of coherent $j^\dagger\mathcal{O}_{]\overline{T}[_\mathcal{T}}$-modules. The same holds for any triple $\mathfrak{T}'' = (T, \overline{T}, \mathcal{T}'')$ separated of finite type over $\mathfrak{T}$ such that $\mathcal{T}'' \to \mathcal{T}$ is smooth around T. Therefore, the condition $(\mathrm{C})^q_{f,\mathfrak{T},E}$ holds. □

3.3 Rigid cohomology as overconvergent isocrystals

Let $\mathfrak{S} = (S, \overline{S}, \mathcal{S})$ be a $\mathcal{V}$-triple separated of finite type and let

$$(X, \overline{X}) \xrightarrow{f} (T, \overline{T}) \xrightarrow{u} (S, \overline{S})$$

be morphisms of pairs separated of finite type over $\mathrm{Spec}\, k$.
As a consequence of Theorem 3.2.1, we have a criterion of the overconvergence of Gauss-Manin connections by the gluing lemma and Proposition 3.2.2.

3.3.1 Theorem

Let E be an overconvergent isocrystal on $(X, \overline{X})/\mathcal{S}_K$ and let q be an integer. Suppose that, for each $q' \leq q$, there exists a triple $\mathfrak{T}' = \coprod (T_i, \overline{T}_i, \mathcal{T}_i)$ separated of finite type over $\mathfrak{S}$ which satisfies the conditions

(i) $\overline{T}' = \coprod \overline{T}_i \to \overline{T}$ *is an open covering;*

(ii) T' *is the pull back of* T *in* $\overline{T}'$*;*

(iii) $\mathcal{T}' = \coprod \mathcal{T}_i \to \mathcal{T}$ *is smooth around* T'*,*

(iv) *the condition* $(\mathrm{C})^{q'}_{f',\mathfrak{T}',E'}$ *holds, where* $f' : (X', \overline{X}') \to (T', \overline{T}')$ *denotes the extension of* f *and* E' *is the inverse image of* E *on* $(X', \overline{X}')/\mathcal{S}_K$*.*

Then the rigid cohomology $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E$ with the Gauss-Manin connection ∇^{GM} is a realization of an overconvergent isocrystal on $(T, \overline{T})/\mathcal{S}_K$. Moreover, the overconvergent isocrystal on $(T, \overline{T})/\mathcal{S}_K$ does not depend on the choice of $\mathfrak{T}'$.

Under the assumption of Theorem 3.3.1, we define the q-th rigid cohomology overconvergent isocrystal $R^q f_{\mathrm{rig}*}E$ as the overconvergent isocrystal on $(T, \overline{T})/\mathcal{S}_K$ in the theorem above.

3.3.2 Proposition

With notation as before, we have the following results.

(1) *Let $E \to F$ be a homomorphism of overconvergent isocrystals on $(X, \overline{X})/\mathcal{S}_K$ and let q be an integer. Suppose that, for each $q' \leq q$ and each E and F, there exists a triple $\mathfrak{T}'$ such that the conditions* (i) - (iv) *in Theorem 3.3.1 holds. Then there is a homomorphism $R^q f_{\mathrm{rig}*}E \to R^q f_{\mathrm{rig}*}F$ of overconvergent isocrystals on $(T, \overline{T})/\mathcal{S}_K$. This homomorphism commutes with the composition.*

(2) *Let $0 \to E \to F \to G \to 0$ be an exact sequence of overconvergent isocrystals on $(X, \overline{X})/\mathcal{S}_K$. Suppose that, for each q and each E, F and G, there exists a triple $\mathfrak{T}'$ such that the conditions* (i) - (iv) *in Theorem 3.3.1 holds. Then there is a connecting homomorphism $R^q f_{\mathrm{rig}*}G \to R^{q+1} f_{\mathrm{rig}*}E$ of overconvergent isocrystals on $(T, \overline{T})/\mathcal{S}_K$. This connecting homomorphism is functorial. Moreover, there is a long exact sequence*

$$\begin{array}{cccccccc} 0 & \to & f_{\mathrm{rig}*}E & \to & f_{\mathrm{rig}*}F & \to & f_{\mathrm{rig}*}G \\ & \to & R^1 f_{\mathrm{rig}*}E & \to & R^1 f_{\mathrm{rig}*}F & \to & R^1 f_{\mathrm{rig}*}G \\ & \to & R^2 f_{\mathrm{rig}*}E & \to & \cdots & & \end{array}$$

of overconvergent isocrystals on $(T, \overline{T})/\mathcal{S}_K$.

Proof. Since the induced homomorphism (resp. the connecting homomorphism) commutes with the isomorphism ϵ in the proof of Theorem 3.2.1 by [9, Propositions 4.2.1, 4.2.3], the assertions hold by [5, Corollaire 2.1.11, Propositions 2.2.7]. □

3.3.3 Proposition

With the situation as in Theorem 3.3.1, assume furthermore that the residue field k of K is perfect, there is a Frobenius endomorphism σ on K, and $\mathfrak{S} =$ $(\mathrm{Spec}\, k, \mathrm{Spec}\, k, \mathrm{Spf}\, \mathcal{V})$. Let E be an overconvergent F-isocrystal on $(X, \overline{X})/K$ and let q be an integer. Suppose that, for each $q' \leq q$, there exists a triple $\mathfrak{T}'$ such that the conditions (i) - (iv) *in Theorem 3.3.1 holds and suppose that, for each closed point t, the Frobenius endomorphism*

$$\sigma_t^* H^q_{\mathrm{rig}}((X_t, \overline{X}_t)/K_t, E_t) \to H^q_{\mathrm{rig}}((X_t, \overline{X}_t)/K_t, E_t)$$

is an isomorphism. Then the rigid cohomology sheaf $R^q f_{\mathrm{rig}*}E$ is an overconvergent F-isocrystal on $(T, \overline{T})/K$. Here $f_t : (X_t, \overline{X}_t) \to (t, t)$ is the fiber of $f : (X, \overline{X}) \to (T, \overline{T})$ at t, K_t is a unramified extension of K corresponding to the residue extension $k(t)/k$, $\sigma_t : K_t \to K_t$ is the unique extension of the Frobenius endomorphism σ and E_t is the inverse image of E on $(X_t, \overline{X}_t)/K_t$.

Proof. Let σ_X (resp. σ_T) be an absolute Frobenius on $(X,\overline{X})$ (resp. $(T,\overline{T})$). Then the Frobenius isomorphism $\sigma_X^*E \to E$ induces a Frobenius homomorphism

$$\sigma_T^* R^q f_{\mathrm{rig}*}E \to R^q f_{\mathrm{rig}*}E$$

of overconvergent isocrystals on $(T,\overline{T})/K$ by Theorem 3.3.1 and Proposition 3.3.2. We have only to prove that the Frobenius homomorphism is an isomorphism when $\overline{T} = T = t$ for a k-rational point t and $K_t = K$ by Proposition 3.2.2 and the same reason as in the proof of Proposition 2.3.1.

Let us put $\mathcal{T} = (t, t, \mathrm{Spf}\,\mathcal{V})$. The realization of the overconvergent isocrystal $R^q f_{\mathrm{rig}*}E$ on t/K with respect to $\mathfrak{T}$ is $H^q_{\mathrm{rig}}((X,\overline{X})/K, E)$. Hence, the assertion follows from the hypothesis. □

3.4 The Leray spectral sequence

We apply the construction of the Leray spectral sequence in [15, Remark 3.3] to our relative rigid cohomology cases.

3.4.1 Theorem

With notation as in 3.3, suppose that, for each integer q, there exists a triple $\mathfrak{T}'$ such that the conditions (i) - (iv) *in Theorem 3.3.1 hold. Then there exists a spectral sequence*

$$\underline{E}_2^{qr} = \mathbb{R}^q u_{\mathrm{rig}\mathfrak{S}*}(\mathbb{R}^r f_{\mathrm{rig}*}E) \Rightarrow \mathbb{R}^{q+r}(u\circ f)_{\mathrm{rig}\mathfrak{S}*}E$$

of sheaves of $j^\dagger\mathcal{O}_{]\overline{S}[_{\mathcal{S}}}$-modules.

Proof. Let $\mathfrak{Y} = (Y, \overline{Y}, \mathcal{Y})$ (resp. $\mathfrak{U} = (U, \overline{U}, \mathcal{U})$) be a Zariski covering of $(X,\overline{X})$ (resp. $(T,\overline{T})$) over $\mathfrak{S}$ with a morphism $\mathfrak{Y} \to \mathfrak{U}$ of triples over $\mathfrak{S}$ such that $\mathcal{Y} \to \mathcal{U}$ is smooth around Y. Let $\mathfrak{Y}_.$ (resp. $\mathfrak{U}_.$) be the Čech diagram as $(X,\overline{X})$-triples (resp. $(T,\overline{T})$-triples) over $\mathfrak{S}$ associated to the $(X,\overline{X})$-triple $\mathfrak{Y}$ (resp. the $(T,\overline{T})$-triple $\mathfrak{U}$) over $\mathfrak{S}$ and let us denote by

$$\mathfrak{Y}_. \xrightarrow{g_.} \mathfrak{U}_. \xrightarrow{v_.} \mathfrak{S}$$

the structure morphisms. The Čech diagram $\mathfrak{Y}_.$ (resp. $\mathfrak{U}_.$) is a universally de Rham descendable hypercovering of $(X,\overline{X})$ (resp. $(T,\overline{T})$) over $\mathfrak{S}$ [9, Sect. 10.1].

Let us consider the filtration $\{\mathrm{Fil}^q\}_q$ of $\mathrm{DR}^\dagger(\mathfrak{Y}_./\mathfrak{S}, (E_{\mathfrak{Y}_.}, \nabla_{\mathfrak{Y}_.}))$ which is defined in 3.2 and take a finitely filtered injective resolution

$$\mathrm{DR}^\dagger(\mathfrak{Y}_./\mathfrak{S}, (E_{\mathfrak{Y}_.}, \nabla_{\mathfrak{Y}_.})) \to I^\cdot_.$$

as complexes of abelian sheaves on $]\overline{Y}_.[_{\mathcal{Y}_.}$, that is, $\mathrm{Fil}^q I^\cdot_.$ (resp. $\mathrm{gr}^q_{\mathrm{Fil}} I^\cdot_.$) is an injective resolution. Let

$$\widetilde{g}_{.*} I^\cdot_. \to M^{\cdot\cdot}_.$$

be a finitely filtered resolution as complexes of abelian sheaves on $]\overline{U}_\cdot[_{\mathcal{U}_\cdot}$ such that

(i) $M_s^{qr} = 0$ if one of q, r and s is less than 0;

(ii) $\mathrm{Fil}^i \widetilde{g}_{\cdot *} I^r_\cdot \to \mathrm{Fil}^i M^{\cdot r}_\cdot$ (resp. $\mathrm{gr}^i_{\mathrm{Fil}} \widetilde{g}_{\cdot *} I^r_\cdot \to \mathrm{gr}^i_{\mathrm{Fil}} M^{\cdot r}_\cdot$) is a resolution by $\widetilde{v}_{\cdot *}$-acyclic sheaves for any r;

(iii) the complex

$$\underline{H}^r(\mathrm{Fil}^i M^{0\cdot}_\cdot) \to \underline{H}^r(\mathrm{Fil}^i M^{1\cdot}_\cdot) \to \underline{H}^r(\mathrm{Fil}^i M^{2\cdot}_\cdot) \to \cdots$$

is a resolution of $\underline{H}^r(\mathrm{Fil}^i I^\cdot_\cdot)$ by $\widetilde{v}_{\cdot *}$-acyclic sheaves for any r, and the same for $\mathrm{gr}^i_{\mathrm{Fil}} I^\cdot_\cdot \to \mathrm{gr}^i_{\mathrm{Fil}} M^{\cdot\cdot}_\cdot$.

One can construct such a resolution $\widetilde{g}_{\cdot *} I^\cdot_\cdot \to M^{\cdot\cdot}_\cdot$ inductively on degrees and it is called a filtered C-E resolution in [15].
Now we define a filtration $\{F^i\}_i$ of $\widetilde{v}_{\cdot,*} M^{\cdot\cdot}_\cdot$ by

$$F^i M^{q\cdot}_\cdot = \mathrm{Fil}^{i-q} M^{q\cdot}_\cdot.$$

Let us consider a spectral sequence

$$(*) \qquad {}^F\underline{E}_1^{qr} = \underline{H}^{q+r}(\mathrm{gr}^q_F \mathrm{tot}(\widetilde{v}_{\cdot *} M^{\cdot\cdot}_\cdot)) \Rightarrow \underline{H}^{q+r}(\mathrm{tot}(\widetilde{v}_{\cdot *} M^{\cdot\cdot}_\cdot))$$

for the total complex of $\widetilde{v}_{\cdot *} M^{\cdot\cdot}_\cdot$ with respect to the filtration $\{F^i\}_i$. Since $\mathfrak{Y}_\cdot$ is a universally de Rham descendable hypercovering of $(X, \overline{X})$ over $\mathfrak{S}$, we have

$$\mathbb{R}^r(u \circ f)_{\mathrm{rig}\mathfrak{S}*} E \cong \underline{H}^r(\mathrm{tot}(\widetilde{v}_{\cdot *} M^{\cdot\cdot}_\cdot)).$$

by the definition of rigid cohomology in [9, Sect 10.4]. Let $({}^{\mathrm{Fil}}\underline{E}_1^{\cdot r}, d_1^{\cdot r})$ be the complex induced by the edge homomorphism of the spectral sequence

$${}^{\mathrm{Fil}}\underline{E}_1^{qr} = \underline{H}^{q+r}(\mathrm{gr}^q_{\mathrm{Fil}} \widetilde{g}_{\cdot *} I^\cdot_\cdot) \Rightarrow \underline{H}^{q+r}(\widetilde{g}_{\cdot *} I^\cdot_\cdot).$$

Then there is a resolution

$$({}^{\mathrm{Fil}}\underline{E}_1^{\alpha r}, d_1^{\alpha r})_\alpha \to \{\underline{H}^{\alpha+r}(\mathrm{gr}^\alpha_{\mathrm{Fil}} M^{\beta\cdot}_\cdot)\}_{\alpha,\beta}$$

by the double complex on $]\overline{U}_\cdot[_{\mathcal{U}_\cdot}$ by the condition (iii). The complex induced by the edge homomorphisms in the ${}^F\underline{E}_1$-stage of the spectral sequence $(*)$ is isomorphic to the total complex of $\{\widetilde{v}_{\cdot *}\underline{H}^{\alpha+r}(\mathrm{gr}^\alpha_{\mathrm{Fil}} M^{\beta\cdot}_\cdot)\}_{\alpha,\beta}$. Hence there is a spectral sequence

$${}^F\underline{E}_2^{qr} = \underline{H}^q(\mathrm{tot}(\mathbb{R}\widetilde{v}_{\cdot *}{}^{\mathrm{Fil}}\underline{E}_1^{\cdot r})) \Rightarrow \underline{H}^{q+r}(\mathrm{tot}(\widetilde{v}_{\cdot,*} M^{\cdot\cdot}_\cdot)).$$

Since the direct image overconvergent isocrystal $\mathbb{R}^r f_{\mathrm{rig}*} E$ on $(T, \overline{T})/\mathcal{S}_K$ exists by Theorem 3.3.1, we have

$$({}^{\mathrm{Fil}}\underline{E}_1^{\cdot r}, d_1^{\cdot r}) \cong \mathrm{DR}^\dagger(\mathfrak{U}_\cdot/\mathfrak{S}, ((\mathbb{R}^r f_{\mathrm{rig}*} E)_{\mathfrak{U}_\cdot}, \nabla^{\mathrm{GM}}_{\mathfrak{U}_\cdot}))$$

by Theorem 3.2.1, Proposition 3.2.2 and the definition of rigid cohomology. Here we also use the fact that an injective sheaf on $]\overline{U}_{\bullet}[_{\mathcal{U}_{\bullet}}$ consists of injective sheaves at each stage [9, Corollary 3.8.7]. Therefore, we have the Leray spectral sequence

$$\underline{E}_2^{qr} = \mathbb{R}^q u_{\mathrm{rig}\mathfrak{S}*}(\mathbb{R}^r f_{\mathrm{rig}*}E) \Rightarrow \mathbb{R}^{q+r}(u \circ f)_{\mathrm{rig}\mathfrak{S}*}E$$

in rigid cohomology. □

4 Examples of coherence

Let $\mathfrak{S} = (S, \overline{S}, \mathcal{S})$ be a $\mathcal{V}$-triple separated of finite type and let $(X, \overline{X}) \xrightarrow{f} (T, \overline{T}) \longrightarrow (S, \overline{S})$ be a sequence of morphisms of pairs separated of finite type over $\mathrm{Spec}\, k$. In order to see the existence of the overconvergent isocrystal $R^q f_{\mathrm{rig}*}E$, one has to show the coherence of direct images. Berthelot's conjecture [4, Sect. 4.3] asserts that, if $\overline{f}$ is proper, $X = \overline{f}^{-1}(T)$ and $\overset{\circ}{f}$ is smooth, then the rigid cohomology overconvergent (F-)isocrystal $R^q f_{\mathrm{rig}*}E$ exists for any q and any overconvergent (F-)isocrystal E on $(X, \overline{X})/\mathcal{S}_K$. In this section we discuss a generic coherence which relates to Berthelot's conjecture.

4.1 Liftable cases

The following Theorems 4.1.1 and 4.1.4 are due to Berthelot [4, Théorème 5]. We give a proof of the theorems along our studies in the previous sections.

4.1.1 Theorem

Suppose that there exists a commutative diagram

$$\begin{array}{ccccc} X & \to & \overline{X} & \to & \mathcal{X} \\ \overset{\circ}{f}\downarrow & & \overline{f}\downarrow & & \downarrow \widehat{f} \\ T & \to & \overline{T} & \to & \mathcal{T} \end{array}$$

of $\mathfrak{S}$-triples such that both squares are cartesian, $\widehat{f} : \mathcal{X} \to \mathcal{T}$ is proper and smooth around X and $\widehat{g} : \mathcal{T} \to \mathcal{S}$ is smooth around T. Then the condition $(\mathrm{C})^q_{f,\mathfrak{T},E}$ holds for any q and any overconvergent isocrystal E on $(X, \overline{X})/\mathcal{S}_K$. In particular, the rigid cohomology overconvergent isocrystal $R^q f_{\mathrm{rig}*}E$ on $(T, \overline{T})/\mathcal{S}_K$ exists. If $\mathfrak{S} = (\mathrm{Spec}\, k, \mathrm{Spec}\, k, \mathrm{Spf}\, \mathcal{V})$ and the relative dimension of X over T is less than or equal to d, then $R^q f_{\mathrm{rig}*}E = 0$ for $q > 2d$.
Moreover, the base change homomorphism is an isomorphism of overconvergent isocrystals for any base extension $(T', \overline{T}') \to (T, \overline{T})$ separated of finite type over $(S, \overline{S})$.

Proof. We may assume that $\mathcal{T}$ is affine. First we prove the coherence of $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E$. By the Hodge-de Rham spectral sequence

$$\underline{E}_1^{qr} = R^r \widetilde{f}_*(E \otimes_{j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}} j^\dagger \Omega^q_{]\overline{X}[_{\mathcal{X}}/]\overline{T}[_{\mathcal{T}}}) \Rightarrow \mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E$$

we have only to prove that $R^r\widetilde{f}_*(E\otimes_{j^\dagger\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}} j^\dagger\Omega^q_{]\overline{X}[_{\mathcal{X}}/]\overline{T}[_{\mathcal{T}}})$ is a sheaf of coherent $j^\dagger\mathcal{O}_{]\overline{T}[_{\mathcal{T}}}$-modules for any q and r. Since the second square is cartesian, the associated analytic map $\widetilde{f}:]\overline{X}[_{\mathcal{X}}\to]\overline{T}[_{\mathcal{T}}$ is quasi-compact. If $\{V\}$ is a filter of a fundamental system of strict neighbourhoods of $]T[_{\mathcal{T}}$ in $]\overline{T}[_{\mathcal{T}}$, then $\{\widetilde{f}^{-1}(V)\}$ is a filter of a fundamental system of strict neighbourhoods of $]\overline{X}[_{\mathcal{X}}$ since both squares are cartesian. Let us take a sheaf $\mathcal{E}$ of coherent $j^\dagger\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules with $E=j^\dagger\mathcal{E}$ ($\mathcal{E}$ is defined only on a strict neighbourhood in general). One can take a filter of a fundamental system $\{V\}$ of strict neighbourhoods of $]T[_{\mathcal{T}}$ in $]\overline{T}[_{\mathcal{T}}$ such that, if $j_V : V \to]\overline{T}[_{\mathcal{T}}$ (resp. $j_{\widetilde{f}^{-1}(V)} : \widetilde{f}^{-1}(V) \to]\overline{X}[_{\mathcal{X}}$) denotes the open immersion, then $R^qj_{V*}\mathcal{E}=0$ (resp. $R^qj_{\widetilde{f}^{-1}(V)*}\widetilde{f}^*\mathcal{E}=0$) by [9, Sect. 2.6, Proposition 5.1.1]. Since the direct limit commutes with cohomological functors by quasi-separatedness and quasi-compactness, we have

$$\begin{aligned}
R^r\widetilde{f}_*(E\otimes_{j^\dagger\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}} j^\dagger\Omega^q_{]\overline{X}[_{\mathcal{X}}/]\overline{T}[_{\mathcal{T}}})&\\
&\cong R^r\widetilde{f}_*j^\dagger(\mathcal{E}\otimes_{\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}}\Omega^q_{]\overline{X}[_{\mathcal{X}}/]\overline{T}[_{\mathcal{T}}})\\
&\cong R^r\widetilde{f}_*(\varinjlim_V j_{\widetilde{f}^{-1}(V)*}j^{-1}_{\widetilde{f}^{-1}(V)}(\mathcal{E}\otimes_{\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}}\Omega^q_{]\overline{X}[_{\mathcal{X}}/]\overline{T}[_{\mathcal{T}}}))\\
&\cong \varinjlim_V R^r\widetilde{f}_*(j_{\widetilde{f}^{-1}(V)*}j^{-1}_{\widetilde{f}^{-1}(V)}(\mathcal{E}\otimes_{\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}}\Omega^q_{]\overline{X}[_{\mathcal{X}}/]\overline{T}[_{\mathcal{T}}}))\\
&\cong \varinjlim_V R^r(\widetilde{f}j_{\widetilde{f}^{-1}(V)})_*j^{-1}_{\widetilde{f}^{-1}(V)}(\mathcal{E}\otimes_{\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}}\Omega^q_{]\overline{X}[_{\mathcal{X}}/]\overline{T}[_{\mathcal{T}}})\\
&\cong \varinjlim_V j_V^{-1}R^r(j_{V*}\widetilde{f})_*(\mathcal{E}\otimes_{\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}}\Omega^q_{]\overline{X}[_{\mathcal{X}}/]\overline{T}[_{\mathcal{T}}})\\
&\cong \varinjlim_V j_V^{-1}j_{V*}R^r\widetilde{f}_*(\mathcal{E}\otimes_{\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}}\Omega^q_{]\overline{X}[_{\mathcal{X}}/]\overline{T}[_{\mathcal{T}}})\\
&\cong j^\dagger R^r\widetilde{f}_*(\mathcal{E}\otimes_{\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}}\Omega^q_{]\overline{X}[_{\mathcal{X}}/]\overline{T}[_{\mathcal{T}}}).
\end{aligned}$$

Here $R^r\widetilde{f}_*(\mathcal{E}\otimes_{\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}}\Omega^q_{]\overline{X}[_{\mathcal{X}}/]\overline{T}[_{\mathcal{T}}})$ is coherent on a strict neighborhood of $]T[_{\mathcal{T}}$ in $]\overline{T}[_{\mathcal{T}}$ by Kiehl's finiteness theorem of cohomology of coherent sheaves [17, Theorem 3.3] and Chow's lemma. Hence, each E_1-term is coherent. The situation is unchanged after any extension $\mathcal{T}'\to\mathcal{T}$ smooth around T. Therefore, the condition $(\mathrm{C})^q_{f,\mathfrak{T},E}$ holds for any q and any E. Hence, the rigid cohomology overconvergent isocrystal $R^qf_{\mathrm{rig}*}E$ exists by Theorem 3.3.1.

Suppose that $\mathfrak{S}=(\operatorname{Spec}k,\operatorname{Spec}k,\operatorname{Spf}\mathcal{V})$ and the relative dimension of X over T is less than or equal to d. Then $R^qf_{\mathrm{rig}*}E$ is an overconvergent isocrystal on $(T,\overline{T})/K$. In order to prove the vanishing of $R^qf_{\mathrm{rig}*}E$ for $q>2d$, we have only to prove the assertion when $\overline{T}=T=t$ for a k-rational point t by Proposition 3.2.2 and the same reason as in the proof of Proposition 2.3.1. Let us put $\mathfrak{T}=(\operatorname{Spec}k,\operatorname{Spec}k,\operatorname{Spf}\mathcal{V})$. The realization of the overconvergent isocrystal $R^qf_{\mathrm{rig}*}E$ on t/K with respect to $\mathfrak{T}$ is $H^q_{\mathrm{rig}}((X,\overline{X})/K,E)$. Hence, the vanishing follows from Lemma 4.1.2 below.

Since the liftable situation is unchanged locally on base schemes, the base change homomorphism for $(T',\overline{T}') \to (T,\overline{T})$ is isomorphic as overconvergent isocrystals by Proposition 2.3.1. $\square$

4.1.2 Lemma

Let X be a smooth separated scheme of finite type over $\operatorname{Spec} k$, let Z be a closed subscheme of X, and let E be an overconvergent isocrystal on X/K. Suppose that X is of dimension d and Z is of codimension greater than or equal to e. Then the rigid cohomology $H^q_{Z,\mathrm{rig}}(X/K,E)$ with supports in Z vanishes for any $q > 2d$ and any $q < 2e$.

Proof. We use double induction on d and e, similar to the proof of [6, Théorème 3.1]. If $d = 0$, then the assertion is trivial. Suppose that the assertion holds for X with dimension less than d and suppose that $e = d$. Then we may assume that Z is a finite set of k-rational points by Proposition 2.2.2. Hence the assertion follows from the Gysin isomorphism

$$H^q_{Z,\mathrm{rig}}(X/K,E) \cong H^{q-2e}_{\mathrm{rig}}(Z/K,E_Z)$$

[20, Theorem 4.1.1]. Suppose that the assertion holds for any closed subscheme with codimension greater than e. By using the excision sequence

$$\begin{array}{llll} \cdots \to H^q_{Z',\mathrm{rig}}(X/K,E) & \to & H^q_{Z,\mathrm{rig}}(X/K,E) & \to & H^q_{Z\setminus Z',\mathrm{rig}}(X\setminus Z'/K,E) \\ \quad \to H^{q+1}_{Z',\mathrm{rig}}(X/K,E) & \to & \cdots, & & \end{array}$$

for a closed subscheme Z' of Z (see [6, Proposition 2.5] for the constant coefficients; the general case is similar), we may assume Z is irreducible. We may also assume that Z is absolutely irreducible by Proposition 2.2.2. Then there is an affine open subscheme U of X such that the inverse image Z_U of Z in U is smooth over $\operatorname{Spec} k$ after a suitable extension of k. Since $Z \setminus Z_U$ is of codimension greater than e, we may assume that Z is smooth over $\operatorname{Spec} k$ by the excision sequence and Proposition 2.2.2. Applying the Gysin isomorphism to $Z \subset X$, we have the assertion by the induction hypothesis if $e > 0$. Now suppose that $e = 0$. We may assume $X = Z$ by induction on the number of generic points of X. We may also assume that $X = Z$ is an affine smooth scheme over $\operatorname{Spec} k$ and we can find an affine smooth lift $\widetilde{X}$ of X over $\operatorname{Spec} \mathcal{V}$ by [10, Théorème 6]. Let $\mathcal{X}$ be the p-adic completion of the Zariski closure of $\widetilde{X}$ in a projective space over $\operatorname{Spec} \mathcal{V}$ and put $\overline{X} = \mathcal{X} \times_{\mathrm{Spf}\,\mathcal{V}} \operatorname{Spec} k$. Then $H^q(]\overline{X}[_{\mathcal{X}}, \mathcal{E}) = 0\,(q > 0)$ for any sheaf $\mathcal{E}$ of coherent $j^\dagger\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules by [9, Corollary 5.1.2]. Hence one can calculate the rigid cohomology by the complex of global sections of the de Rham complex associated to a realization of E. Therefore, $H^q_{\mathrm{rig}}(X/K,E) = 0$ for any $q > 2d$. This completes the proof. $\square$

Since the situation in Theorem 4.1.1 is unchanged by any extension $\mathcal{V} \to \mathcal{W}$ of complete discrete valuation rings, we have

4.1.3 Proposition

Under the assumption in Theorem 4.1.1, the condition $(\mathrm{F})^q_{f,\mathcal{W}/\mathcal{V},E}$ *holds for any* q*, any extension* $\mathcal{W}$ *of complete discrete valuation ring over* $\mathcal{V}$ *and any overconvergent isocrystal* E *on* $(X, \overline{X})/\mathcal{S}_K$*.*

We mention overconvergent F-isocrystals.

4.1.4 Theorem

With the situation as in Theorem 4.1.1, suppose $\mathfrak{S} = (\operatorname{Spec} k, \operatorname{Spec} k, \operatorname{Spf} \mathcal{V})$ *and let* σ *be a Frobenius endomorphism on* K*. Then, for an overconvergent* F*-isocrystal* E *on* $(T, \overline{T})/K$*, the rigid cohomology overconvergent isocrystal* $R^q f_{\mathrm{rig}*}E$ *is an overconvergent* F*-isocrystal on* $(T, \overline{T})/K$ *for any* q*.*
Moreover, the base change homomorphism is an isomorphism as overconvergent F*-isocrystals for any base extension* $(T', \overline{T}') \to (T, \overline{T})$ *separated of finite type over* $(S, \overline{S})$*.*

Proof. Let us put $\mathcal{V}' = \varinjlim(\mathcal{V} \xrightarrow{\sigma} \mathcal{V} \xrightarrow{\sigma} \cdots)$. Then $\mathcal{V}'$ is a complete discrete valuation ring over $\mathcal{V}$ whose residue field k' is the perfection of k. Applying Proposition 4.1.3 to the base extension by $\operatorname{Spec} k' \to \operatorname{Spec} k$, we may assume that k is algebraically closed. Let $\mathcal{W}$ be the ring of Witt vectors with coefficients in k and let L be the quotient field of $\mathcal{W}$. Since the residue field of L is algebraically closed, the restriction of σ on L is the canonical Frobenius endomorphism. Regarding an overconvergent F-isocrystal E on $(X, \overline{X})/K$ as an overconvergent F-isocrystal E on $(X, \overline{X})/L$, we may assume that $K = L$ by the argument of [20, Sect. 5.1].
Let t be a closed point of T. Since the associated convergent F-isocrystal E_t on X_t/K comes from an F-crystal on $X_t/\mathcal{W}$ [5, Théorème 2.4.2], the Frobenius homomorphism

$$\sigma_t^* H^q_{\mathrm{rig}}(X_t/K, E_t) \to H^q_{\mathrm{rig}}(X_t/K, E_t)$$

is an isomorphism by the Poincaré duality of the crystalline cohomology theory [3, Théorème 2.1.3] and the comparison theorem between the rigid cohomology and the crystalline cohomology [6, Proposition 1.9] [20, Theorem 5.2.1]. Therefore, the assertion follows from Proposition 3.3.3. □

4.2 Generic coherence

First we present a problem concerning an existence of smooth liftings.

4.2.1 Problem

Let $\mathcal{V}$ be a complete discrete valuation ring of mixed characteristics with residue field k and let X be a proper smooth connected scheme over $\operatorname{Spec} k$. Are there a finite extension $\mathcal{W}$ of complete discrete valuation ring over $\mathcal{V}$ and a projective smooth scheme Y over $\operatorname{Spec}\mathcal{W}$ with a proper surjective and generically finite morphism

$$Y \times_{\operatorname{Spec}\mathcal{W}} \operatorname{Spec} l \to X$$

over $\operatorname{Spec} k$? Here l is the residue field of $\mathcal{W}$.

4.2.2 Remark

We give two remarks on the problem above.

(1) When X is a proper smooth curve over $\operatorname{Spec} k$, one can take a projective smooth scheme Y over $\operatorname{Spec}\mathcal{V}$ such that $Y \times_{\operatorname{Spec}\mathcal{V}} \operatorname{Spec} k = X$ [12, Exposé III, Corollaire 7.4].

(2) In Proposition 4.2.3 and its application to Theorem 4.2.4 and Corollary 4.2.5 we use only the smoothness of the formal completion $\widehat{Y} \to \operatorname{Spf}\mathcal{W}$. Hence, it is sufficient to resolve a weaker version of the problem which asks for the existence of a formal $\mathcal{W}$-scheme $\mathcal{Y}$ which is projective smooth over $\operatorname{Spf}\mathcal{W}$ with a proper surjective and generically finite morphism

$$\mathcal{Y} \times_{\operatorname{Spec}\mathcal{W}} \operatorname{Spec} l \to X$$

over $\operatorname{Spec} k$. If such a $\mathcal{Y}$ exists, then there is a projective scheme over $\operatorname{Spec}\mathcal{W}$ whose formal completion over $\operatorname{Spf}\mathcal{W}$ is $\mathcal{Y}$ by [13, Chap. 3, Corollaire 5.1.8].

Let $\boldsymbol{m}$ be the maximal ideal of $\mathcal{V}$ and let

$$\begin{array}{ccc} X & \to & \overline{X} \\ \overset{\circ}{f}\downarrow & & \downarrow\overline{f} \\ T & \to & \overline{T} \end{array}$$

be a cartesian square of k-schemes such that $\overline{T}$ is an affine and integral scheme which is smooth over $\operatorname{Spec} k$, $\overline{f}$ is proper and $\overset{\circ}{f}$ is smooth with a connected generic fiber X_η, where η denotes the generic point of $\overline{T}$. Let us take a smooth lift $\mathcal{T} = \operatorname{Spec} R$ of $\overline{T}$ over $\operatorname{Spec}\mathcal{V}$ and put $\mathfrak{T} = (T, \overline{T}, \mathcal{T})$ with $\mathcal{T} = \operatorname{Spf}\widehat{R}$. Here $\widehat{Z}$ (resp. $\widehat{A}$) means the p-adic formal completion for a $\mathcal{V}$-scheme Z (resp. a $\mathcal{V}$-algebra A). Let us denote by $R_{\boldsymbol{m}}$ the localization of R at the prime ideal $\boldsymbol{m}R$. Note that $\widehat{R}$ is integral and $\widehat{R_{\boldsymbol{m}}}$ is a complete discrete valuation ring over $\mathcal{V}$ with special point η.

4.2.3 Proposition

With notation as above, suppose that there is a projective smooth scheme Y over $\operatorname{Spec}\widehat{R_{\boldsymbol{m}}}$ whose reduction is the generic fiber X_η of X. Then there exist

(i) *an open dense subscheme U of T (we define the associated $\mathcal{V}$-triple $\mathfrak{U} = (U, \overline{T}, \mathcal{T})$);*

(ii) *a formal $\mathcal{V}$-scheme $\mathcal{T}'$ which is finite flat over $\mathcal{T}$ ($\mathfrak{U}' = (U', \overline{T}', \mathcal{T}')$, where U' (resp. $\overline{T}'$) is the inverse image of U (resp. $\overline{T}$) in $\mathcal{T}'$);*

(iii) *a formal $\mathcal{V}$-scheme $\mathcal{T}''$ separated of finite type over $\mathcal{T}'$ with a closed immersion $\overline{T}' \to \mathcal{T}''$ over $\mathcal{T}'$ such that $\mathcal{T}'' \to \mathcal{T}'$ is etale around U' ($\mathfrak{U}'' = (U', \overline{T}', \mathcal{T}'')$);*

(iv) *a formal $\mathcal{V}$-scheme $\mathcal{X}''$ projective over $\mathcal{T}''$ with a natural isomorphism*

$$\mathcal{X}'' \times_{\mathcal{T}''} U' \cong X \times_T U'$$

such that $\mathcal{X}'' \to \mathcal{T}''$ is smooth around $\mathcal{X}'' \times_{\mathcal{T}''} U'$.

Proof. Since Y is projective over $\operatorname{Spec} \widehat{R_{\boldsymbol{m}}}$, one can take $a_1, \cdots, a_n \in \boldsymbol{m}\widehat{R_{\boldsymbol{m}}}$ for any i such that there exists a projective scheme Z over the n-dimensional $\widehat{R}$-affine space $\mathbb{A}^n_{\widehat{R}}$ with the natural cartesian square

$$\begin{array}{ccc} Y & \to & Z \\ \downarrow & & \downarrow \\ \operatorname{Spec} \widehat{R_{\boldsymbol{m}}} & \to & \mathbb{A}^n_{\widehat{R}} \end{array} \qquad a_i \leftarrow x_i.$$

Indeed, if one considers the sheaf $\mathcal{I}$ of ideals of definition of Y in a projective space over $\operatorname{Spec} \widehat{R_{\boldsymbol{m}}}$, then the Serre twist $\mathcal{I}(r)$ of $\mathcal{I}$ is generated by global sections for sufficiently large r. Then all coefficients which appear in the generators belong to $R + \boldsymbol{m}\widehat{R_{\boldsymbol{m}}}$.

Let $I \subset \widehat{R}[x]$ be an ideal of definition of the image of $\operatorname{Spec} \widehat{R_{\boldsymbol{m}}}$ in $\mathbb{A}^n_{\widehat{R}}$. Choose $b \in \widehat{R_{\boldsymbol{m}}}^n$ with $|b| < 1$ such that $g(b) = 0$ for all $g \in I$ and denote by Z_b the pull back of $Z \to \mathbb{A}^n_{\widehat{R}}$ by the natural morphism $\operatorname{Spec} \widehat{R}[b]^{\mathrm{nor}} \to \mathbb{A}^n_{\widehat{R}}$, where $\operatorname{Spec} \widehat{R}[b]^{\mathrm{nor}}$ is the normalization of $\operatorname{Spec} \widehat{R}[b]$. Note that $\widehat{R}[b]^{\mathrm{nor}}$ is finitely generated over $\widehat{R}$ since the characteristic of the field of fractions of $\widehat{R}$ is 0. The map defined by $b \mapsto 0$ determines a closed immersion $\overline{T} \to \operatorname{Spf} \widehat{\widehat{R}[b]^{\mathrm{nor}}}$ over $\operatorname{Spf} \widehat{R}$ since $\widehat{R}[b]^{\mathrm{nor}}$ is included in $\widehat{R_{\boldsymbol{m}}}$. $\overline{T}$ is a connected component of $\operatorname{Spf} \widehat{\widehat{R}[b]^{\mathrm{nor}}} \times_{\operatorname{Spf} \mathcal{V}} k$. The generic fiber of $\widehat{Z}_b \times_{\operatorname{Spf} \widehat{\widehat{R}[b]^{\mathrm{nor}}}} \overline{T} \to \overline{T}$ is X_η by our construction of Z. Hence, there are an open dense subset of X containing the generic fiber and an open dense subset of $\widehat{Z}_b \times_{\operatorname{Spf} \widehat{\widehat{R}[b]^{\mathrm{nor}}}} T$ containing the generic fiber which are isomorphic to each other.

Put $a = (a_1, \cdots, a_n)$. Since smoothness is an open condition, $\widehat{Z}_a \to \operatorname{Spf} \widehat{\widehat{R}[a]^{\mathrm{nor}}}$ is smooth around X_η by applying the Jacobian criterion to the cartesian squares

$$\begin{array}{ccccc} X_\eta & \to & \widehat{Y} & \to & \widehat{Z}_a \\ \downarrow & & \downarrow & & \downarrow \\ \eta & \to & \operatorname{Spf} \widehat{R_{\boldsymbol{m}}} & \to & \operatorname{Spf} \widehat{\widehat{R}[a]^{\mathrm{nor}}}. \end{array}$$

Now we consider the associated analytic morphism to $\widehat{Z} \to \widehat{\mathbb{A}}^n_R$. Since smoothness is an open condition for morphisms of rigid analytic spaces, there exists $0 < \lambda < 1$ such that $\widehat{Z}_b \to \mathrm{Spf}\, \widehat{\widehat{R}[b]^{\mathrm{nor}}}$ is smooth around X_η for any $b \in \widehat{R_{\boldsymbol{m}}}^n$ with $|b-a| = \max_i |b_i - a_i| \leq \lambda$ such that $g(b) = 0$ for any $g \in I$ by the quasi-compactness of $\widehat{Z}^{\mathrm{an}}_K$.

Let us take an element $b \in (\mathrm{Frac}(\widehat{R})^{\mathrm{alg}} \cap \widehat{R_{\boldsymbol{m}}})^n$ with $|b-a| \leq \lambda$ such that $g(b) = 0$ for any $g \in I$. Here $\mathrm{Frac}(\widehat{R})^{\mathrm{alg}}$ is the algebraic closure of $\mathrm{Frac}(\widehat{R})$ in an algebraic closure $\widehat{R_{\boldsymbol{m}}}[p^{-1}]^{\mathrm{alg}}$ of $\widehat{R_{\boldsymbol{m}}}[p^{-1}]$. It is possible to choose such a b using Noether's normalization theorem and the approximation by Newton's method. Consider an extension $\widehat{R}[b]^{\mathrm{nor}} \widehat{\otimes}_{\widehat{R}} \widehat{R_{\boldsymbol{m}}}$ over $\widehat{R_{\boldsymbol{m}}}$. Then $\widehat{R}[b]^{\mathrm{nor}} \widehat{\otimes}_{\widehat{R}} \widehat{R_{\boldsymbol{m}}}$ is finite over $\widehat{R_{\boldsymbol{m}}}$ for $|b| < 1$ and it has no p-torsion. We denote by $\widehat{R_{\boldsymbol{m}}}'$ a finite extension as a complete discrete valuation ring over $\widehat{R_{\boldsymbol{m}}}$ which contains $\widehat{R}[b]^{\mathrm{nor}}$. Then there are a sequence $\widehat{R} = \widehat{R}_0, \widehat{R}_1, \cdots, \widehat{R}_s$ of finite extensions of $\widehat{R}$ in $\mathrm{Frac}(\widehat{R})^{\mathrm{alg}}$ and a sequence $q_1(z), \cdots, q_s(z)$ of monic polynomials with $q_i(z) \in \widehat{R}_{i-1}[z]$ such that $\widehat{R}_i \cong \widehat{R}_{i-1}[z]/(q_i(z))$ and $\widehat{R_{\boldsymbol{m}}}'$ is generated by $\widehat{R}_s$ over $\widehat{R_{\boldsymbol{m}}}$ using the approximation by Newton's method.

Now we define $\mathcal{V}$-formal schemes separated of finite type

$$\begin{array}{rcl}
\mathcal{T}' & = & \mathrm{Spf}\, \widehat{R}_s \\
\mathcal{T}'' & = & \text{the Zariski closure of the image of the diagonal morphism} \\
 & & \mathrm{Spf}\, \widehat{R_{\boldsymbol{m}}}' \to \mathcal{T}' \times_{\mathcal{T}} \mathrm{Spf}\, \widehat{\widehat{R}[b]^{\mathrm{nor}}} \text{ as a } \mathcal{V}\text{-formal scheme} \\
\mathcal{X}'' & = & \widehat{Z}_b \times_{\mathrm{Spf}\, \widehat{\widehat{R}[b]^{\mathrm{nor}}}} \mathcal{T}''
\end{array}$$

and define the k-scheme $\overline{T}' = \overline{T} \times_{\mathcal{T}} \mathcal{T}'$. Then there is a natural closed immersion $\overline{T}' \to \mathcal{T}''$ over $\mathcal{T}'$. By our construction of $\mathcal{T}''$, $\mathcal{T}'' \to \mathcal{T}'$ is etale around all generic points of $\mathcal{T}'' \times_{\mathcal{V}} k$ above on $\eta \times_T T'$ since all the localizations of $\widehat{R}[b]^{\mathrm{nor}} \widehat{\otimes}_{\widehat{R}} \widehat{R_{\boldsymbol{m}}}$ at the prime ideal above $\boldsymbol{m}$ is contained in $\widehat{R_{\boldsymbol{m}}}'$. Hence, $\mathcal{T}'' \to \mathcal{T}'$ is etale around a dense open subscheme of $\overline{T}'$. By the property of $\widehat{Z}_b$, $\mathcal{X}'' \to \mathcal{T}''$ is smooth around $X_\eta \times_{\overline{T}} \overline{T}'$. Moreover, there are an open dense subset of $X \times_{\overline{T}} \overline{T}'$ containing all of the generic fibers over $\overline{T}'$ and an open dense subset of $\mathcal{X}'' \times_{\mathcal{T}''} \overline{T}'$ containing all of the generic fibers over $\overline{T}'$ which are isomorphic to each other. Therefore, there is an open dense subscheme U of T such that $\mathfrak{U} = (U, \overline{T}, \mathcal{T})$, $\mathfrak{U}' = (U', \overline{T}', \mathcal{T}')$ with $U' = U \times_{\overline{T}} \overline{T}'$, $\mathfrak{U}'' = (U', \overline{T}', \mathcal{T}'')$ and $\mathcal{X}''$ are the desired objects. □

Now we apply the study in 4.1. Let us keep the notation as in the beginning of this section. Suppose that the diagram

$$\begin{array}{ccc}
X & \to & \overline{X} \\
\overset{\circ}{f} \downarrow & & \downarrow \overline{f} \\
T & \to & \overline{T}
\end{array}$$

is a cartesian square of k-schemes such that $\overline{f}$ is proper and $\overset{\circ}{f}$ has smooth generic fibers.

4.2.4 Theorem

Under the assumption as above, assume furthermore that $\overline{T}$ is an affine integral scheme which is smooth over $\mathrm{Spec}\, k$ and the generic fiber of $\overset{\circ}{f} : X \to T$ is connected. Suppose that there exists a projective smooth lift of the generic fiber of $\overset{\circ}{f}$ over the spectrum of a complete discrete valuation ring which is induced by the localization of a smooth lift of $\overline{T}$ over $\mathrm{Spec}\, \mathcal{V}$. Then there exists an open dense subscheme U of T and a formal $\mathcal{V}$-scheme $\mathcal{T}$ over $\mathcal{S}$ which is smooth around T such that, if one denotes by $f_U : (\overset{\circ}{f}{}^{-1}(U), \overline{X}) \to (U, \overline{T})$ the restriction of f and puts $\mathfrak{U} = (U, \overline{T}, \mathcal{T})$, then the condition $(\mathrm{C})^q_{f_U, \mathfrak{U}, E}$ holds for any q and any overconvergent isocrystal E on $(\overset{\circ}{f}{}^{-1}(U), \overline{X})/\mathcal{S}_K$. In particular, the rigid cohomology overconvergent isocrystal $R^q f_{U\mathrm{rig}*}E$ on $(U, \overline{T})/\mathcal{S}_K$ exists for any q and any E.

Proof. Let us take U, $\mathfrak{U}$, $\mathfrak{U}'$, $\mathfrak{U}''$ and $\mathcal{X}''$ as in Proposition 4.2.3 except for the formal schemes $\mathcal{U}$, $\mathcal{U}'$, $\mathcal{U}''$ and $\mathcal{X}''$. We replace $\mathcal{U}$, $\mathcal{U}'$, $\mathcal{U}''$ and $\mathcal{X}''$ by $\mathcal{U} \times_{\mathrm{Spf}\, \mathcal{V}} \mathcal{S}$, $\mathcal{U}' \times_{\mathrm{Spf}\, \mathcal{V}} \mathcal{S}$, $\mathcal{U}'' \times_{\mathrm{Spf}\, \mathcal{V}} \mathcal{S}$ and $\mathcal{X}'' \times_{\mathrm{Spf}\, \mathcal{V}} \mathcal{S}$. Let us denote by $f'_U : (X \times_T U', \overline{X} \times_{\overline{T}} \overline{T}') \to (U', \overline{T}')$ (resp. $f''_U : (\mathcal{X}'' \times_{\mathcal{T}''} U', \mathcal{X}'' \times_{\mathcal{T}''} \overline{T}') \to (U', \overline{T}')$) the induced morphism from the conclusion of Proposition 4.2.3, and by E' (resp. E'') the inverse image of E on $(X \times_T U', \overline{X} \times_{\overline{T}} \overline{T}')/\mathcal{T}'_K$ (resp. $(\mathcal{X}'' \times_{\mathcal{T}''} U', \mathcal{X}'' \times_{\mathcal{T}''} \overline{T}')/\mathcal{T}'_K$). Then the condition $(\mathrm{C})^q_{f_U, \mathfrak{U}, E}$ is equivalent to the condition $(\mathrm{C})^q_{f'_U, \mathfrak{U}', E'}$ by the finite flat base change theorem (Theorem 2.1.1) and the faithfully flat descent theorem for finitely generated modules. Since the rigid cohomology is independent of the choices of compactification [4, Sect. 2, Théorème 2], the condition $(\mathrm{C})^q_{f'_U, \mathfrak{U}', E'}$ is equivalent to the condition $(\mathrm{C})^q_{f''_U, \mathfrak{U}', E''}$. Then the assertion follows from Proposition 3.3.1. □

With the situation of Theorem 4.2.4, a smooth lift of $\overline{T}$ over $\mathrm{Spec}\, \mathcal{V}$ always exists by [10, Theorem 6] since $\overline{T}$ is affine and smooth.

4.2.5 Corollary

With notation as above, suppose that Problem 4.2.1 admits an affirmative solution in general. Then there exists an open dense subscheme U of T such that, if $f_U : (\overset{\circ}{f}{}^{-1}(U), \overline{X}) \to (U, \overline{T})$ denotes the morphism of pairs induced by f, the rigid cohomology overconvergent isocrystal $R^q f_{U\mathrm{rig}*}E$ on $(U, \overline{T})/\mathcal{S}_K$ exists for any q and any overconvergent isocrystal E on $(\overset{\circ}{f}{}^{-1}(U), \overline{X})/\mathcal{S}_K$. If

$\mathfrak{S} = (\mathrm{Spec}\, k, \mathrm{Spec}\, k, \mathrm{Spf}\, \mathcal{V})$ and the relative dimension X over T is d, then $R^q f_{U\mathrm{rig}*}E = 0$ for any $q > 2d$.
Moreover, the base change homomorphism is an isomorphism of overconvergent isocrystals for any base extension $(T', \overline{T}') \to (T, \overline{T})$ separated of finite type over $(S, \overline{S})$ (resp. any extension $\mathcal{W}$ of complete discrete valuation ring over $\mathcal{V}$).
Suppose that $\mathfrak{S} = (\mathrm{Spec}\, k, \mathrm{Spec}\, k, \mathrm{Spf}\, \mathcal{V})$ and there exists a Frobenius endomorphism on K. Then $R^q f_{U\mathrm{rig}*}E$ is an overconvergent F-isocrystal on $(U, \overline{T})/K$.

Proof. Note that the rigid cohomology is determined by the reduced structures of $\overline{X}$ and $\overline{T}$. We may assume that $\overline{T}$ is affine. Since we can replace K by finite extensions of K by Proposition 2.2.1 and faithfully flat descent of finitely generated modules, we may assume that there exists a generically etale and proper surjective morphism $\overline{T}' \to \overline{T}$ such that $\overline{T}'$ is smooth over $\mathrm{Spec}\, k$ by applying an alteration [14, Theorem 4.1]. Let us denote by T' the inverse image of T in $\overline{T}'$. Shrinking T, we can find a finite flat extension $\overline{T}''$ of $\overline{T}$ such that T' is an open subscheme of $\overline{T}''$. Indeed, such $\overline{T}''$ exists if one considers an open dense subscheme of $\overline{T}'$ which is standard etale over $\overline{T}$. Applying the finite flat base change theorem (Theorem 2.1.1) to the extension $(T', \overline{T}'')/(T, \overline{T})$, we may assume that $T' = T$ by the faithfully flat descent of finitely generated modules. Since the category of overconvergent isocrystals is independent of the choices of compactification [5, Théorème 2.3.5], we may assume that $\overline{T}' = \overline{T}$. Hence, we may assume that $\overline{T}$ is an affine and integral scheme which is smooth over $\mathrm{Spec}\, k$.
Let $\mathcal{T}$ be a formal $\mathcal{V}$-scheme separated of finite type with closed immersion $\overline{T} \to \mathcal{T}$ over $\mathcal{S}$ such that $\mathcal{T} \to \mathcal{S}$ is smooth around T. Then one can take a decreasing sequence $\{T^{(r)}\}_{r \geq 0}$ of open dense subschemes of T and an r-truncated proper hypercovering

$$f_{\textstyle .}^{(r)} : (X_{\textstyle .}^{(r)}, \overline{X}_{\textstyle .}^{(r)}) \to (X^{(r)}, \overline{X})$$

of pairs over $(T^{(r)}, \overline{T})$ for each r such that

(i) $X^{(r)} = \overset{\circ}{f}{}^{-1}(T^{(r)})$ and $X_n^{(r)}$ is smooth over $T^{(r)}$;

(ii) if we put $\mathfrak{T}^{(r)} = (T^{(r)}, \overline{T}, \mathcal{T})$, the condition $(\mathrm{C})^q_{f_n^{(r)}, \mathfrak{T}^{(r)}, G}$ holds for any q and any overconvergent isocrystal G on $(X_n^{(r)}, \overline{X}_n^{(r)})/\mathcal{T}_K$;

for any $n \leq r$. Indeed, one can construct a proper hypercovering inductively on n by [21, Lemma 4.2.3] such that the generic fiber of $\overline{X}_n^{(r)}$ is projective smooth over the generic point of T after taking a finite extension $\overline{T}' \to \overline{T}$ such that $\Gamma(\overline{T}', \mathcal{O}_{\overline{T}'})$ is free over $\Gamma(\overline{T}, \mathcal{O}_{\overline{T}})$ by Problem 4.2.1. Take a lift $\mathcal{T}'$ of $\overline{T}'$ over $\mathcal{T}$ such that $\mathcal{T}' \to \mathcal{T}$ is finite flat and $\overline{T}' = \overline{T} \times_{\mathcal{T}} \mathcal{T}'$. Then there is an open dense subscheme $T_n^{(r)'}$ of $\overline{T}'$ such that, if we put $\overline{X}_n^{(r)'} = \overline{X}_n^{(r)} \times_{\overline{T}} \overline{T}'$ (resp. $X_n^{(r)'}$ to be the inverse image of $T_n^{(r)'}$ in $\overline{X}_n^{(r)'}$, resp. $f_n^{(r)'} : (X_n^{(r)'}, \overline{X}_n^{(r)'}) \to$

$(T_n^{(r)'}, \overline{T}')$ to be the induced morphism, resp. $\mathfrak{T}^{(r)'} = (T_n^{(r)'}, \overline{T}', \mathcal{T}_n^{(r)'}))$, then the condition $(\mathrm{C})^q_{f_n^{(r)'}, \mathfrak{T}^{(r)'}, G'}$ holds for any q and any overconvergent isocrystal G' on $(X_n^{(r)'}, \overline{X}_n^{(r)'})/\mathcal{T}'_K$ by Theorem 4.2.4. By the finite flat base change theorem and faithfully flat descent of finitely generated modules, there is an open dense subscheme $T_n^{(r)}$ of $\overline{T}'$ such that the condition $(\mathrm{C})^q_{f_n^{(r)}, \mathfrak{T}^{(r)}, G}$ holds for a suitable choice of $f_n^{(r)}$ and $\mathfrak{T}^{(r)}$. Hence, by shrinking T, such a proper hypercovering exists by Proposition 3.1.1.

Let E be an overconvergent isocrystal on $(X^{(r)}, \overline{X})/\mathcal{S}_K$ and let $f^{(q)} : (\overline{f}^{-1}(U^{(q)}), \overline{X}) \to (U^{(q)}, \overline{T})$ be the induced structure morphism. Completing the truncated proper hypercovering $(X_\bullet^{(r)}, \overline{X}_\bullet^{(r)}) \to (X^{(r)}, \overline{X})$ as a full simplicial proper hypercovering [21, Proposition 4.3.1] and using the spectral sequence for the proper hypercovering

$$\underline{E}_1^{qs} = \mathbb{R}^s f^{(r)}_{q\mathrm{rig}\mathfrak{T}^{(r)}*}(f_q^{(r)*}E) \Rightarrow \mathbb{R}^{q+s} f^{(r)}_{\mathrm{rig}\mathfrak{T}^{(r)}*}E$$

[21, Theorem 4.1.1], the sheaf $\mathbb{R}^q f^{(r)}_{\mathrm{rig}\mathfrak{T}^{(r)}*}E$ is a sheaf of coherent $j^\dagger\mathcal{O}_{]\overline{T}[_{\mathcal{T}^{(r)}}}$-modules if $q \leq (r-1)/2$ since the category of coherent sheaves is abelian and any extension of a coherent sheaf by a coherent sheaf in the category of sheaves of $j^\dagger\mathcal{O}_{]\overline{T}[_{\mathcal{T}^{(r)}}}$-modules is coherent.

Since the condition $(\mathrm{C})^q_{f_n^{(r)}, \mathfrak{T}^{(r)}, G}$ holds for $n \leq r$, it also holds after any base extension by a morphism $\mathcal{T}' \to \mathcal{T}$ separated of finite type over $\mathcal{T}$ with a closed immersion $\overline{T} \to \mathcal{T}'$ such that $\mathcal{T}' \to \mathcal{T}$ is smooth around T. Hence, for any q, there exists an open dense subscheme $U^{(q)}$ of T such that the rigid cohomology overconvergent isocrystal $R^q f^{(q)}_{\mathrm{rig}*}E$ on $(U^{(q)}, \overline{T})/\mathcal{S}_K$ exists for any overconvergent isocrystal E on $(\overline{f}^{-1}(U^{(q)}), \overline{X})/\mathcal{S}_K$ by Theorem 3.3.1.

For an open subscheme W of T, we define a morphism $f_W : (\overline{f}^{-1}(W), \overline{X}) \to (W, \overline{T})$ of pairs by the induced structure morphism. By the proposition below, there exists an integer q_0 such that $R^q f_{V\mathrm{rig}\mathfrak{V}*}E = 0$ for any open subscheme V of T, any $q > q_0$ and any overconvergent isocrystal E on $(\overline{f}^{-1}(V), \overline{X})/\mathcal{S}_K$, where $\mathfrak{V} = (V, \overline{T}, \mathcal{T})$. Hence, there exists an open dense subscheme U of T such that the rigid cohomology overconvergent isocrystal $R^q f_{U\mathrm{rig}*}E$ on $(U, \overline{T})/\mathcal{S}_K$ exists for any overconvergent isocrystal E on $(\overline{f}^{-1}(U), \overline{X})/\mathcal{S}_K$. Indeed, we can shrink T by the vanishing of rigid cohomology sheaves.

The rest is same as in Theorems 4.1.1 and 4.1.4. □

4.2.6 Proposition

Let $\mathfrak{S} = (S, \overline{S}, \mathcal{S})$ be a $\mathcal{V}$-triple separated of finite type and let $\overline{f} : \overline{X} \to \overline{T}$ be a morphism of k-schemes separated of finite type over $\overline{S}$. Then there exists an integer q_0 such that, for

(i) *any open subscheme* X *(resp.* T*) of* $\overline{X}$ *(resp.* $\overline{T}$*) with* $\overline{f}^{-1}(T) = X$ *(we denote by* $f : (X, \overline{X}) \to (T, \overline{T})$ *the structure morphism of k-pairs and put* $\mathfrak{T} = (T, \overline{T}, \mathcal{T})$*);*

(ii) *any* $\mathcal{V}$*-formal scheme* $\mathcal{T}$ *separated of finite type over* $\mathcal{S}$ *with an* $\mathcal{S}$*-closed immersion* $\overline{T} \to \mathcal{T}$ *which is smooth over* $\mathcal{S}$ *around* T*;*

(iii) *any overconvergent isocrystal* E *on* $(X, \overline{X})/\mathcal{T}_K$*,*

the rigid cohomology $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E$ *vanishes for any* $q > q_0$*.*

Proof. We may assume that $\mathcal{T}$ is affine. Since $\overline{X}$ is of finite type over $\operatorname{Spec} k$, there is a finite open covering $\{\overline{X}_\alpha\}_\alpha$ of $\overline{X}$ such that there exists a smooth formal scheme separated $\mathcal{X}_\alpha$ of finite type over $\operatorname{Spf}\mathcal{V}$ with a $\mathcal{V}$-closed immersion $\overline{X}_\alpha \to \mathcal{X}_\alpha$ for any α. We use induction on the minimal cardinality n of such a finite open covering of $\overline{X}$.

Suppose that $n = 1$. Let us take a finite affine open covering $\{\mathcal{U}_\beta\}_\beta$ of $\mathcal{X}$, put $\overline{U}_\beta$ (resp. U_β) to be the inverse image of $\overline{X}$ (resp. X) in $\mathcal{U}_\beta$ for each β, and denote by $\mathfrak{U}_\beta = (U_\beta, \overline{U}_\beta, \mathcal{U}_\beta \times_{\operatorname{Spf}\mathcal{V}} \mathcal{T})$ (resp. $f_\beta : \mathfrak{U}_\beta \to \mathfrak{T}$) the associated triple (resp. the structure morphism of triples). For a multi-index $\underline{\beta} = (\beta_0, \beta_1, \cdots, \beta_r)$ ($\beta_0 < \beta_1 < \cdots < \beta_r$), let $\mathfrak{U}_{\underline{\beta}}$ be the intersection of $\mathfrak{U}_{\beta_0}, \mathfrak{U}_{\beta_1}, \cdots, \mathfrak{U}_{\beta_r}$ and let $f_{\underline{\beta}} : \mathfrak{U}_{\underline{\beta}} \to \mathfrak{T}$ be the structure morphism. Let $\mathcal{E}$ be a sheaf of coherent $j^\dagger\mathcal{O}_{]\overline{X}[_\mathcal{X}}$-modules. We denote by $\mathcal{C}^\bullet_{\mathrm{alt}}(\{\mathfrak{U}_\beta\}_\beta, \mathcal{E})$ the alternating sheaf-valued Čech complex of $\mathcal{E}$ with respect to the covering $\{\mathfrak{U}_\beta\}_\beta$ of $\mathcal{X}$ [9, 2.11]. Then we have a natural isomorphism

$$R^q\widetilde{f}_*\mathcal{E} \cong \mathbb{R}^q\widetilde{f}_*\mathcal{C}^\bullet_{\mathrm{alt}}(\{\mathfrak{U}_\beta\}_\beta, \mathcal{E})$$

for any q by [9, Lemma 2.11.1]. Let us fix a multi-index $\underline{\beta}$. Since $\overline{f}_{\underline{\beta}}^{-1}(T) = U_{\underline{\beta}}$, there exists an admissible affinoid covering $\{W_\gamma\}_\gamma$ of $]\overline{T}[_\mathcal{T}$ such that

$$H^q(\widetilde{f}_{\underline{\beta}}^{-1}(W_\gamma), \mathcal{E}) = 0 \quad (q > 1)$$

for any γ. Indeed, we take an admissible affinoid covering by [9, Sect. 2.6] and prove the vanishing by a method similar to [21, Proposition 3.2.3] using [9, Proposition 5.1.1]. Hence, the direct image sheaf $R^q\widetilde{f}_{\underline{\beta}*}\mathcal{E}|_{]\overline{U}_{\underline{\beta}}[_{\mathcal{U}_{\underline{\beta}}}}$ vanishes for $q > 1$. By the Čech spectral sequence $R^q\widetilde{f}_*\mathcal{E} = 0$ for $q > \operatorname{card}(\{\mathfrak{U}_\beta\}_\beta)$. By the Hodge-de Rham spectral sequence there exists an integer q_0 such that $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E$ vanishes for any $q > q_0$. This q_0 is independent of the choices of an open subscheme T of X, a smooth formal scheme $\mathcal{T}$ and an overconvergent isocrystal E on $(X, \overline{X})/\mathcal{T}_K$.

Suppose that n is general. Let us put $\overline{X}' = \cup_{\alpha=2}^n \overline{X}_\alpha$ and $\overline{X}_1' = \overline{X} \cap \overline{X}'$, denote by X_1 (resp. X', resp X_1') the inverse image of T in $\overline{X}_1$ (resp. $\overline{X}'$, resp. $\overline{X}_1'$) and define $f_1 : (X_1, \overline{X}_1) \to (T, \overline{T})$ (resp. $f' : (X', \overline{X}') \to (T, \overline{T})$, resp. $f_1' : (X_1', \overline{X}_1') \to (T, \overline{T})$) as the structure morphism. Then, for any overconvergent isocrystal on $(X, \overline{X})/\mathcal{T}_K$, there exists a natural commutative diagram

$$\begin{array}{ccccccc} \mathbb{R}^q f'_{\mathrm{rig}\mathfrak{T}*}E' & \to & C^{q+1} & \to & \mathbb{R}^{q+1} f_{\mathrm{rig}\mathfrak{T}*}E & \to & \mathbb{R}^{q+1} f'_{\mathrm{rig}\mathfrak{T}*}E' \\ \downarrow & & \downarrow\cong & & \downarrow & & \downarrow \\ \mathbb{R}^q f'_{1\mathrm{rig}\mathfrak{T}*}E'_1 & \to & C^{q+1} & \to & \mathbb{R}^{q+1} f_{1\mathrm{rig}\mathfrak{T}*}E_1 & \to & \mathbb{R}^{q+1} f'_{1\mathrm{rig}\mathfrak{T}*}E'_1 \end{array}$$

of exact rows with a vertical isomorphism in the second terms for any q since $\{\overline{X}_1, \overline{X}'\}$ is an open covering of $\overline{X}$ with $\overline{X}_1 \cap \overline{X}' = \overline{X}'_1$. Here E_1 (resp. E', resp. E'_1) is the inverse image of E on $(X_1, \overline{X}_1)/\mathcal{T}_K$ (resp. $(X', \overline{X}')/\mathcal{T}_K$, resp. $(X'_1, \overline{X}'_1)/\mathcal{T}_K$). Since $\overline{X}'_1$ is an open subscheme of $\overline{X}_1$, it is embedded into a smooth formal scheme separated of finite type over $\mathrm{Spf}\,\mathcal{V}$. Therefore, the assertion follows from the induction hypothesis. □

In the case of families of curves one can take a lift of the generic fiber without any extension (Remark 4.2.2 (1)). Hence, we do not need to take a proper hypercovering.

4.2.7 Theorem

If X is a proper smooth family of curves over T, then the conclusions of Theorem 4.2.4 hold.

References

[1] André, Y., Baldassarri, F., *Geometric theory of G-functions*, Arithmetic Geometry edited by Catanese, F., Cambridge Univ. Press 1997.

[2] Baldassarri, F., Chiarellotto, B., *Algebraic versus rigid cohomology with logarithmic coefficients*, Barsotti Symposium in Algebraic Geometry, Perspective in Math. 15, Academic press (1994), 11–50.

[3] Berthelot, P., *Cohomologie Cristalline des schémas de caractéristique* $p > 0$, Lecture Notes in Math. 407, Springer-Verlag 1974.

[4] Berthelot, P., *Géométrie rigide et cohomologie des variétés algébriques de caractéristique p*, Bull. Soc. Math. France, Mémoire 23 (1986), 7-32.

[5] Berthelot, P., *Cohomologie rigide et cohomologie rigide à supports propres. Première Partie*, preprint, Université de Rennes (1996).

[6] Berthelot, P., *Finitude et pureté cohomologique en cohomologie rigide (avec un appendice par Aise Johan de Jong)*, Invent. math. 128 (1997), 329-377.

[7] Bosch, S., Güntzer U., Remmert, R., *Non-Archimedean Analysis*, Grundlehren der math. Wissenschaften 261, Springer-Verlag 1984.

[8] Crew, R., *Finiteness theorem for the cohomology of an overconvergent isocrystal on a curve*, Ann. Scient. École Norm. Sup. 31 (1998), 717-763.

[9] Chiarellotto, B., Tsuzuki, N., *Cohomological descent of rigid cohomology for etale coverings*, preprint.

[10] Elkik, R., *Solutions d'équations à coefficients dans un anneau hensélian*, Ann. Scient. École Norm. Sup. 6 (1973), 553-604.

[11] Grosse-Klönne, E., *Finiteness of de Rham cohomology in rigid analysis*, Duke Math. J. 113 (2002), 57-91.

[12] Grothendieck, A., *Revêtements Etales et Groupe Fondamental*, SGA1, Lecture Notes in Math. 224, Springer-Verlag 1971.

[13] Grothendieck, A., Dieudonné, J., *Eléments de Géométrie Algébrique III*, Publ. Math. I.H.E.S. 11 (1961).

[14] de Jong, A.J., *Smoothness, semi-stability and alterations*, Publ. Math. I.H.E.S. 83 (1996), 51-93.

[15] Katz, N.M., *Nilpotent connections and the monodromy theorem : application of a result of Turrittin*, Publ. Math. I.H.E.S. 39 (1970), 175-232.

[16] Katz, N.M., Oda, T., *On the differential of de Rham cohomology classes with respect to parameters*, J. Math. Kyoto Univ. 8 (1968), 199-213.

[17] Kiehl, R., *Der Endlichkeitssatz für Abbildungen in der nichtarchimedischen Funktionentheorie*, Invent. math. 2 (1967), 191-214.

[18] Kedlaya, K.S., *Finiteness of rigid cohomology with coefficients*, preprint.

[19] Shiho, A., *Crystalline fundamental groups. II. Log convergent cohomology and rigid cohomology*, J. Math. Sci. Univ. Tokyo 9 (2002), 1-163.

[20] Tsuzuki, N., *On the Gysin isomorphism of rigid cohomology*, Hiroshima Math. J. 29 (1999), 479-527.

[21] Tsuzuki, N., *Cohomological descent of rigid cohomology for proper coverings*, Invent. math. 151 (2003), 101-133.

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