3. Mahler's Work and Algebraic Dynamical Systems

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After Furstenberg had provided a first glimpse of remarkable rigidity phenomena associated with the joint action of several commuting automorphisms (or endomorphisms) of a compact Abelian group, further key examples motivated the development of an extensive theory of such actions.

Two of Mahler's achievements, the recognition of the significance of Mahler measure of multivariate polynomials in relating the lengths and heights of products of polynomials in terms of the corresponding quantities for the constituent factors, and his work on additive relations in fields, have unexpectedly played important roles in the study of entropy and higher order mixing for these actions.

This article briefly surveys these connections between Mahler's work and dynamics. It also sketches some of the dynamical outgrowths of his work that are very active today, including the investigation of the Fuglede–Kadison determinant of a convolution operator in a group von Neumann algebra as a noncommutative generalisation of Mahler measure, as well as Diophantine questions related to the growth rates of periodic points and their relation to entropy.

1 Dynamical background

In order to describe the connections between Mahler's work and dynamical systems, we have to recall some background information.

Let X be a compact Abelian group, and let μ denote the normalised Haar measure on X, so that $\mu(X) = 1$. We write $\mathrm{Aut}(X)$ for the group of continuous algebraic automorphisms of X. Halmos [14] observed 75 years ago that, if $A \in \mathrm{Aut}(X)$, then the measure ν defined by $\nu(E) = \mu(A(E))$ is also a normalised, translation-invariant measure on X, and hence $\nu = \mu$ by uniqueness of Haar measure. In other words, A preserves the measure μ.

Example 1.1 (Toral automorphisms). *Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and let $\mathbb{T}^n$ denote the n-dimensional torus. Then every $A \in \mathrm{GL}(n, \mathbb{Z})$ gives an automorphism of $\mathbb{T}^n$, and all continuous group automorphisms of $\mathbb{T}^n$ arise this way. Hence* $\mathrm{Aut}(\mathbb{T}^n) \cong \mathrm{GL}(n, \mathbb{Z})$.
An explicit example to keep in mind is the matrix $A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ acting on $\mathbb{T}^2$. Together with its square, the Arnold cat map $A^2 = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right]$ (cf., e.g., [1, 9]), this

toral automorphism has given rise to a vast amount of literature — mathematical, numerological and phenomenological — which we cannot explore here, but which certainly makes for fascinating reading.

Rather than exploring intricacies of individual toral automorphisms, we shall concentrate here on dynamical systems arising from the simultaneous action of *several* automorphisms of a compact Abelian group X, and on the somewhat surprising properties of such systems.

Let Γ be a countable discrete group (not necessarily Abelian). An *algebraic Γ-action* is a homomorphism $\alpha\colon \Gamma \to \mathrm{Aut}(X)$ for some compact Abelian group X. It is convenient to use exponential notation for α, writing α^γ instead of $\alpha(\gamma)$. In Example 1.1, $\Gamma = \mathbb{Z}$ and $\alpha^k = A^k$.

The interest in algebraic actions of groups other (i.e., bigger) than $\mathbb{Z}$ has its roots in two examples: *Furstenberg's example* [11], consisting of the $\mathbb{N}^2$-action α on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ generated by the commuting endomorphisms $\times 2$ and $\times 3$, and *Ledrappier's example* [21], which will play quite an important role in this article.

Example 1.2 (Ledrappier's example). *Consider the compact Abelian group $Y = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$ with coordinate-wise addition. Each element $x \in Y$ has the form $x = (x_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^2}$, where each $x_{\boldsymbol{n}} \in \mathbb{Z}/2\mathbb{Z}$, and can be thought of as a two-dimensional array of 0's and 1's. There is a natural $\mathbb{Z}^2$-shift action σ on Y defined by $(\sigma^{\boldsymbol{m}}x)_{\boldsymbol{n}} = x_{\boldsymbol{n}-\boldsymbol{m}}$. Let $\boldsymbol{e}_1 = (1,0)$ and $\boldsymbol{e}_2 = (0,1)$ be the standard basis for $\mathbb{Z}^2$. Define a subgroup X_L of Y by*

$$X_L = \{x \in Y : x_{\boldsymbol{n}} + x_{\boldsymbol{n}+\boldsymbol{e}_1} + x_{\boldsymbol{n}+\boldsymbol{e}_2} = 0 \text{ for all } \boldsymbol{n} \in \mathbb{Z}^2\}. \tag{1}$$

This additive condition is clearly shift-invariant, so that we can define an algebraic $\mathbb{Z}^2$-action α_L on X by restricting σ to X_L.

Both these examples are deceptively simple. In Furstenberg's example, the existence of nonatomic α-invariant probability measures ν on $\mathbb{T}$ other than Lebesgue measure has remained unresolved since 1967 and has led to a major new direction of research on *measure rigidity* of algebraic group actions. In Ledrappier's example it was the *higher order mixing* properties of the system which provided the original focus of work by Ledrappier and others. Another avenue of research opened up with the replacement of the 'alphabet' $\mathbb{Z}/2\mathbb{Z}$ in (1) by $\mathbb{T}$, leading to the notion of a 'principal algebraic $\mathbb{Z}^d$-action' (cf. Example 2.1) and, beyond that, to the exploration of *algebraic actions* of arbitrary countable groups Γ.

However, before starting to explore algebraic group actions at that level of generality, we return to the more familiar ground of algebraic $\mathbb{Z}^d$-actions with its wealth of examples (see [32] for a detailed account of that theory).

2 Algebraic $\mathbb{Z}^d$-actions

For $\Gamma = \mathbb{Z}^d$, the integer group ring $\mathbb{Z}\Gamma$ is isomorphic to the ring

$$R_d := \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$$

of Laurent polynomials in the commuting variables $u_1, \dots, u_d$. We write a typical element $f \in R_d$ as $\sum_{\boldsymbol{m}\in\mathbb{Z}^d} f_{\boldsymbol{m}} \boldsymbol{u}^{\boldsymbol{m}}$, where $\boldsymbol{u}^{\boldsymbol{m}} = u_1^{m_1} \cdots u_d^{m_d}$ and $f_{\boldsymbol{m}} \in \mathbb{Z}$ with $f_{\boldsymbol{m}} = 0$ for all but finitely many $\boldsymbol{m} \in \mathbb{Z}^d$. When $d = 2$, for notational simplicity, we use variables u and v rather than u_1 and u_2 throughout, so that $R_2 = \mathbb{Z}[u^{\pm 1}, v^{\pm 1}]$.

Example 2.1 (Principal and cyclic $\mathbb{Z}^d$-actions). *There is a natural shift action σ of $\mathbb{Z}^d$ on $\mathbb{T}^{\mathbb{Z}^d}$ given by $(\sigma^{\boldsymbol{m}} x)_{\boldsymbol{n}} = x_{\boldsymbol{n}-\boldsymbol{m}}$ for every $x \in \mathbb{T}^{\mathbb{Z}^d}$ and $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{Z}^d$. This definition of the shift map is the opposite of the more traditional one, but is consistent with how shifts must be defined when the acting group is noncommutative. For $f \in R_d$ define*

$$X_f = \left\{ x \in \mathbb{T}^{\mathbb{Z}^d} : \sum_{\boldsymbol{m}\in\mathbb{Z}^d} f_{\boldsymbol{m}} x_{\boldsymbol{n}+\boldsymbol{m}} = 0 \text{ for all } \boldsymbol{n} \in \mathbb{Z}^d \right\} \subset \mathbb{T}^{\mathbb{Z}^d}. \tag{2}$$

As in Ledrappier's example, this condition on x is invariant under the shift action σ on $\mathbb{T}^{\mathbb{Z}^d}$, and so the restriction α_f of σ to X_f gives an algebraic $\mathbb{Z}^d$-action on X_f, called the principal algebraic $\mathbb{Z}^d$-action defined by f.

For principal actions there is a convenient and very explicit way to describe the Pontryagin dual $\widehat{X_f}$ and the $\mathbb{Z}^d$-action dual to α_f. The dual group of the Cartesian product $\mathbb{T}^{\mathbb{Z}^d}$ is the direct sum $\bigoplus_{\boldsymbol{m}\in\mathbb{Z}^d} \mathbb{Z}$, which as an additive group is just R_d. The automorphism dual to the shift-transformation $\sigma^{\boldsymbol{m}}$ is left multiplication by $\boldsymbol{u}^{\boldsymbol{m}}$ on R_d. The dual of the subgroup X_f of $\mathbb{T}^{\mathbb{Z}^d}$ is the quotient of R_d by the annihilator of X_f, which is the principal ideal fR_d. Thus $\widehat{X_f} = R_d/fR_d$, which explains the terminology 'principal action'.

If we replace the principal ideal fR_d by an arbitrary ideal $I \subset R_d$, we obtain the cyclic R_d-module $M = R_d/I$ and the corresponding cyclic algebraic $\mathbb{Z}^d$-action $\alpha_{R_d/I}$ on $X_{R_d/I} = \widehat{R_d/I}$. When $I = fR_d$ is principal, we abbreviate these to α_f on X_f, as above.

Example 2.2 (A toral automorphism as a principal $\mathbb{Z}$-action). *Let $\Gamma = \mathbb{Z}$. Then $\mathbb{Z}\Gamma$ is isomorphic to $\mathbb{Z}[u^{\pm 1}]$, the ring of Laurent polynomials in a single variable u. If $f(u) = u^2 - u - 1$, it is an instructive little exercise to show that the principal $\mathbb{Z}$-action (X_f, α_f) is isomorphic to the toral automorphism $A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ in Example 1.1.*

The Arnold cat map $B = A^2 = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right]$ in Example 1.1 is, of course, also of the form $B = \alpha_M$ for some R_1-module M. Show that this module M is again cyclic. However, the third power $C = A^3 = \left[\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix}\right]$ is of the form $C = \alpha_N$ for some R_1-module N which is not cyclic (cf. [32] Example 5.3 (2)). What about A^n with $n > 3$? Are any of the corresponding R_1-modules cyclic?

Example 2.3 (Furstenberg's example, revisited). *Put $\Gamma = \mathbb{Z}^2$ so that $\mathbb{Z}\Gamma = \mathbb{Z}[u^{\pm 1}, v^{\pm 1}]$. Let $I = \langle u-2, v-3 \rangle = (u-2)R_2 + (v-3)R_2 \subset R_2$ be the nonprincipal ideal generated by $u-2$ and $v-3$. As in Example 2.1, we* *see that the cyclic $\mathbb{Z}^2$-action $\alpha = \alpha_{R_2/I}$ is the restriction of the shift-action σ on $\mathbb{T}^{\mathbb{Z}^2}$ to the closed, shift-invariant subgroup*

$$X = X_{R_2/I} = \bigl\{x \in \mathbb{T}^{\mathbb{Z}^2} : x_{\boldsymbol{n}+\boldsymbol{e}_1} = 2x_{\boldsymbol{n}} \text{ and } x_{\boldsymbol{n}+\boldsymbol{e}_2} = 3x_{\boldsymbol{n}} \text{ for every } \boldsymbol{n} \in \mathbb{Z}^2\bigr\}.$$

If $\pi_{\mathbf{0}} \colon X \to \mathbb{T}$ is the projection which sends each $x = (x_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^2} \in X$ to its zero coordinate $x_{\mathbf{0}}$, then the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\alpha^{\boldsymbol{e}_1}} & X \\ {\scriptstyle \pi_{\mathbf{0}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathbf{0}}} \\ \mathbb{T} & \xrightarrow[\times 2]{} & \mathbb{T} \end{array} \quad \textit{and} \quad \begin{array}{ccc} X & \xrightarrow{\alpha^{\boldsymbol{e}_2}} & X \\ {\scriptstyle \pi_{\mathbf{0}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathbf{0}}} \\ \mathbb{T} & \xrightarrow[\times 3]{} & \mathbb{T} \end{array}$$

commute, so that we obtain Furstenberg's example as a factor of $\alpha_{R_2/I}$.

Example 2.4 (Ledrappier's example, revisited). *If we identify the additive group $Y = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$ in Example 1.2 with $\{0, \frac{1}{2}\}^{\mathbb{Z}^2} \subset \mathbb{T}^{\mathbb{Z}^2} = \widehat{R_2}$, then $\widehat{Y} = R_2/2R_2$, and the group $\widehat{X}$ dual to (1) is the cyclic R_2-module $R_2/\langle 2, f\rangle$, where $f(u,v) = 1+u+v \in R_2$, and where $\langle 2, f\rangle = 2R_2 + fR_2$ is the nonprincipal ideal generated by 2 and f.*

For explicit calculations with Ledrappier's example, it is convenient to rewrite this R_2-module by viewing f as an element $\tilde{f}$ of the ring $R_2^{(2)} := \mathbb{F}_2[u^{\pm1}, v^{\pm1}]$ of Laurent polynomials in u, v with coefficients in the prime field $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$, and by identifying $\widehat{X}$ with the R_2-module $R_2^{(2)}/\tilde{f}R_2^{(2)}$.

Example 2.5 (Ledrappier's example with continuous alphabet). *If we replace the alphabet $\mathbb{Z}/2\mathbb{Z}$ in Ledrappier's Example 1.2 by $\mathbb{T}$, we obtain the closed, shift-invariant subgroup*

$$X' = \{x \in \mathbb{T}^{\mathbb{Z}^2} : x_{\boldsymbol{n}} + x_{\boldsymbol{n}+\boldsymbol{e}_1} + x_{\boldsymbol{n}+\boldsymbol{e}_2} = 0 \text{ for every } \boldsymbol{n} \in \mathbb{Z}^2\}. \tag{3}$$

It is easy to check that the shift-action of $\mathbb{Z}^2$ on $\widehat{X'}$ coincides with the principal $\mathbb{Z}^2$-action α_f on X_f, where $f(u,v) = 1+u+v$.

3 Mixing properties of Algebraic $\mathbb{Z}^d$-actions

A little experimentation with (1) in Ledrappier's Example 1.2 yields

$$x_{\boldsymbol{n}} + x_{\boldsymbol{n}+2^k\boldsymbol{e}_1} + x_{\boldsymbol{n}+2^k\boldsymbol{e}_2} \equiv 0 \pmod 2, \tag{4}$$

so that the coordinates $x_{\boldsymbol{n}}$ and $x_{\boldsymbol{n}+2^k\boldsymbol{e}_1}$ together determine the coordinate $x_{\boldsymbol{n}+2^k\boldsymbol{e}_2}$ of x for every $x \in X$, $\boldsymbol{n} \in \mathbb{Z}^d$ and $k \geqslant 0$.

Having identified the dual group of Ledrappier's example with $R_2^{(2)}/\tilde{f}R_2^{(2)}$ in Example 2.4, Eq. (4) comes as no surprise; since $(g+h)^2 = g^2 + h^2$ for all

$g, h \in R_2^{(2)}$, it follows that $\tilde{f}(u,v)^{2^k} = 1 + u^{2^k} + v^{2^k}$ lies in the annihilator $\tilde{f}R_2^{(2)}$ of X for every $k \geqslant 1$, precisely the content of (4).

In order to appreciate the significance of (4), we recall that a measure-preserving action $T\colon \boldsymbol{n} \mapsto T^{\boldsymbol{n}}$ of $\mathbb{Z}^d$ on a probability space $(Y, \mathcal{T}, \nu)$ is *mixing* if

$$\lim_{\|\boldsymbol{n}\|\to\infty} \nu(B_1 \cap T^{\boldsymbol{n}} B_2) = \nu(B_1)\nu(B_2)$$

for all sets $B_1, B_2 \in \mathcal{T}$, where $\|\boldsymbol{n}\|$ denotes the euclidean norm of $\boldsymbol{n}$. More generally, the action T is *r-mixing* with $r \geqslant 2$ if, for all $B_1, \dots, B_r \in \mathcal{T}$,

$$\nu\Bigl(\bigcap_{i=1}^{r} T^{\boldsymbol{n}_i} B_i\Bigr) \longrightarrow \prod_{i=1}^{r} \nu(B_i) \text{ as } \|\boldsymbol{n}_i - \boldsymbol{n}_j\| \to \infty \text{ for } 1 \leqslant i < j \leqslant r. \tag{5}$$

For single measure-preserving automorphisms of probability spaces, the question of whether mixing implies mixing of every order has been open for well over 50 years [15, p. 99]. However, Ledrappier's example shows that the answer is negative for $\mathbb{Z}^d$-actions with $d \geqslant 2$, which we now explain.

It is relatively simple to show that there are no long-range correlations between pairs of coordinates, so that Ledrappier's example (X, α) is mixing. However, (4) shows that, if $B = \{x \in X : x_{\mathbf{0}} = 0\}$, then for all $k \geqslant 1$ we have

$$B \cap \alpha^{2^k \boldsymbol{e}_1}(B) \cap \alpha^{2^k \boldsymbol{e}_2}(B) = B \cap \alpha^{2^k \boldsymbol{e}_1}(B).$$

Hence

$$\mu\bigl(B \cap \alpha^{2^k \boldsymbol{e}_1}(B) \cap \alpha^{2^k \boldsymbol{e}_2}(B)\bigr) = \mu\bigl(B \cap \alpha^{2^k \boldsymbol{e}_1}(B)\bigr) = \frac{1}{4} \neq \frac{1}{8} = \mu(B)^3 \tag{6}$$

for all $k \geqslant 1$. Thus Ledrappier's example is not 3-mixing.

In order to reflect the particularly regular way in which higher-order mixing breaks down in Ledrappier's example, we introduce a definition.

Definition 3.1. *Let $T\colon \boldsymbol{n} \mapsto T^{\boldsymbol{n}}$ be a measure-preserving $\mathbb{Z}^d$-action on a probability space $(Y, \mathcal{T}, \nu)$. A nonempty finite set $F \subset \mathbb{Z}^d$ is mixing if*

$$\lim_{k\to\infty} \nu\Bigl(\bigcap_{\boldsymbol{n}\in F} T^{k\boldsymbol{n}}(B_{\boldsymbol{n}})\Bigr) = \prod_{\boldsymbol{n}\in F} \nu(B_{\boldsymbol{n}})$$

for every collection of Borel sets $B_{\boldsymbol{n}} \in \mathcal{T}$, $\boldsymbol{n} \in F$. A nonempty finite set $F \subset \mathbb{Z}^d$ is called nonmixing if it is not mixing.

According to (6), Ledrappier's example has the nonmixing set $F = \{\mathbf{0}, \boldsymbol{e}_1, \boldsymbol{e}_2\}$ of size 3.

More generally, suppose $p \geqslant 2$ is a rational prime, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ the corresponding prime field, and $R_d^{(p)} = \mathbb{F}_p[u_1^{\pm1}, \dots, u_d^{\pm1}]$ is the ring of Laurent polynomials in $u_1, \dots, u_d$ with coefficients in $\mathbb{F}_p$. We write a typical element

$\tilde{f} \in R_d^{(p)}$ as $\tilde{f} = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} \tilde{f}_{\boldsymbol{n}} u^{\boldsymbol{n}}$ with $\tilde{f}_{\boldsymbol{n}} \in \mathbb{F}_p$ for every $\boldsymbol{n} \in \mathbb{Z}^d$ and denote by $\mathcal{S}(\tilde{f}) = \{\boldsymbol{n} \in \mathbb{Z}^d : \tilde{f}_{\boldsymbol{n}} \neq 0\}$ the *support* of $\tilde{f}$. Exactly as in the brief discussion of Ledrappier's example at the beginning of this section, one obtains the following result.

Proposition 3.2. *Let $I \subset R_d^{(p)}$ be an ideal and let $\alpha_{R_d^{(p)}/I}$ be the cyclic R_d-action on the group $X_{R_d^{(p)}/I}$ defined by the ring $R_d^{(p)}/I$, viewed as an R_d-module. For every $\tilde{f} \in I$, the set $\mathcal{S}(\tilde{f})$ is nonmixing for $\alpha_{R_d^{(p)}/I}$.*

Perhaps surprisingly, such an action $\alpha_{R_d^{(p)}/I}$ may have nonmixing sets which do *not* originate from elements of the ideal I. For example, if

$$\tilde{g}(u, v) = 1 + u + u^2 + uv + v^2 \in R_2^{(2)},$$

then $\alpha_{R_2^{(2)}/\tilde{g}R_2^{(2)}}$ also has the nonmixing set $F = \{\boldsymbol{0}, \boldsymbol{e}_1, \boldsymbol{e}_2\}$ appearing in Ledrappier's example, although the ideal $\tilde{g}R_2^{(2)}$ does not contain any element whose support has cardinality 3. For explanation and details we refer to [19] and [32, Section 28].

The question of existence – or nonexistence – of nonmixing sets for general algebraic $\mathbb{Z}^d$-actions turns out to be intimately connected with a result by Kurt Mahler in his paper [M31] on Taylor coefficients of rational functions. The results in [18, 30, 31] show that a mixing algebraic $\mathbb{Z}^d$-action α on a compact Abelian group X has nonmixing sets if and only if the dual R_d-module $\widehat{X}$ has an associated prime ideal $I \subset R_d$ with the following properties: if $\mathbb{F} = \mathrm{Quot}(R_d/I)$ is the field of fractions of the integral domain R_d/I, and if $G \subset \mathbb{F}$ is the multiplicative subgroup generated by the images in R_d/I of the monomials $u_1, \ldots, u_d \in R_d$, then there exist finitely many elements $a_1, \ldots, a_r$ in G and a nonzero element $(c_1, \ldots, c_r) \in \mathbb{F}^r$, such that

$$c_1 a_1^m + \cdots + c_r a_r^m = 1 \text{ for infinitely many } m \geqslant 1. \tag{7}$$

If the group X is connected, the field $\mathbb{F} = \mathrm{Quot}(R_d/I)$ in (7) has characteristic zero for every prime ideal I associated with $\widehat{X}$, and if X is totally disconnected, $\mathbb{F} = \mathrm{Quot}(R_d/I)$ always has positive characteristic. For Ledrappier's example, the prime ideal in question is $I = \langle 2, 1 + u + v\rangle$, and the field $\mathbb{F} = \mathrm{Quot}(R_d/I)$ has characteristic 2.

In the former case, when $\mathbb{F}$ has characteristic zero, a beautiful p-adic argument by Kurt Mahler in [M31, p. 57] shows that (7) implies the existence of integers $1 \leqslant k < l \leqslant r$ and $b > 0$ such that

$$a_k^b = a_l^b. \tag{8}$$

By translating this back into our dynamical setting one obtains a contradiction to the hypothesis that α is mixing. This leads to the following conclusion. See [30] and [32, p. 268].

Corollary 3.3. *Let α be a mixing algebraic $\mathbb{Z}^d$-action on a compact connected Abelian group X. Then, every nonempty finite subset $F \subset \mathbb{Z}^d$ is mixing for α.*

In the latter case, when $\mathbb{F}$ has positive characteristic, David Masser [19] proved that (7) implies a more complicated relationship between $a_1, \dots, a_r$; if $\mathbb{F}$ has characteristic $p > 1$, and if $\overline{\mathbb{F}}_p$ is the algebraic closure of the prime field $\mathbb{F}_p$, then there exist elements $b_1, \dots, b_r$ in the algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$ and $k, l \geqslant 1$ such that $a_i = b_i^k$ for $i = 1, \dots, r$ and $\{b_1^l, \dots, b_r^l\}$ is linearly dependent over $\overline{\mathbb{F}}_p$.

This result allows one in principle to determine the nonmixing sets of algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact Abelian groups.

The story of the connection between mixing properties of algebraic $\mathbb{Z}^d$-actions and additive relations in fields, which turns out to have begun with Mahler's Theorem [M31, p. 57], doesn't end with Corollary [30, Cor. 2.3]. Subsequent developments, based on remarkable work on S-unit equations both in characteristic zero (cf., e.g., [36]) and in positive characteristic (cf., e.g., [5, 6, 19, 29]), clarified the connection between nonmixing sets and the order of mixing. The following result gives a brief and incomplete summary of these later developments.

Theorem 3.4. *Let α be a mixing algebraic $\mathbb{Z}^d$-action on a compact Abelian group X.*

(1) If X is connected, α is mixing of every order [33, Cor. 3.3].

(2) If X is totally disconnected, α has nonmixing sets if and only if it does not have completely positive entropy. Moreover, if $r \geqslant 2$, then α is r-mixing if and only if every subset $F \subset \mathbb{Z}^d$ of cardinality r is mixing for α [29, p. 190].

4 Entropy and Mahler measure

Entropy is a numerical invariant of dynamical systems which can be defined for measure-preserving as well as continuous actions. Here, our focus will on 'topological' entropy, which provides a rough measure of the distortion of the topology of a space under a group action by homeomorphisms of that space.

The exact calculation of entropy, or even a numerical approximation, is generally difficult. However, computing entropy for algebraic actions is easier for a very important reason: the homogeneity of an algebraic action means that the calculation of the amount of 'distortion' of the space can be reduced to measuring the distortion of small neighbourhoods of the identity of the group under the action.

In this section, we consider an algebraic $\mathbb{Z}$-action α on a compact Abelian group X equipped with Haar measure μ.

Let $\mathcal{U}$ be an open neighbourhood of the identity 0_X of X. The set of points in X that remain within $\mathcal{U}$ for the first n iterates of α is $\bigcap_{j=0}^{n-1} \alpha^{-j}\mathcal{U}$, and the rate of decay of the measure of this set measures how $\mathcal{U}$ changes under the first few elements $\alpha^1, \dots, \alpha^{n-1}$ of the action α. In order to obtain a scale invariant quantity, we consider decreasing sequences of neighbourhoods of the identity and define the *entropy* $\mathsf{h}(\alpha)$ *of* α as

$$\mathsf{h}(\alpha) := \lim_{\mathcal{U}\searrow\{0_X\}} \limsup_{n\to\infty} -\frac{1}{n}\log\mu\Bigl(\bigcap_{j=0}^{n-1}\alpha^{-j}\mathcal{U}\Bigr). \tag{9}$$

What is not apparent from this definition is the crucial property that entropy is invariant under measure-preserving conjugacy: if (X,α) and (Y,β) are algebraic $\mathbb{Z}$-actions, and if $\phi\colon X \to Y$ is an invertible measurable map which preserves Haar measure and is *equivariant* in the sense that $\phi\circ\alpha^n = \beta^n\circ\phi$, then $\mathsf{h}(\alpha) = \mathsf{h}(\beta)$. The proof of this invariance involves, in particular, establishing the equality of topological and Haar measure-theoretic entropy for algebraic actions (see [37] for details).

Example 4.1 (k-shift). *Let $X_k = (\mathbb{Z}/k\mathbb{Z})^{\mathbb{Z}}$, and σ_k be the shift on X_k, which is called the k-shift. To compute entropy, it is enough to consider neighbourhoods $\mathcal{U}_r = \{x \in X_k\colon x_j = 0 \text{ for } -r \leqslant j \leqslant r\}$ for large r. Since Haar measure here is product measure,*

$$\mu\Bigl(\bigcap_{j=0}^{n-1}\sigma_k^{-j}\mathcal{U}_r\Bigr) = \mu\bigl(\{x\in X_k\colon x_j = 0 \text{ for } -r\leqslant j\leqslant n-1+r\}\bigr) = \Bigl(\frac{1}{k}\Bigr)^{n+2r},$$

and hence

$$-\frac{1}{n}\log\mu\Bigl(\bigcap_{j=0}^{n-1}\sigma_k^{-j}\mathcal{U}_r\Bigr) \xrightarrow{n\to\infty} \log(k)$$

for every $r \geqslant 1$, and so $\mathsf{h}(\sigma_k) = \log(k)$.

Example 4.2 (Toral automorphism). *Let $A \in \mathrm{GL}(r,\mathbb{Z}) = \mathrm{Aut}(\mathbb{T}^r)$. Then, the eigenvalues $\lambda_1,\dots,\lambda_r$ of A, listed with multiplicity, are all nonzero. The eigenvalues λ_i for which $|\lambda_i| > 1$ control the volume decrease in (9). If $\mathcal{U}$ is a 'nice' small neighbourhood of 0 in $\mathbb{T}^r$ (like $\prod_{i=1}^r(-\varepsilon,\varepsilon)$ for small $\varepsilon > 0$) then, knowing the Jordan form of A, it is relatively easy to show that we can find positive constants c_1 and c_2 such that*

$$c_1 n^{-r}\Bigl(\prod_{|\lambda_i|>1}|\lambda_i|^{-n}\Bigr) \leqslant \mu\Bigl(\bigcap_{j=0}^{n-1}A^{-j}\mathcal{U}\Bigr) \leqslant c_2 n^{r}\Bigl(\prod_{|\lambda_i|>1}|\lambda_i|^{-n}\Bigr).$$

It then follows from the definition (9) that

$$\mathsf{h}(A) = \sum_{|\lambda_i|>1}\log|\lambda_i|. \tag{10}$$

This formula for entropy was established by Sinai in 1959 [34, 35], shortly after the introduction of entropy.

For a polynomial $f(u) \in \mathbb{C}[u]$, Mahler [M143] defined its 'measure' to be

$$\mathsf{M}(f) \;=\; \exp\left[\int_0^1 \log|f(e^{2\pi \mathrm{i}t})|\,\mathrm{d}t\right].$$

It is convenient to introduce the *logarithmic Mahler measure* of f to be

$$\mathsf{m}(f) \;=\; \log \mathsf{M}(f) \;=\; \int_0^1 \log|f(e^{2\pi \mathrm{i}t})|\,\mathrm{d}t.$$

If we write $f(u) = s\prod_{j=1}^{r}(u-\lambda_j)$, then, as Mahler observed, Jensen's formula yields that

$$\mathsf{m}(f) \;=\; \log(s) \;+\; \sum_{|\lambda_j|>1} \log|\lambda_j|.$$

For $A \in \mathrm{GL}(r,\mathbb{Z})$ as in Example 4.2, let $\chi_A(u) = \det[uI - A]$ be its (monic) characteristic polynomial. The calculation of entropy in this example can then be expressed as $\mathsf{h}(A) = \mathsf{m}(\chi_A)$, that is, *entropy equals the logarithmic Mahler measure of the related characteristic polynomial.*

The first author (DL) observed this around 1970, but viewed it as merely a curiosity. It does, however, provide a link between dynamics and the famous (and still open) Lehmer Problem: does $\inf\{\mathsf{m}(f)\colon f \in \mathbb{Z}[u] \text{ and } \mathsf{m}(f) > 0\} = 0$? The article [3] by David Boyd in this *Selecta* discusses Lehmer's Problem in detail. As shown in [23], this question is equivalent to asking whether there are toral automorphisms of arbitrarily small positive entropy, and also equivalent to asking whether there is an ergodic automorphism of $\mathbb{T}^{\mathbb{Z}}$ with finite entropy.

The next example shows that p-adic fields arise naturally in the study of algebraic actions, leading to a p-adic version of Mahler measure that is used in calculating entropy for certain actions.

For each rational prime p, recall that $\mathbb{Q}_p$ denotes the completion of $\mathbb{Q}$ with respect to the p-adic valuation $|\cdot|_p$, normalised so that $|p|_p = p^{-1}$. We use the convention that $|\cdot|_\infty$ is the usual absolute value on $\mathbb{Q}$, so that $\mathbb{Q}_\infty = \mathbb{R}$. Each $\mathbb{Q}_p$ is a locally compact field, and has a normalised Haar measure μ_p.

Example 4.3. *Let $f(u) = 2u - 3$, and consider the principal algebraic $\mathbb{Z}$-action α_f on X_f defined in (2). It is easy to check that $\widehat{X_f} \cong \mathbb{Z}[1/6]$ and that α_f is dual to multiplication by $3/2$ on $\mathbb{Z}[1/6]$. Then, locally, X_f is $\mathbb{Q}_2 \times \mathbb{Q}_3 \times \mathbb{R}$, and in this local view, α_f is the 3×3 diagonal matrix $\frac{3}{2} \cdot I$.*

However, here $3/2$ has different sizes in each component:

$$\left|\frac{3}{2}\right|_2 = 2, \quad \left|\frac{3}{2}\right|_3 = \frac{1}{3}, \quad \left|\frac{3}{2}\right|_\infty = \frac{3}{2}.$$

Reasoning as in Example 4.2, only those 'eigenvalues' with size greater that 1 contribute to entropy, and so

$$\mathsf{h}(\alpha_f) = \log(2) + \log\left(\frac{3}{2}\right) = \log(3).$$

Observe also that $\mathsf{m}(f) = \log(2) + \log\left(\frac{3}{2}\right)$*, so that*

$$\mathsf{h}(\alpha_f) = \mathsf{m}(f).$$

This example combines geometric expansion in $\mathbb{Q}_\infty$ *and arithmetic expansion in* $\mathbb{Q}_2$ *to calculate entropy.*

In order to describe a general setting for such results, we introduce the full solenoid group $\Sigma = \widehat{\mathbb{Q}_d}$, where $\mathbb{Q}_d$ denotes the rationals with the discrete topology. If $A \in \mathrm{GL}(r, \mathbb{Q})$, then A acts via duality on the compact Abelian group Σ^r. After a series of papers by several authors dealing with special cases, Yuzvinskii [38] gave a general formula for the entropy of A.

Theorem 4.4 (Yuzvinskii). *Let* $A \in \mathrm{GL}(r, \mathbb{Q})$ *have complex eigenvalues* $\lambda_1, \dots, \lambda_r$ *listed with multiplicity, and let* s *be the smallest positive integer such that* $s\chi_A(u) \in \mathbb{Z}[u]$*. Then*

$$\mathsf{h}(A; \Sigma^r) = \log s + \sum_{|\lambda_j|>1} \log|\lambda_j|.$$

Yuzvinskii's proof relied heavily on complicated algebra. The role of the p-adics, and the resulting conceptual simplification, was spelled out in [28], which we now briefly describe.

Let $A \in \mathrm{GL}(n, \mathbb{Q})$, and $\overline{\mathbb{Q}}_p$ denote the algebraic closure of $\mathbb{Q}_p$. Then, $\chi_A(u)$ factors in $\overline{\mathbb{Q}}_p[u]$ as $\chi_A(u) = (u - \lambda_1^{(p)}) \dots (u - \lambda_r^{(p)})$. We define the *$p$-adic logarithmic Mahler measure of* $\chi_A(u)$ to be

$$\mathsf{m}_p(\chi_A) = \sum_{|\lambda_j^{(p)}|_p>1} \log|\lambda_j^{(p)}|_p,$$

where, here, $|\cdot|_p$ is the (unique) extension of the p-adic absolute value from $\mathbb{Q}$ to $\overline{\mathbb{Q}}_p$.

We will express the 'global' entropy of A acting on Σ^r as the sum of 'local' entropies, one for each $p \leqslant \infty$. Roughly speaking, Σ^r is locally the product $\prod_{p\leqslant\infty} \mathbb{Q}_p^r$, and each factor is preserved by A. Since entropy adds over products, it follows that $\mathsf{h}(A; \Sigma^r) = \sum_{p\leqslant\infty} h(A; \mathbb{Q}_p^r)$, where each summand is the Bowen entropy of a linear map. It is shown in [28] that $\mathsf{h}(A; \mathbb{Q}_p^r) = \mathsf{m}_p(\chi_A)$ for each $p \leqslant \infty$ and that this quantity vanishes for all but finitely many p. Hence

$$\mathsf{h}(A; \Sigma^r) = \sum_{p\leqslant\infty} \mathsf{m}_p(\chi_A).$$

Thus the somewhat mysterious term $\log(s)$ in Yuzvinskii's formula is simply the sum of the p-adic entropies over $p < \infty$, while the remaining term is the local entropy at $p = \infty$.

Example 4.5. *Let* $A = [3/2] \in \mathrm{GL}(1,\mathbb{Q})$*. Then* $s = 2$ *and* $\lambda_1 = 3/2$*, and so* $\mathsf{h}(A;\Sigma) = \log(3)$.

Example 4.6. *Let* $B = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 6/5 \end{smallmatrix}\right] \in \mathrm{GL}(2,\mathbb{Q})$*. The complex eigenvalues of* B *both have absolute value 1, and so there is no geometric contribution to entropy. The only nonzero contribution happens in* $\mathbb{Q}_5$*, and* $\mathsf{h}(B,\Sigma^2) = \log(5)$*, providing an interesting example where the only expansion is arithmetic.*

Remark 4.7. *Consider the polynomials* $f(u) = 2u - 3$*,* $g(u) = 5u^2 - 6u + 5$ *in* $\mathbb{Z}[u^{\pm 1}]$ *and define the principal* $\mathbb{Z}$*-actions* α_f *on* X_f *and* α_g *on* X_g *as in (2). Then, the same calculations as in Example 4.3 show that* $\mathsf{h}(\alpha_f) = \mathsf{m}(f) = \mathsf{h}(A,\Sigma)$ *and* $\mathsf{h}(\alpha_g) = \mathsf{m}(g) = \mathsf{h}(B,\Sigma^2)$*. Here the principal algebraic actions* (X_f,α_f) *and* (X_g,α_g) *are equal entropy factors of the actions* (Σ, A) *and* (Σ^2, B)*, respectively, appearing in Examples 4.5 and 4.6.*

We note that the group X_f *coincides with the group* $X_{R_2/I}$ *from Furstenberg's Example 2.3, although the* $\mathbb{Z}^2$*-action is quite different.*

5 Algebraic $\mathbb{Z}^d$-actions and Mahler measure

In this section, we discuss entropy for algebraic $\mathbb{Z}^d$-actions and the discovery of its connection with the Mahler measure of polynomials in several variables.

Let α be an algebraic $\mathbb{Z}^d$-action on X with Haar measure μ. To define entropy, we simply replace the iterates $\{0,1,\dots,n-1\}$ used for $\mathbb{Z}$-actions in (9) with the n-cube $F_n = \{0,1,\dots,n-1\}^d$ and set

$$\mathsf{h}(\alpha) := \lim_{\mathcal{U}\searrow\{0_X\}} \limsup_{n\to\infty} -\frac{1}{|F_n|}\log\mu\Big(\bigcap_{\boldsymbol{j}\in F_n}\alpha^{-\boldsymbol{j}}\mathcal{U}\Big).$$

The crucial property of the F_n is that their boundaries are small compared with their volumes. More precisely, they obey the *Følner condition* that for every $\boldsymbol{k} \in \mathbb{Z}^d$,

$$\frac{|(F_n + \boldsymbol{k})\triangle F_n|}{|F_n|} \to 0 \quad \text{as } n \to \infty, \tag{11}$$

where $|\cdot|$ denotes cardinality and $\triangle$ denotes symmetric difference.

Example 5.1. *Recall Ledrappier's example* (X,μ) *from Example 1.2. Now, let* $F_n = \{0,1,\dots,n-1\}^2$ *and* $\mathcal{U} = \{x \in X : x_{(0,0)} = 0\}$*. Since* μ *is shift-invariant,* $\mu\big(\bigcap_{\boldsymbol{j}\in F_n}\alpha^{-\boldsymbol{j}}\mathcal{U}\big) = \mu\big(\bigcap_{\boldsymbol{j}\in F_n}\alpha^{\boldsymbol{j}}\,\mathcal{U}\big)$.

Consider the map $\phi_n \colon X \to (\mathbb{Z}/2\mathbb{Z})^{F_n}$ *given by the restriction* $\phi_n(x) = x|_{F_n}$*. Its image* $\phi_n(X)$ *is a subgroup of* $(\mathbb{Z}/2\mathbb{Z})^{F_n}$ *and its kernel is* $\bigcap_{\boldsymbol{j}\in F_n}\alpha^{\boldsymbol{j}}\,\mathcal{U}$*. Hence*

$$\mu\Big(\bigcap_{\boldsymbol{j}\in F_n}\alpha^{\boldsymbol{j}}\mathcal{U}\Big) = \frac{1}{|\phi_n(X)|}.$$

Next, we observe that the defining relation $x_{(i,j)} + x_{(i,j+1)} + x_{(i+1,j)} = 0$ *shows that the coordinates of* x *in* F_n *determine its coordinates in* $\{(0,0),(1,0),\dots,(2n-1,0)\}$ *and conversely, and that the coordinates in the latter range may be chosen freely. Hence* $|\phi_n(X)| = 2^{2n}$*. Thus, as* $n \to \infty$*, we have that*

$$-\frac{1}{n^2}\log\mu\Bigl(\bigcap_{\boldsymbol{j}\in F_n}\alpha^{-\boldsymbol{j}}\mathcal{U}\Bigr) = -\frac{1}{n^2}\log\mu\Bigl(\bigcap_{\boldsymbol{j}\in F_n}\alpha^{\boldsymbol{j}}\mathcal{U}\Bigr) = -\frac{1}{n^2}\log 2^{-2n} \to 0.$$

A similar argument works for arbitrarily small neighbourhoods $\mathcal{U}$ *of* 0_X*, showing that* $\mathsf{h}(\alpha) = 0$*.*

In the spring of 1988, the second author (KS) visited the Institute for Advanced Study at Princeton. Before leaving for Princeton, KS had been discussing examples of principal $\mathbb{Z}^2$-actions (α_f, X_f) for $f(u,v) \in R_2$ with Tom Ward, who was a PhD student at Warwick at the time, and who had observed positivity of entropy for some of these examples. In Princeton, KS started thinking about positivity of entropy for the 'continuous' version (3) of Ledrappier's example, but was unable to resolve the question.

After Princeton, KS visited Seattle and discussed this problem with DL, who observed that if the state group is $C_n = \{0, 1/n, 2/n, \dots, (n-1)/n\} \subset \mathbb{T}$ (so that C_2 gives Ledrappier's original example), then each of these 'finite approximations' has zero entropy for the same reason as in the preceding example. Since $C_n \to \mathbb{T}$ in some sense, the continuous Ledrappier example is a limit of zero entropy approximations, and so should also have zero entropy. DL was so convinced by this reasoning that he bet KS a Japanese dinner that this was correct.

However, the attempt to turn this intuition into something rigorous ran into serious difficulties. After fruitless efforts, it began to occur to the authors that the entropy might be positive after all. It was then that DL remembered that for principal algebraic $\mathbb{Z}$-actions α_f, entropy equals the logarithmic Mahler measure $\mathsf{m}(f)$ of f. He wrote a note to KS which concluded, 'And here is a really crazy conjecture: for general f the entropy should be $\log \mathsf{M}(f)$, where $\mathsf{M}(f)$ is the Mahler measure of f.' Motivated by this conjecture, the authors were subsequently able to obtain the equality $\mathsf{h}(\alpha_f) = \mathsf{m}(f) \approx 0.3230$ for $f(u,v) = 1+u+v$, resulting in a delicious Japanese dinner for KS. The equality $\mathsf{h}(\alpha_f) = \mathsf{m}(f)$ conjectured by DL was subsequently proved in full generality in [27] and provided the crucial step for computing entropy for general algebraic $\mathbb{Z}^d$-actions.

Theorem 5.2 ([27]). *Let* $0 \neq f \in \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$*. Then, the entropy of the associated principal algebraic* $\mathbb{Z}^d$*-action* α_f *is given by* $\mathsf{h}(\alpha_f) = \mathsf{m}(f)$*.*

There are several ways to make this result plausible. We will use the growth rate of periodic points, which leads to some current research and open problems.

To motivate what follows, first consider the toral automorphism $A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ acting on $\mathbb{T}^2$. It is an instructive exercise to show that the toral automorphism $(\mathbb{T}^2, \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right])$ is the principal algebraic $\mathbb{Z}$-action α_f, where $f(u) = u^2 - u - 1$. Let $P_n(A) = \{x \in \mathbb{T}^2 \colon A^n x = x\}$, the subgroup of points in $\mathbb{T}^2$ having period n under A. To compute $|P_n(A)|$, observe that a point $x \in \mathbb{T}^2$ is in $P_n(A)$ iff its lift $\tilde{x} \in [0,1)^2 \subset \mathbb{R}^2$ satisfies $(A^n - I)\tilde{x} \in \mathbb{Z}^2$. Thus $|P_n(A)|$ equals the number of lattice points in the parallelogram $(A^n - I)\big([0,1)^2\big)$, and this is well-known to be $|\det(A^n - I)|$. Let Ω_n denote the set of nth roots of unity, which is a cyclic subgroup of $\mathbb{S} := \{z \in \mathbb{C} \colon |z| = 1\}$. Let λ_1 and λ_2 be the eigenvalues of A. Then

$$\begin{aligned} |P_n(A)| \;=\; |\det(A^n - I)| \;&=\; |(\lambda_1^n - 1)(\lambda_2^n - 1)| \\ &=\; \Bigg| \prod_{\zeta \in \Omega_n} (\lambda_1 - \zeta)(\lambda_2 - \zeta) \Bigg| \;=\; \prod_{\zeta \in \Omega_n} |f(\zeta)|. \end{aligned}$$

Thus, we can view

$$\frac{1}{n} \log |P_n(A)| \;=\; \frac{1}{n} \sum_{\zeta \in \Omega_n} \log |f(\zeta)| \tag{12}$$

as the logarithmic Mahler measure of f over the subgroup Ω_n of $\mathbb{S}$, which we will denote by $\mathsf{m}_{\Omega_n}(f)$. Notice that the right-hand side of (12) is a Riemann sum approximation to $\int_0^1 \log |f(e^{2\pi i s})| \, \mathrm{d}s = \mathsf{m}(f)$. Since $\log |f|$ is continuous on $\mathbb{S}$, we see that $\mathsf{m}_{\Omega_n}(f) \to \mathsf{m}(f) = \mathsf{h}(\alpha_f)$ as $n \to \infty$, so that the growth rate of periodic points exists as a limit and equals entropy.

The convergence of $\mathsf{m}_{\Omega_n}(f)$ to $\mathsf{m}(f)$ is much more delicate if f has roots on $\mathbb{S}$, for example if $f(u) = u^4 + 4u^3 - 2u^2 + 4u + 1$. Then $\log |f|$ has logarithmic singularities on $\mathbb{S}$, and the value of $\log |f(\zeta)|$ for some $\zeta \in \Omega_n$ could be extremely negative should ζ be very close to a root λ of f, or equivalently, if $|\lambda^n - 1|$ is very small. However, a deep Diophantine result of Gelfond [12] says that if $\lambda \in \mathbb{S}$ is an algebraic number, then for every $\varepsilon > 0$ there is a constant $C > 0$ such that $|\lambda^n - 1| > C e^{-\varepsilon n}$, and using this one can show the convergence $\mathsf{m}_{\Omega_n}(f) \to \mathsf{m}(f)$ for all $f \in \mathbb{Z}[u^{\pm 1}]$ that have no roots that are roots of unity.

Let us use this approach on the continuous Ledrappier example, i.e., the principal algebraic $\mathbb{Z}^2$-action α_f, where $f(u,v) = 1 + u + v$. For simplicity, we start with 'square' sublattices $n\mathbb{Z} \times n\mathbb{Z} = n\mathbb{Z}^2 \subset \mathbb{Z}^2$. Define

$$P_{n \times n}(\alpha_f) \;:=\; \{x \in X_f \colon \alpha^{n\boldsymbol{j}} x = x \quad \text{for all } \boldsymbol{j} \in \mathbb{Z}^2\}$$

to be the subgroup of all points in X_f fixed by iterates in $n\mathbb{Z}^2$. A calculation similar to the 1-dimensional case above suggests that

$$|P_{n \times n}(\alpha_f)| \;=\; \prod_{(\xi,\zeta) \in \Omega_n^2} |f(\xi,\zeta)|, \tag{13}$$

so that

$$\frac{1}{n^2}\log|P_{n\times n}(\alpha_f)| = \frac{1}{n^2}\sum_{(\xi,\zeta)\in\Omega_n^2}\log|f(\xi,\zeta)| \tag{14}$$

is a Riemann sum approximation to $\mathsf{m}(f)$, and we would therefore expect that $(1/n^2)\log|P_{n\times n}(\alpha_f)| \to \mathsf{h}(\alpha_f)$ as $n\to\infty$.

However, there is a serious problem. If $\omega = e^{2\pi i/3}$, then $f(\omega,\omega^2) = 0 = f(\omega^2,\omega)$. Thus, if $3 \mid n$, two of the summands in the right-hand side of (14) are $\log(0) = -\infty$, and the product in (13) equals 0. Dynamically, what is happening is that when $3 \mid n$, the subgroup $P_{n\times n}(\alpha_f)$ is no longer finite, but rather a finite union of cosets of a 2-dimensional torus. The solution to this situation is to count the *connected components* of $P_{n\times n}(\alpha_f)$, which corresponds to ignoring those points in Ω_n^2 where f vanishes. Thus we *define*

$$\mathsf{m}_{\Omega_n^2}(f) := \frac{1}{n^2}\sum_{(\xi,\zeta)\in\Omega_n^2,\ f(\xi,\zeta)\neq 0}\log|f(\xi,\zeta)|.$$

With this convention, one can prove that $\mathsf{m}_{\Omega_n^2}(f)\to\mathsf{h}(\alpha_f)$, i.e., that the growth rate of periodic components exists as a limit and is equal to entropy. For more information about $\mathsf{m}(1+u+v)$; see [3] in this *Selecta*.

For general $f\in\mathbb{Z}[u_1^{\pm1},\dots,u_d^{\pm1}]$, a crucial role is played by the *unitary variety* of f, defined as $\mathsf{U}(f) := \{\boldsymbol{s}\in\mathbb{S}^d\colon f(\boldsymbol{s}) = 0\}$. As suggested by the discussion above, if K is a finite subgroup of $\mathbb{S}^d$, we define

$$\mathsf{m}_K(f) := \frac{1}{|K|}\sum_{\boldsymbol{s}\in K\smallsetminus\mathsf{U}(f)}\log|f(\boldsymbol{s})|. \tag{15}$$

We use the notation $K\to\infty$ to mean that Haar measures on the finite subgroups K converge weakly to Haar measure on $\mathbb{S}^d$.

Problem 5.3. *If* $0\neq f\in\mathbb{Z}[u_1^{\pm1},\dots,u_d^{\pm1}]$, *does*

$$\mathsf{m}_K(f)\to\mathsf{m}(f) \quad \textit{as} \quad K\to\infty? \tag{16}$$

In other words, do the Riemann sums for $\log|f|$ over finite subgroups of $\mathbb{S}^d$ (modified to avoid values of $-\infty$) converge to $\int_{\mathbb{S}^d}\log|f|$?

In [32], KS had shown that the answer to Problem 5.3 is 'yes' if $\mathsf{h}(\alpha_f)<\infty$, and if one replaces the limit in (16) by 'lim sup'. In [25], it was shown that the answer is 'yes' if $\mathsf{U}(f)$ is finite, and then in [26] that the answer is also 'yes' if the real dimension $\dim\mathsf{U}(f)$ of $\mathsf{U}(f)$ is less than or equal to $d-2$. Both papers use dynamical ideas, in particular they use homoclinic points for the action to create sufficiently many periodic components.

Example 5.4. *Consider a three variable version of Ledrappier's example with continuous alphabet, defined by* $f(u,v,w) = 1+u+v+w$. *Here, the unitary variety* $\mathsf{U}(f)\subset\mathbb{S}^3$ *is a union of three circles, each given by setting one of the* *variables equal to* -1*. Since these circles are cosets of 1-dimensional subgroups, the dimension of the connected component of the identity for points fixed by all iterates in* $n\mathbb{Z}^3$ *grows linearly in* n*. A subtlety in the proof of convergence in [26], as illustrated in this example, is that although the dimension of the connected components grows, the linear constraint here can be used to show that these components do not contribute to entropy.*

Convergence in (16) is also a Diophantine problem, essentially asking how close points in K can come to $U(f)$. Using ideas involving Diophantine analysis, Vesselin Dimitrov [8] gave a completely different proof of (16) in his 2017 PhD thesis from Yale, again under the assumption that $\dim \mathsf{U}(f) \leqslant d-2$. Very recent work of Habegger [13] on special points near definable sets, whose proof relies on logic and O-minimal sets, was used by Dimitrov to provide yet a third proof, quite different from the previous two, again only in the case $\dim \mathsf{U}(f) \leqslant d-2$.

All three of these proofs fail when $\dim \mathsf{U}(f) = d-1$. The following example illustrates the difficulties.

Example 5.5. *Let* $f(u,v) = 3 - u - u^{-1} - v - v^{-1}$*. Here,* $\mathsf{U}(f)$ *is a 1-dimensional oval in* $\mathbb{S}^2$*, so that* $\log|f|$ *has a 1-dimensional set of logarithmic singularities. There are only four points on* $\mathsf{U}(f)$ *both of whose coordinates are roots of unity,* $(\omega^{\pm 1}, 1)$ *and* $(1, \omega^{\pm 1})$*, where* $\omega = e^{\pi i/3}$ *(see [26, Ex. 4.4]). Whether* $\mathsf{m}_K(f) \to \mathsf{m}(f)$ *as* $K \to \infty$ *is still open. But for 'square' subgroups* $K = \Omega_n^2$*, Dimitrov [7] has very recently shown convergence using quite difficult Diophantine arguments, and that this holds for all nonzero* $f \in \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ *using 'square' subgroups* Ω_n^d*.*

We can put Problem 5.3 into a more general context as follows. Let $\mathcal{K}$ denote the set of all compact subgroups of $\mathbb{S}^d$. For $K \in \mathcal{K}$, let μ_K be Haar measure on K. In analogy with (15), define the logarithmic Mahler measure of $0 \neq f \in \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ over K to be

$$\mathsf{m}_K(f) := \int_{K \smallsetminus \mathsf{U}(f)} \log|f(\boldsymbol{s})| \, \mathrm{d}\mu_K(\boldsymbol{s}),$$

which agrees with our earlier definition when K is finite.

Now $\mathcal{K}$ is a compact metric space with respect to the Hausdorff metric on compact subsets of $\mathbb{S}^d$. Lawton [20] showed that the function $K \mapsto \mathsf{m}_K(f)$ is continuous on the closed subset of $\mathcal{K}$ consisting of all subgroups having dimension at least 1. For example, the 1-dimensional subgroups $K_n = \{(s, s^n) \colon s \in \mathbb{S}\}$ of $\mathbb{S}^2$ converge to $\mathbb{S}^2$ in the Hausdorff metric, and so $\mathsf{m}\big(f(u,u^n)\big) \to \mathsf{m}\big(f(u,v)\big)$ as $n \to \infty$, where $f(u,u^n)$ is considered as a polynomial in $\mathbb{Z}[u^{\pm 1}]$. Boyd [3] discusses the rate of this convergence, and in particular how fast $\mathsf{m}(1+u+u^n)$ converges to $\mathsf{m}(u,v)$. We can therefore reformulate Problem 5.3 in terms of continuity of Mahler measures on compact subgroups of $\mathbb{S}^d$.

Problem 5.6. *Fix* $0 \neq f \in \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$*. Is the function* $K \mapsto \mathsf{m}_K(f)$ *continuous on* $\mathcal{K}$*?*

6 Algebraic actions of noncommutative groups

For a general countable group Γ, we denote by $\mathbb{Z}\Gamma$ the integral group ring of Γ, where Γ is written multiplicatively. A typical element in $\mathbb{Z}\Gamma$ has the form $f = \sum_{\gamma\in\Gamma} f_\gamma \gamma$, where each $f_\gamma \in \mathbb{Z}$, and where all but finitely many of the f_γ vanish. Multiplication in $\mathbb{Z}\Gamma$ is carried out in the obvious way to extend multiplication in Γ. If M is a countable left module over $\mathbb{Z}\Gamma$, regarded as a discrete Abelian group under addition, the Pontryagin dual $X_M = \widehat{M}$ is a compact Abelian group, and we obtain an algebraic Γ-action α_M on X_M by setting, for every $\gamma \in \Gamma$, α_M^γ equal to the automorphism of X_M dual to left multiplication by γ on M.

Since every algebraic Γ-action arises in this manner, there is a 1-1 correspondence between left $\mathbb{Z}\Gamma$-modules and algebraic Γ-actions, exactly as for $\Gamma = \mathbb{Z}^d$ (except that we now have to be careful about 'left' and 'right'). However, if Γ is noncommutative, much less is known about (left) ideals in and (left) modules over $\mathbb{Z}\Gamma$ than is the case for $\Gamma = \mathbb{Z}^d$, and even for principal algebraic Γ-actions (defined by complete analogy with the principal $\mathbb{Z}^d$-actions Example 2.1) our understanding of their dynamical properties is rudimentary.

In order to fix notation, we again write σ for the *left* shift-action $(\sigma^\theta x)_\gamma = x_{\theta^{-1}\gamma}$ of Γ on $\mathbb{T}^\Gamma$, and consider, for $f \in \mathbb{Z}\Gamma$, the closed, shift-invariant subgroup

$$X_f = \Big\{x \in \mathbb{T}^\Gamma \colon \sum\nolimits_{\gamma\in\Gamma} f_\gamma x_{\theta\gamma} = 0 \text{ for all } \theta \in \Gamma\Big\} \subset \mathbb{T}^\Gamma. \qquad (17)$$

As in Example 2.1, we call the restriction α_f of σ to X_f the *principal algebraic* Γ*-action defined by* f. The shift-transformation σ^γ on $\mathbb{T}^\Gamma$ is again dual to left multiplication by γ on $\mathbb{Z}\Gamma$, and the dual of the subgroup $X_f \subset \mathbb{T}^\Gamma$ is the quotient of $\mathbb{Z}\Gamma$ by the left principal ideal $\mathbb{Z}\Gamma f$, that is, $\widehat{X_f} = \mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$.

One afternoon about fifteen years ago, Wolfgang Lück presented a copy of his new book *L^2-Invariants: Theory and Applications to Geometry and K-Theory* to his colleague Christopher Deninger at the University of Münster. Not quite knowing what to do with a thick book on unfamiliar topics, Deninger started flipping through its pages randomly. By chance, he stumbled on an example showing that the Mahler measure of $f \in \mathbb{Z}[u_1^{\pm1}, \dots, u_d^{\pm1}]$ equals the Fuglede–Kadison determinant of an associated convolution operator ρ_f on $\ell^2(\mathbb{Z}^d)$. Knowing the connection between entropy of algebraic $\mathbb{Z}^d$-actions and Mahler measure, Deninger realised that the convolution operator approach, which is easily generalised to arbitrary countable groups Γ, might give a way to compute entropy for principal algebraic Γ-actions. He was able to show in [4] that his idea works for *expansive* principal actions of at least a restricted class of amenable groups, a breakthrough that initiated the serious study of algebraic actions of arbitrary countable discrete groups. His insight can be viewed as a way to define Mahler measure for polynomials in noncommuting variables.

Before 2010, it was widely believed that entropy theory for group actions was restricted to actions by amenable groups. This changed when Lewis Bowen [2] introduced radically new ideas that ultimately allowed an extension of entropy theory to a much larger class of groups called sofic groups, those having a certain kind of finite approximation. To date, there is no known example of a countable group that is not sofic.

In this section, we briefly sketch these two major developments in dynamics, and how they combined recently in a comprehensive entropy theory for algebraic actions.

Let $0 \neq f \in \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$. Define $f^*(u_1, \dots, u_d) := f(u_1^{-1}, \dots, u_d^{-1})$. Regard f as a function on $\mathbb{S}^d$, so that $f^* = \bar{f}$, the complex conjugate of f. Consider the multiplication operator T_f on $L^2(\mathbb{S}^d)$ given by $T_f(\phi) = f \cdot \phi$ for $\phi \in L^2(\mathbb{S}^d)$. Then

$$T_f^* T_f \;=\; T_{f^*} T_f \;=\; T_{|f|^2} \;=\; T_{|f|}^2,$$

and the spectral measure $\mu_{|f|}$ of $T_{|f|}$ is the push-forward of Lebesgue measure on $\mathbb{S}^d$ under the map $|f|$, so that $\mu_{|f|}$ is supported on the real interval $[0, \|f\|_\infty]$. Fuglede and Kadison [10] introduced a notion of determinant for certain classes of operators that include $T_{|f|}$. We then calculate, using their definition and change of variables, that

$$\det T_{|f|} \;:=\; \exp\left[\int_0^\infty \log t \,\mathrm{d}\mu_{|f|}(t)\right] \;=\; \exp\left[\int_{\mathbb{S}^d} \log |f(\boldsymbol{s})| \,\mathrm{d}\mu(\boldsymbol{s})\right] \;=\; \mathsf{M}(f).$$

This is the fact that Deninger came across in Lück's book.

The Fourier transform gives an isomorphism from $L^2(\mathbb{S}^d)$ to $\ell^2(\mathbb{Z}^d)$, and under this isomorphism the multiplication operator T_f is mapped to the convolution operator ρ_f on $\ell^2(\mathbb{Z}^d)$. Concretely, if we view $w \in \ell^2(\mathbb{Z}^d)$ as a formal sum $w = \sum_{\boldsymbol{n} \in \mathbb{Z}^d} w_{\boldsymbol{n}} \boldsymbol{u}^{\boldsymbol{n}}$, then $\rho_f(w) = w \cdot f$, extending the customary multiplication of polynomials. The connection with α_f is provided by the observation that if points $t \in \mathbb{T}^{\mathbb{Z}^d}$ are similarly regarded as formal sums, $t = \sum_{\boldsymbol{n} \in \mathbb{Z}^d} t_{\boldsymbol{n}} \boldsymbol{u}^{\boldsymbol{n}}$, then $X_f = \ker \rho_{f^*}$. Deninger realised that, since

$$\mathsf{h}(\alpha_f) \;=\; \mathsf{m}(f) \;=\; \log \det T_{|f|} \;=\; \log \det \rho_f,$$

the calculation of entropy could be phrased entirely in terms of convolution operators. This avoids the use of Fourier transforms, and suggests a general way to deal with principal actions of noncommutative groups.

Now let Γ be a general discrete countable group. As above, consider points $w \in \ell^2(\Gamma)$ as formal sums $\sum_{\gamma \in \Gamma} w_\gamma \gamma$. For $f \in \mathbb{Z}\Gamma$, there is the convolution operator ρ_f on $\ell^2(\Gamma)$ given by $\rho_f(w) = w \cdot f$. The weak operator closure of the set of complex combinations of these convolutions operators is called the *group von Neumann algebra* $\mathcal{L}\Gamma$ of Γ. For $U \in \mathcal{L}\Gamma$, the functional calculus gives an operator $|U| \in \mathcal{L}\Gamma$ with $U^*U = |U|^2$. The positive self-adjoint operator $|U|$ has a spectral measure $\mu_{|U|}$ supported on $[0, \|U\|]$. Fuglede and Kadison defined $\det U$ by

$$\det(U) \;:=\; \int_0^\infty \log(t) \,\mathrm{d}\mu_{|U|}(t).$$

One deep result in this theory is that $\det(UV) = (\det U)(\det V)$. This definition applies to the operators ρ_f, and we abbreviate $\det \rho_f$ to $\det f$.

Deninger [4] showed that his idea worked for amenable groups having special kinds of Følner sequences. A series of improvements by several authors culminated in the definitive result for principal algebraic actions of amenable groups by Hanfeng Li and Andreas Thom [22]; Let Γ be an amenable group and $f \in \mathbb{Z}\Gamma$; if ρ_f is injective on $\ell^2(\Gamma)$, then $\mathsf{h}(\alpha_f) = \log \det f$, and otherwise $\mathsf{h}(\alpha_f) = \infty$. A consequence is that $\mathsf{h}(\alpha_{f^*}) = \mathsf{h}(\alpha_f)$, which is highly nontrivial since there is no obvious dynamical connection between α_f and α_{f^*} when Γ is noncommutative.

A concrete example of noncommutative Γ is the discrete Heisenberg group $\mathbb{H}$, the group generated by u, v, and w with relations $uw = wu$, $vw = wv$, and $vu = wuv$. Even for this simplest infinite noncommutative group, there are many open problems, e.g., characterise those $f \in \mathbb{Z}\mathbb{H}$ for which $\mathsf{h}(\alpha_f) = 0$, or determine the higher order mixing properties of principal $\mathbb{H}$-actions. For a comprehensive survey of what is currently known about algebraic $\mathbb{H}$-actions; see [24].

At roughly the same time, the extension of entropy theory to sofic groups was undergoing vigorous development, with algebraic actions providing important and guiding examples. A lucid and systematic account is contained in the recent book by David Kerr and Hanfeng Li [17]. This book describes a profound shift in viewing Γ-actions, from the traditional 'internal' view using objects within the space being acted upon to an 'external' view using finite models of the action. In this way, Følner sets and amenability are avoided, but at the cost of more abstract and complicated machinery, whose implications are still being worked out.

With the ability to define entropy for principal Γ-actions for sofic Γ, and a viable candidate $\log \det f$ for its value, these two strands of dynamical progress culminated in the definitive theorem by Ben Hayes [16], who showed that if Γ is sofic and $f \in \mathbb{Z}\Gamma$, then $\mathsf{h}(\alpha_f) = \log \det f$ provided that ρ_f is injective on $\ell^2(\Gamma)$, and is equal to ∞ otherwise.

The chain of events set in motion by the discovery in [27] that entropy equals logarithmic Mahler measure for algebraic $\mathbb{Z}^d$-actions has led to a remarkable level of generality in the entropy theory of algebraic actions. However, other dynamical properties of such actions, like mixing, positivity of entropy, or the Bernoulli property, still remain rather mysterious as soon as one leaves the comfortable world of $\mathbb{Z}^d$-actions.

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