Preface

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Kurt Mahler was born at Krefeld am Rhein in Germany in 1903, and died in Canberra, Australia in 1988. Although he lived outside of Germany from 1933 onwards, his mathematical roots were always in the great school of number theory which flourished in Germany between the two World Wars. Thus it is very fitting that "Documenta Mathematica" should publish this extra volume about the continuing impact of his work today.

Mahler had very broad interests within number theory. At the same time, he was unmoved by current mathematical fashion, and always followed his own path through different aspects of the subject. Thus many of his results only became widely used long after his original papers were written. The articles in this volume reflect both the diversity of his interests, and how widely his ideas have influenced current research. Almost half are devoted to Mahler's lifetime fascination with questions of Diophantine approximation and transcendental number theory. Within this broad area, he was always motivated by the most basic arithmetic questions about classical numbers like e, π , and γ, but he understood very well that new ideas were needed to attack such questions. Much of his mathematical life can be seen as a search for these new ideas. His general classification of complex (and later p-adic numbers) seems to have been motivated by his desire to prove the algebraic independence of certain very classical numbers like e and π. Mahler's two beautiful papers published in "Crelle" in 1932 provided a highly original and powerful reworking of Hermite's method for proving the transcendence of e, which had been previously overlooked despite the great interest in this question immediately after the publication of Hermite's proof. Mahler was always fascinated by Siegel's proof of the transcendence of the values at algebraic points of Bessel functions, and gave several lecture courses on Shidlovsky's powerful generalisation of it to a certain class of solutions of higher order linear differential equations. In yet another direction, during a bout of illness while a student in Göttingen, Mahler invented a completely new method for studying the transcendence properties of the values at algebraic points of a class of transcendental functions satisfying simple non-differential functional equations. All of these ideas of Mahler led to very extensive development in the work of later mathematicians, as is fully explained in the articles by Amou and Bugeaud, Adamczweski, and Nesterenko in the present volume. Mahler was also always greatly interested in p-adic numbers. As far as the Diophantine approximation of p-adic numbers is concerned, he proved the first generalisation of the Thue–Siegel theorem to p-adic algebraic numbers, and immediately used his result to show that one could generalise Siegel's celebrated theorem on the finiteness of the number of integer points on an affine curve of genus ≥ 1 defined over $\mathbb{Q}$ to the finiteness of the set of S-integral points on such a curve, where S is an arbitrary finite set of primes (in fact, Mahler only gave the proof for curves of genus 1, probably because of a lack of patience in working through the machinery used by Siegel to handle curves of genus > 1). The article by Evertse, Győry, and Stewart explains very well some of the subsequent developments which have grown out of Mahler's ideas on p-adic Diophantine approximation. Mahler often lamented in his lectures about the lack of interest of the 20th century mathematical community with the arithmetic properties of decimal expansions. He himself, with typical high originality, used his p-adic generalisation of the Thue–Siegel theorem to prove the transcendence of all decimals of the form $0.f(1)f(2)f(3)\ldots$, where f is any non-constant polynomial taking non-negative integer values at the positive integers. It should also be mentioned that, quite outside questions of Diophantine approximation and transcendence of p-adic numbers, Mahler's beautiful theorem characterising the continuous p-adic valued functions defined on the ring of p-adic integers $\mathbb{Z}_p$ is today a basic tool in the construction of p-adic L-functions and their relationship to Iwasawa theory. Yet another area of number theory in which Mahler made fundamental contributions is the geometry of numbers. His compactness theorem for lattices is arguably the most important result in the subject since the initial work of Minkowski, and its wide subsequent use is discussed in the article by Evertse. Mahler found his measure for polynomials of several variables from his desire to simplify some of the rather inelegant arguments in transcendental number theory involving the naive definition of the height of a polynomial. The articles by Boyd, and Lind and Schmidt, explain the astonishing variety of questions in which his measure has since proven to be a basic tool.

The University of Newcastle, in Newcastle, Australia, should be congratulated on making all of Mahler's published papers permanently available for future generations to read online in "The Kurt Mahler Archive". His published papers and books are always written with great clarity and precision. Equally well, his lectures were lucid, very precise, and somehow communicated to his audience his enthusiasm and interest for the subject he was lecturing on. I myself had the great good fortune to attend three of Mahler's beautiful courses in the years 1962-64 whilst I was an undergraduate in Canberra, and he then guided my first frail footsteps in mathematical research in 1965 during my final honours year. If I had not been taught by Mahler in my formative years, I would never have ended up myself as a number theorist, or probably even as a mathematician.

Cambridge, England, 1 December 2018