2. Mahler's Measure for Polynomials in Several Variables

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If $P(x_1, \ldots, x_k)$ is a polynomial with complex coefficients, the Mahler measure of P, $M(P)$, is defined to be the geometric mean of $|P|$ over the k-torus, $\mathbb{T}^k$. We briefly describe Mahler's motivation for defining this function and his applications of it to polynomial inequalities. We then describe how this function occurs naturally in the study of Lehmer's problem concerning the set of all measures of one-variable polynomials with integer coefficients. We describe work of Deninger which shows how Mahler measure arises in the study of the far-reaching Beĭlinson conjectures and leads to surprising conjectural explicit formulas for some measures of multivariable polynomials. Finally we describe some of the recent work of many authors proving some of these formulas by a variety of different methods.

1 Introduction

Let $P(x_1, \ldots, x_k)$ be a polynomial with complex coefficients. If P is not identically zero, the Mahler measure of P is defined by

$$M(P) = \exp\left(\int_0^1 \cdots \int_0^1 \log|P(\exp(2\pi i t_1), \ldots, \exp(2\pi i t_k))| \, \mathrm{d}t_1 \cdots \mathrm{d}t_k\right). \quad (1)$$

So $M(P)$ is the geometric mean of $|P|$ over the k-torus, $\mathbb{T}^k$. We define $M(0) = 0$. An obvious but important property of M is that

$$M(PQ) = M(P)M(Q). \quad (2)$$

For many questions, it is more natural to consider the quantity

$$m(P) = \int_0^1 \cdots \int_0^1 \log|P(\exp(2\pi i t_1), \ldots, \exp(2\pi i t_k))| \, \mathrm{d}t_1 \cdots \mathrm{d}t_k,$$

which we call the *logarithmic Mahler measure of P*.

In [M148], Mahler introduced $M(P)$ in order to give a simple proof of the so-called Gel'fond–Mahler inequality for multivariable polynomials. This is a result that has important applications in transcendental number theory. Let $H(P)$, the *height of* P be defined to be the maximum of the absolute value of the coefficients of P. Then (2) together with some straightforward upper bounds for the coefficients of P in terms of $M(P)$ give the inequality

$$H(P)H(Q) \leq 2^n H(PQ),$$

where n is the total degree of PQ. One should compare the simple and elegant proof of this inequality in [M148] with the elaborate proof given in [27].

Of course Mahler did not call the quantity $M(P)$ the *Mahler measure* or *Mahler's measure*, but simply referred to it as *the measure*, one of the ways to measure the size of P. The term "Mahler's measure" seems to have first been adopted in the early 1980s, see e.g. [8, 25].

It should be noted that for single variable polynomials, $M(P)$ had appeared in the literature before Mahler began to call it the "measure". For example, it appears in an influential paper of Lehmer [33] where it is called $\Omega(P)$. Lehmer did not use the integral definition (1), but defined $\Omega(P)$ in terms of the zeros of $P(x)$. The relationship between the two definitions is a famous formula of Jensen, that if $P(x) = a_0 \prod_{j=1}^{k}(x - \alpha_j)$, then

$$M(P) = |a_0| \prod_{j=1}^{k} \max(|\alpha_j|, 1).$$

If $P(x)$ is monic and has all its roots on the unit circle then clearly $M(P) = 1$. In this case, a theorem of Kronecker shows that all the roots of $P(x)$ are roots of unity, polynomials that were not interesting for the questions that concerned Lehmer. So he searched for polynomials for which $M(P)$ is small, but greater than 1. For non-reciprocal polynomials the smallest value he was able to find was for $P(x) = x^3 - x - 1$ which has one root θ_0 outside the unit circle $\theta_0 = 1.3247\ldots$, so $M(P) = \theta_0$. Its reciprocal, $Q(x) = x^3 + x^2 - 1$ has a pair of complex roots outside the unit circle each of modulus $\sqrt{\theta_0}$ and also has $M(Q) = \theta_0$. The proof that θ_0 is indeed the smallest measure of non-reciprocal polynomials was provided by Smyth [42] in his PhD thesis.

In contrast to the case of non-reciprocal polynomials, Lehmer produced the remarkable reciprocal polynomial

$$P(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

which has one zero σ_1 outside the unit circle, with $\sigma_1 = 1.1762808\ldots$ ("Lehmer's constant"), so $M(P) = 1.1762808\ldots$. Lehmer stated that he was unable to find a smaller value of the measure. Given that this example was found in 1933, it would seem likely that a smaller value would have been found in subsequent years, but this record still stands today in spite of extensive computation [7, 10, 36, 37]. Lehmer asked whether one could do any better than this, a question now called "Lehmer's problem", and the optimistic answer that indeed σ_1 is the smallest measure of a one-variable polynomial is known as "Lehmer's conjecture", even though Lehmer never made this conjecture explicitly. It has been proved by Mossinghoff, Rhin and Wu [37] that no polynomial of degree ≤ 56 achieves a smaller value of $M(P) > 1$. John Brillhart, one of Lehmer's students, told me that he once asked Lehmer how he had found *Lehmer's constant* and he replied "Oh, just fooling around". I suspect there was more to it than that!

Interestingly, a number of years before Smyth's paper [42], in a paper that went unnoticed for many years, Breusch [20] showed that for P non-reciprocal, one has

$$M(P) \geq M(x^3 - x^2 - 1/4) = 1.1796\ldots,$$

which proves Lehmer's conjecture for non-reciprocal polynomials. In contrast to Smyth's result, this result is not sharp since the polynomial that gives the lower bound does not have integer coefficients. However, note that the lower bound Breusch obtains is slightly larger than Lehmer's constant which does show that one must look to reciprocal polynomials to obtain small measures. Breusch's paper was rediscovered in 2005 by Wladyslaw Narkiewicz while he was searching for another paper in the same volume in which that paper appears. As it happened, at the time I learned of this I was visiting Chris Smyth in Edinburgh and had the dubious privilege of breaking the news to him about Breusch's result. Smyth's approach to the question is quite different from Breusch's and it is amusing to speculate whether he would have discovered the approach that led him to obtain his best possible result if he had known of Breusch's paper.

My own interest in Mahler measure and Lehmer's problem began with my study of the papers of Dufresnoy and Pisot [24] on a systematic enumeration of small Pisot numbers by using their association with certain rational functions of one complex variable. The Pisot (or Pisot–Vijayaraghavan) numbers S are those algebraic integers $\theta > 1$ all of whose other algebraic conjugates lie inside the unit circle. So, for example, θ_0 is a Pisot number[1] and is known to be the smallest such number by a result of Siegel [41]. Now this result is a special case of Smyth's theorem. There is a famous algorithm due to Schur for characterizing the coefficient sequences of holomorphic functions bounded by 1 on the unit circle. Dufresnoy and Pisot extended this to certain meromorphic functions. They systematically applied this to the study of the Pisot numbers smaller than the golden ratio. In fact their paper contains most of the ingredients necessary for designing an algorithm [6] to produce all the Pisot numbers in an interval $[a, b]$ which contains no limit points of the set S of Pisot numbers.

[1] The number θ_0 is now often referred to as the "plastic number".

While studying this algorithm, we were led to the question of whether the sequence $M(1+x+x^n)$ is bounded by $\sqrt{2} = 1.4142\ldots$. Computation suggested this sequence in fact converges to

$$\beta = M(1+x+y) = 1.3813564445\ldots.$$

Our numerical computation of β was accomplished by expanding the integral into a geometrically convergent series,

$$\log M(1+x+y) = \frac{2}{\pi}\sum_{m=0}^{\infty}\binom{-1/2}{m}(-1/4)^m(2m+1)^{-2} = 1.014941606\ldots$$

We were able to settle the question of the convergence of $M(1+x+x^n)$ to $M(1+x+y)$ in [5], obtaining an error term that was sufficient to prove the desired result about $M(1+x+x^n)$. This showed that β is a limit point of Mahler measures of one-variable non-reciprocal polynomials and it was natural to ask whether this might be the smallest such limit point. When I brought this question up at a workshop in Oberwohlfach in the summer of 1979, Michel Waldschmidt informed me that Mahler had already defined the quantity $M(P)$ for multi-variable polynomials in the paper [M148] and then I began calling it "Mahler measure" or "Mahler's measure".

As luck would have it, around the time I was working on such matters I had invited C. J. Smyth to visit me at UBC on a sabbatical from the James Cook University in Townsville, Australia. When I showed him the interesting case of $M(1+x+y)$, he quickly saw how to expand the integrand in a different series than the one that I had used, not as quickly convergent, but giving instead the elegant and intriguing formula [8, 42]

$$\log M(1+x+y) = \frac{3\sqrt{3}}{4\pi}L(2,\chi_{-3}) = L'(-1,\chi_3). \tag{3}$$

This formula of Smyth, expressing a logarithmic Mahler measure as the special value of an L-function, was the inspiration for much subsequent research in the study of Mahler's measure.

The constant β appears in a number of apparently unrelated contexts. Amusingly, it appears in a paper of Mahler [M153], without being identified as a 2-variable Mahler measure. It is also the best constant in an inequality for the size of the largest factor of a one-variable polynomial related to the Gel'fond–Mahler inequality [11] and in an asymptotic formula for the so-called binomial circulant determinant, conjectured by Frame [26] and proved in [9].

In the unpublished paper [5], I gave a proof for a more general result:

$$\lim_{n\to\infty} M(P(x,x^n)) = M(P(x,y)) \tag{4}$$

which differs from the proof for $M(1+x+y)$ in lacking an explicit error term. This proof is reproduced in an appendix of [8]. In our computations of small Salem numbers [5] using Salem's construction, and in our numerical survey [7, 10] of polynomials with measure at most 1.3, we had obtained many reciprocal polynomials of one variable with small measure. Multiplying these by various cyclotomic polynomials, we observed that a lot of the resulting polynomials fell into patterns of the form $P(\pm x^m, \pm x^n)$, where $P(x, y)$ was one of the two polynomials

$$P_1(x, y) = xy(xy + y + x + 1 + 1/x + 1/y + 1/(xy))$$

and

$$P_2(x, y) = xy(1 + x + 1/x + y + 1/y).$$

The polynomial $P_2(x, y)$ was also discovered at about the same time by Stewart and te Riele and independently by Smyth. Numerically,

$$\alpha_1 = M(P_1(x, y)) \approx 1.2554338662666087457$$

and

$$\alpha_2 = M(P_2(x, y)) \approx 1.2857348642919862749.$$

So there are infinitely many non-reciprocal polynomials with measure arbitrarily close to these two numbers and hence here seem to be two rather small limit points of measures of one-variable polynomials. We should admit that the term "limit point" is used here optimistically since although we know that $M(P(x, x^n)) - M(P(x, y))$ converges to zero, we do not have an explicit error term so there remains open the (remote) possibility that for some $P(x, y)$ this quantity is *identically zero* for all sufficiently large n. This question is discussed by Mossinghoff and the author in [18], a paper in which we compute a list of 48 two-variable polynomials with measures at most 1.37. The two polynomials P_1 and P_2 are the only known two-variable polynomials with measure smaller than 1.3. There are two more smaller than θ_0.

The difficulty of producing an explicit error term in (4) can be seen from the discussion in a recent paper of Condon [22]. In that paper, he derives a complete asymptotic expansion for $m(1+x+x^n) - m(1+x+y)$ in inverse integer powers of n and describes some experiments with the polynomial $P_2(x, y)$ that shows the subtlety of the question. The difference is that $1 + x + y = 0$ vanishes only at two points on the torus $\mathbb{T}^2$ while the polynomial $P_2(x, y)$ vanishes on a one-dimensional subset of the torus. The question is not at all delicate in the case that $P(x, y)$ does not vanish on the torus since then $\log |P(x, y)|$ is a bounded smooth function and it is easy to see that $m(P(x, x^n))$ is a Riemann sum for $m(P(x, y))$. In this case, Condon's analysis gives a complete asymptotic expansion of the difference $m(P(x, x^n)) - m(P(x, y))$.

The formula (4) can be generalized to polynomials in k variables, i.e.

$$\lim M(P(x^{a_1}, x^{a_2}, \ldots, x^{a_k})) = M(P(x_1, \ldots, x_k)), \tag{5}$$

where the limit is interpreted either as an iterated limit in which $k-1$ of the variables a_j tend to ∞ successively [8] or, more generally, tend to infinity simultaneously in a suitably controlled manner as shown by Lawton [32]. The result (5) suggested that the set $\mathbb{L}$ of measures of polynomials in arbitrarily many variables is an interesting object of study and led to the conjectures put forward in [8].

2 Mahler's measure is an entropy

An important advance in the study of the multi-variable Mahler measure was the paper of Lind, Schmidt and Ward [34] in which they showed that $m(P)$ is more than a tool in proving useful inequalities for polynomials, but has an intrinsic meaning in terms of certain discrete dynamical systems called *subshifts of finite type*. Given a (Laurent) polynomial $P(x_1,\dots,x_k)$ in k variables, one can define such a dynamical system acting on $\mathbb{Z}^n$ thought of as the dual to the torus $\mathbb{T}^n$. Their main result is that the logarithmic Mahler measure $m(P)$ is exactly the *entropy* of this dynamical system.

3 Deninger's interpretation of Mahler's measure

For many years, I had tried unsuccessfully to find a formula for $m(P_2(x,y))$ analogous to Smyth's elegant formula (3). So I was delighted when I was told by Henri Cohen in December of 1995 that, at the Journées Arithmétiques in Barcelona earlier that year, Christopher Deninger had announced such a formula for $m(P)$ for *all* P in terms of L-function values. When I contacted Deninger, he explained that his formula was based on a conjecture of Beĭlinson and had been proved in general only for P which do not vanish on the torus. So his general theory did not immediately apply to $1+x+y+1/x+1/y$. However, he was intrigued by the question and within a day had proved a formula expressing $m(P_2(x,y))$ as a Kronecker–Eisenstein series from which he could predict that

$$m(1+x+y+1/x+1/y) = rL'(0,E_{15}), \tag{6}$$

where r is an unspecified rational number, and $L(s,E_{15})$ is the L-function of an elliptic curve of conductor $N=15$. The initially mysterious appearance of this elliptic curve is explained by the fact that the algebraic curve given by $1+x+y+1/x+1/y=0$ defines just such an elliptic curve. Using the system PARI/GP it is easy to compute $L'(0,E_{15})$ and comparing with the value of $m(P_2)$ that we had previously computed to many decimal places, we found that $r=1$ to the precision of the computation.

Following this remarkable prediction we embarked on an extensive computation starting from a dozen or so values of $m(P(x,y))$ that we had previously computed accurately. This led eventually to an infinite number of conjectures analogous to (6) all of the form $m(P(x,y)) = rL'(0,E)$ where r is an explicit rational number of small height and E is an elliptic curve which is a factor of the Jacobian of the algebraic curve $P(x, y) = 0$; see [12]. In particular, we produced conjectures for families of polynomials such as

$$m(k + x + y + 1/x + 1/y) = r_k L'(0, E_k), \tag{7}$$

where E_k is (for $k \neq 0, 4$) the elliptic curve defined by $k+x+y+1/x+1/y = 0$.

Fernando Rodriguez Villegas took an immediate interest in this and suggested computations for a much wider variety of polynomials than I had initially contemplated. He also clarified the necessary conditions for such formulas to hold and showed how they were connected to Beĭlinson's conjecutures. Indeed, for families of polynomials such as the family in (7) he showed that $m(P)$ is essentially the regulator map $r(\{x, y\})$ for the K-group $K_2(E_k)$. He pointed out that (7) should hold if k^2 is an integer and proved it in case $k^2 = 8, 18$ and 32 using known cases of Beĭlinson's conjectures (for CM curves).

An intriguing feature of families such as (7) is that the same conductor may appear for different values of the parameter k since the curves in question are isogenous. For example, in (7), $N_k = 15$ for $k = 1, 5$ and 16 suggesting the identities $m(5+x+y+1/x+1/y) = 6m(1+x+y+1/x+1/y)$ and $m(16+x+y+1/x+1/y) = 11m(1+x+y+1/x+1/y)$. Rodriguez Villegas [38], Bertin [1] and Lalín–Rogers [30] were able to prove many such non-obvious identities between logarithmic Mahler measures by using computations in $K_2(E)$.

Recently, formulas of the type (7) for various values of k and analogous formulas for other polynomials began to be proved by a variety of methods. For the family of polynomials (7), we now have proofs for $k = 2$ and 8 by Lalín and Rogers [30], $k^2 = -4, -1$ and 2 by Rogers and Zudilin [39], $k = 1$ by Rogers and Zudilin [40], $k = 3$ and 12 by Brunault [21], and a different proof for $k = 3$ by Lalín, Samart and Zudilin [31]. In particular, Rogers and Zudilin [40] have now a complete proof of Deninger's conjecture (6). Their proof is one of the more intricate of the proofs so far obtained for the family (7). There are also a number of results for polynomials $P(x, y)$ not in the family (7). In particular the explicit formula $m(P_1(x, y)) = L'(0, E_{14})$, where E_{14} is an elliptic curve of conductor 14, conjectured in [12] was given an elegant proof by Mellit [35].

However, it seems that all the methods of proof discovered so far apply to only a finite set of polynomials, so, for example, we do not yet have a proof for (7) for all values of k.

4 Polynomials in more than 2 variables

Prior to Deninger's paper [23], there were almost no formulas for $m(P)$ for polynomials in more than two variables, other than an early formula due to Smyth [8, 43]:

$$\pi^2\, m(1 + x + y + z) = (7/2)\zeta(3).$$

However, some of the insights supplied by Deninger and Rodriguez Villegas led to the elegant result proved by Lalín in her thesis [28],

$$\pi^4\, m((1+u)(1+v)(1+x)+(1-u)(1-v)(y+z)) = 93\,\zeta(5).$$

Also, as predicted by Rodriguez Villegas in [38], Bertin [2] has been able to produce formulas for certain 3-variable polynomials defining $K3$-surfaces, especially the 3-variable generalization of (7).

$$P_k(x,y,z) = k + x + 1/x + y + 1/y + z + 1/z, \tag{8}$$

For example, in [2], she proves

$$\pi^3\, m(6+x+y+z+1/x+1/y+1/z) = 24\sqrt{6}\, L_F(\phi,3),$$

where $F = \mathbb{Q}(\sqrt{-6})$ and $L_F(\phi,3)$ is a Hecke L-series. The recent multi-author paper [3] extends this work to singular $K3$-surfaces defined by (8) for further values of k.

In the report of an early BIRS workshop [16] we mention an interesting lecture given by Vincent Maillot concerning the Mahler measure of non-reciprocal polynomials in many variables which suggests a refinement of Deninger's theory. Based on this prediction, the author and Rodriguez Villegas independently and jointly computed a few examples of polynomials in 3 or more variables whose Mahler measure should be given by a special value of an appropriate L-function (sometimes with some explicit lower order corrections). A few of these examples are mentioned in [16] and in the slides from a lecture given by the author in 2006 [14]. Lalín [29] has been able to prove some results in this direction for some polynomials in 3 variables.

5 Expository articles on Mahler's measure and related matters

We refer the reader to a few informative expository articles for more information about some recent work: The report [16] on the 2003 BIRS Workshop "The Many Aspects of Mahler's Measure" gives some indication of work up to that point on a number of aspects of Mahler's measure including its connections with knot theory and hyperbolic geometry, an important topic that we have not discussed above. Smyth's survey [44] is devoted to results concerning the Mahler measure of one-variable polynomials, and a companion article [45] surveys the history of the Salem numbers, a closely related subject. The paper of Bertin and Lalín [4] is an interesting survey of work on the Mahler measure of many-variable polynomials.

The account [13] of one of the author's lectures relates another aspect of this story, namely the connection between Mahler measure and the geometry of hyperbolic 3-space via the so-called A-polynomial of a hyperbolic manifold (The proof of (6) by Rogers and Zudilin mysteriously uses one of these results).

The papers by the author, Rodriguez Villegas and Dunfield [17, 19] explore this question more systematically. Slides for the author's lecture at the 2015 Pacific Northwest Number Theory Conference in Eugene, Oregon [15] give an expanded version of some of the material discussed above, particularly concerning the family (7), and contain photos of some of the dramatis personae.

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