4. Mahler's Classification of Complex Numbers

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We present Mahler's work on his classification of numbers and discuss subsequent works closely related to this classification.

1 Introduction

In 1932, Mahler [M11] introduced his classification of numbers, in which the complex numbers are divided into four disjoint classes A, S, T and U. Numbers belonging to each class are called A-, S-, T- and U-numbers, respectively. However, in a previous paper [M9] published in 1930, Mahler already gave the definition of 'S-numbers', which correspond to the union of A-numbers and S-numbers in [M11]. Here and below, 'S-number' always refers to the definition given in [M9], while S-number has its now usual meaning. As we shall see below, in these papers, Mahler's aim is to prove algebraic independence of certain numbers, such as e or π, and a Liouville number. This led him to introduce his classification, where two numbers belonging to distinct classes should be algebraically independent, or equivalently, two algebraically dependent numbers should belong to the same class.

In this survey, we briefly explain Mahler's original papers on his classification of numbers as well as subsequent related works by other mathematicians. Sections 2, 3 and 4 are devoted to a presentation of Mahler's papers [M9], [M11] and [M12], respectively. In Section 4, we also discuss several subsequent works. In these sections, we follow Mahler's style, using his notation to be consistent with the original papers. In Section 5, we introduce Koksma's classification of numbers, which is closely related to Mahler's classification. In Section 6, we discuss various results on the existence of numbers in classes S, T and U, and also present an open problem on the existence of numbers in given subclasses of Mahler's and Koksma's classifications. In Section 7, we introduce a third classification of numbers due to Sprindžuk, and give several results on this classification. In Section 8, we gather several results on the transcendence of values of certain functions from the point of view of Mahler's classification. In Section 9, we briefly mention Mahler's papers [M27] and [M179] as well as works related to these papers.

Finally, we remark that the books [8], [16], [38] and [43] treat Mahler's and Koksma's classifications; in particular, [16] includes many references on this topic.

2 The origin

The main purpose of [M9] was to show the algebraic independence of e and a Liouville number. Recall that a real number u is called a Liouville number if, for any positive real number ω, there exists a rational number x/y with $y > 0$, such that

$$0 < |u - x/y| < y^{-\omega}. \tag{1}$$

From its continued fraction expansion, we deduce that e is not a Liouville number. Moreover, as was shown in 1929 by Popken [33], for any positive integer m, there exists a positive number $c(m)$, depending only on m, such that

$$\left|\sum_{h=0}^{m} a_h \mathrm{e}^h\right| \geq a^{-m-c(m)/(\log\log a)}, \quad a_h \in \mathbb{Z}, \quad a = \max(|a_0|, \ldots, |a_m|), \tag{2}$$

provided that a is sufficiently large.

In 1930, Mahler [M9] introduced the notion of 'S-number' and observed that Popken's result [33] shows that e is an example of an 'S-number'.

Definition 2.1 (Mahler [M9]). *A number s is called an 'S-number' if there exist a positive number γ and, for each positive integer m, a positive number $\Gamma(m)$ with the following property: for any integers $a_0, a_1, \ldots, a_m$, we have either*

$$\sum_{h=0}^{m} a_h s^h = 0,$$

or

$$\left|\sum_{h=0}^{m} a_h s^h\right| \geq \Gamma(m) a^{-\gamma m}, \quad a = \max(|a_0|, \ldots, |a_m|). \tag{3}$$

According to Stolarsky [44], the terminology 'S-number' was chosen to honour Siegel. The fundamental result in [M9] is the following statement.

Theorem 2.2 (Mahler [M9]). *Let t be an 'S-number' and s be a number such that there are integer polynomials $C_0(X), \ldots, C_f(X)$ with $C_f(t) \neq 0$ and*

$$C(s|t) := \sum_{i=0}^{f} C_i(t) s^i = 0.$$

Then s is also an 'S-number'.

Theorem 2.2 implies that, if a number u is not an 'S-number', then u and any transcendental 'S-number' are algebraically independent. In particular, e and a Liouville number are algebraically independent.

Mahler's proof of Theorem 2.2 includes more precise information on s, namely:

(a) If s is an algebraic number of degree f and $a_0, \dots, a_m$ are integers, not all zero, then

$$\left|\sum_{h=0}^{m} a_h s^h\right| \geq \Gamma(m) a^{-(f-1)}, \quad a = \max(|a_0|, \dots, |a_m|). \tag{4}$$

(b) If s is a transcendental number and $a_0, \dots, a_m$ are integers, not all zero, then

$$\left|\sum_{h=0}^{m} a_h s^h\right| \geq \Delta(m) a^{-\delta m}, \quad a = \max(|a_0|, \dots, |a_m|), \tag{5}$$

with $\delta = \gamma f g + f - 1$, where γ is a positive real number for which (3) holds with t instead of s and g is the maximum degree of the polynomials $C_i(X)$.

Let us sketch the proof of (b). We may assume that $C(X|Y)$ in $\mathbb{Z}[X, Y]$ is irreducible. Then the zeros $s_0 = s, s_1, \dots, s_{f-1}$ of $C(X|t)$ are all transcendental numbers. Let $a_0, \dots, a_m$ be integers, not all zero. Set

$$L(t|a_0 \cdots a_m) = C_f(t)^m \prod_{\nu=0}^{f-1} \left(\sum_{h=0}^{m} a_h s_\nu^h\right). \tag{6}$$

(Note that, instead of the factor $C_f(t)^m$ on the right-hand side of (6), the factor $C_f(t)^{fm}$ was used in [M9].) It follows from the fundamental theorem of symmetric functions that

$$L(t|a_0 \cdots a_m) = \sum_{\ell=0}^{gm} A_\ell t^\ell, \quad A_\ell \in \mathbb{Z}, \ |A_\ell| \leq \alpha a^f,$$

with a positive constant α depending only on $C(X|Y)$. Hence we have, by (3) applied to the 'S-number' t, that

$$|L(t|a_0 \cdots a_m)| \geq \Gamma(gm)(\alpha a^f)^{-\gamma g m}.$$

This implies the desired assertion with

$$\Delta(m) = |C_f(t)|^{-m} \prod_{\nu=1}^{f-1} \left(\sum_{h=0}^{m} |s_\nu|^h\right)^{-1} \Gamma(gm) \alpha^{-\gamma g m}.$$

3 Mahler's classification

Mahler introduced his classification of numbers in [M11]. The main purpose of that paper is to prove the following results.

Theorem 3.1 (Mahler [M11]). *Let $\vartheta_1, \dots, \vartheta_N$ be N algebraic numbers which are linearly independent over the rationals and λ be a Liouville number. Then the numbers*

$$\mathrm{e}^{\vartheta_1}, \dots, \mathrm{e}^{\vartheta_N}, \lambda$$

are algebraically independent over the field of algebraic numbers.

Theorem 3.2 (Mahler [M11]). *Let z be a nonzero real logarithm of a positive rational number or $z=\pi$, and λ a Liouville number. Then the numbers z and λ are algebraically independent over the field of algebraic numbers.*

The validity of these results rely on a quantitative refinement of the Lindemann–Weierstrass theorem and of the Lindemann theorem, respectively. More precisely, Mahler proved the following results.

Theorem 3.3 (Mahler [M11]). *Let $\vartheta_1,\dots,\vartheta_N$ be N algebraic numbers in an algebraic number field of degree n which are linearly independent over the rationals. Then*

$$\left|\sum_{i_1=0}^{M_1}\cdots\sum_{i_N=0}^{M_N} a_{i_1\dots i_N}\mathrm{e}^{i_1\vartheta_1+\cdots+i_N\vartheta_N}\right| \geq a^{-T_{N,n}M_1\cdots M_N}$$

for

$$a_{i_1\dots i_N}\in\mathbb{Z},\quad a:=\max|a_{i_1\dots i_N}|,$$

provided that a is sufficiently large, where $T_{N,n}$ is a positive number depending only on N and n.

Theorem 3.4 (Mahler [M11]). *Let z be a nonzero real logarithm of a positive rational number or $z=2\pi\mathrm{i}$. Then there exist a real number $c>1$ and, for every positive integer m, a positive number $C(m)$ such that for all integers $a_0,\dots,a_m$, not all zero, we have*

$$\left|\sum_{i=0}^{m} a_i z^i\right| \geq C(m)a^{-c^m},\quad a:=\max\{|a_0|,\dots,|a_m|\},$$

provided that a is sufficiently large.

Though Theorem 3.3 (resp., Theorem 3.4) does not directly imply Theorem 3.1 (resp., Theorem 3.2), it is easily seen that an argument similar to that given in Section 2, by using Theorem 3.3 (resp., Theorem 3.4) instead of Popken's result, works well to prove Theorem 3.1 (resp., Theorem 3.2). It follows from Theorem 3.4 that the number z there is not a Liouville number. But Theorem 3.4 does not ensure that z is an 'S-number'. (To determine whether z is an 'S-number' or not is still an open problem.) Therefore, the main result in [M9] is not sufficient to prove Theorem 3.2. This fact motivated Mahler to classify numbers other than 'S-numbers' in a suitable way.

We now explain Mahler's classification of numbers. Let z be a real or complex number. For positive integers m and a, we define

$$\omega_m(a)=\omega_m(a|z):=\min\left|\sum_{k=0}^{m} a_k z^k\right|,$$

where the minimum is taken over all nonzero sums with integers a_k satisfying $|a_k| \leq a$. Then we further define

$$\omega_m = \omega_m(z) := \limsup_{a\to\infty} \frac{\log(1/\omega_m(a|z))}{\log a}, \quad \omega = \omega(z) := \limsup_{m\to\infty} \frac{\omega_m(z)}{m}.$$

If $\omega_m(z) < \infty$ for all m, we set $\mu(z) = \infty$ and, otherwise, we set $\mu(z) = \mu$, where μ is the smallest m for which $\omega_m(z) = \infty$. Note that z is a Liouville number if and only if $\mu(z) = 1$.

Definition 3.5 (Mahler's classification). *A number z is called an*
A-number, if $\omega = 0$;
S-number, if $0 < \omega < \infty$;
T-number, if $\omega = \infty$ and $\mu = \infty$;
U-number, if $\omega = \infty$ and $\mu < \infty$.

Set $\sigma = \sigma(z) = 1$ or 2 if z is real or complex, respectively. Mahler gave the following properties on his classification:

a) For any algebraic number z of degree n, we have

$$\begin{aligned} \omega_m &\geq \frac{m+1}{\sigma} - 1 \ \text{ for } m \leq n-1, \\ \omega_m &\leq \frac{n}{\sigma} - 1 \ \text{ for all } m, \\ \omega &= 0, \ \ \mu = \infty; \end{aligned}$$

b) For any transcendental number z, we have

$$\omega_m \geq \frac{m+1}{\sigma} - 1 \ \text{ for all } m, \quad \omega \geq \frac{1}{\sigma};$$

c) If z_1 and z_2 are algebraically dependent transcendental numbers satisfying

$$\sum_{i=0}^{f}\sum_{j=0}^{g} C_{ij} z_1^i z_2^j = 0, \quad C_{ij} \in \mathbb{Z}, \quad C_{fg} \neq 0,$$

we have

$$\omega_m(z_1) \leq f - 1 + f\omega_{mg}(z_2), \quad \omega(z_1) \leq fg\omega(z_2), \quad \mu(z_1) \leq f\mu(z_2).$$

In fact, the first inequalities in a) and b) follow from the pigeonhole principle. The second inequality in a) follows from a result of Liouville [28]. Finally, c) follows from the argument given in the paper [M9]; see also Section 2. Mahler proved the following fundamental results.

Theorem 3.6 (Mahler [M11]). *The set of A-numbers is the set of algebraic numbers.*

Theorem 3.7 (Mahler [M11]). *Two algebraically dependent numbers belong to the same class.*

Since Theorem 3.4 asserts that the number z there is not a U-number, it follows from Theorem 3.7 that z and a U-number are algebraically independent, generalising Theorem 3.2. From Theorem 3.3 it is possible to generalise Theorem 3.1 replacing a Liouville number λ there by a U-number.

As a special case of Theorem 3.3 for $N = 1$, Mahler gave the estimates

$$\omega_m(\mathrm{e}^{\vartheta}) \leq 2n(2n-1)m + (2n-1) \text{ for all } m, \quad \omega(\mathrm{e}^{\vartheta}) \leq 2n(2n-1),$$

where ϑ is an algebraic number of degree n. In the special case $\vartheta = 1$, it follows from Popken's result (2) that

$$\omega_m(\mathrm{e}) = m \text{ for all } m, \quad \omega(\mathrm{e}) = 1.$$

In this respect, Mahler proved that we can take $c(m) = cm^2 \log m$ in (2), where c is a positive real number [M11, Satz 3].

4 Mahler's conjecture on S-numbers

A little after having defined his classification of numbers, in 1932 Mahler [M12] proved that almost all real and almost all complex numbers are S-numbers. Throughout this paper, 'almost all' refers to the (linear or planar) Lebesgue measure. To state his result more precisely, for $f(X)$ in $\mathbb{Z}[X]$, we denote by $\partial(f)$ its degree and by $H(f)$ its height, that is, the maximum of the absolute values of its coefficients.

Theorem 4.1 (Mahler [M12]). *Let m be a positive integer and ε be a positive number. Then, for almost all real and for almost all complex numbers x,*

$$|f(x)| \geq H(f)^{-(4+\varepsilon)m}$$

holds for all $f(X)$ in $\mathbb{Z}[X]$ with $\partial(f) \leq m$ and sufficiently large $H(f)$. Therefore, almost all real and almost all complex numbers x are S-numbers satisfying $\omega_m(x) \leq 4m$ for all m.

It easily follows from a covering argument that (using Mahler's terminology $\omega_1(x)$) the set of real numbers x with $\omega_1(x) > 1$ has linear Lebesgue measure zero. As an extension of this fact, at the end of his paper [M12], Mahler stated the following conjecture.

Mahler's conjecture 4.2 (1st form). *Let m be a positive integer, and ε be a positive number. Then, for almost all real (resp. complex) numbers x, we have*

$$|f(x)| \geq H(f)^{-\gamma m}, \tag{7}$$

with $\gamma = 1 + \varepsilon$ (resp. $\gamma = 1/2 + \varepsilon$), for all $f(X)$ in $\mathbb{Z}[X]$ with $\partial(f) \leq m$ and sufficiently large $H(f)$.

In view of the property b) given in Section 2, we may rewrite this conjecture in the following form, where the assertion for complex numbers is slightly refined.

Mahler's conjecture 4.3 (2nd form). *For almost all real (resp. complex) numbers x and for all m,*

$$\omega_m(x) = m \quad (\textit{resp. } \omega_m(x) = (m-1)/2). \tag{8}$$

After partial results were obtained by several authors, Mahler's conjecture (2nd form) was finally confirmed in 1965 by Sprindžuk [42]; see also [43]. Since then a refined version of Mahler's conjecture for real numbers was stated and proved by A. Baker [7], where $H(f)^{-\gamma m}$ in the right-hand side of (7) is replaced by $\psi(H)^m$ with a positive monotonic decreasing function $\psi(H)$ of the positive integer variable H such that $\sum \psi(H)$ converges; see also [8, Ch. 9] and [16, Ch. 4]. A. Baker also conjectured that the function $\psi(H)^m$ can be replaced by $H^{-m+1}\psi(H)$. This conjecture was confirmed by Bernik [14]; see also [15, Ch. 2.4]. Furthermore, Beresnevich's result in [13] implies that the convergence of $\sum \psi(H)$ is necessary in Bernik's result; see also [16, Ch. 6.6]. Kleinbock and Margulis [23] gave an alternative proof of Sprindžuk's theorem, along with a stronger version.

5 Koksma's classification

In 1939, Koksma [24] introduced a classification of numbers which is closely related to Mahler's classification. In what follows, for an algebraic number α, we denote by $H(\alpha)$ its height, that is, the maximum of the absolute values of the coefficients of its minimal polynomial over $\mathbb{Z}$.

Let ξ be a real or complex number. For positive integers n and H, we define

$$\omega_n^*(H) = \omega_n^*(\xi, H) := \min |\xi - \alpha|,$$

where the minimum is taken over all algebraic numbers $\alpha \neq \xi$ with $\deg \alpha \leq n$ and $H(\alpha) \leq H$. We further define

$$\omega_n^* = \omega_n^*(\xi) := \limsup_{H \to \infty} \frac{\log(1/\omega_n^*(\xi, H))}{\log H}, \quad \omega^* = \omega^*(\xi) := \limsup_{n \to \infty} \frac{\omega_n^*(\xi)}{n}.$$

If $\omega_n^*(\xi) < \infty$ for all m, then we set $\mu^*(\xi) = \mu^* = \infty$ and, otherwise, $\mu^*(\xi) = \mu^*$ is the smallest n for which $\omega_n^*(\xi) = \infty$.

Definition 5.1 (Koksma's classification). *A number ξ is called an*

A^*-number, if $\omega^* = 0$;
S^*-number, if $0 < \omega^* < \infty$;
T^*-number, if $\omega^* = \infty$ and $\mu^* = \infty$;
U^*-number, if $\omega^* = \infty$ and $\mu^* < \infty$.

Note that Koksma classified only transcendental numbers, where ξ is defined to be an S^*-number if $\omega^* < \infty$. Under this setting, Koksma proved that the class of S^*-numbers (resp., of T^*-numbers, of U^*-numbers) is the same as the class of S-numbers (resp., of T-numbers, of U-numbers). Since it is easily shown that $\omega_n^*(\xi) \leq \omega_n(\xi)$, we see that $\omega(\xi) = 0$ implies $\omega^*(\xi) = 0$. On the other hand, Wirsing [45] proved that, for a real (resp., complex) transcendental number ξ and for all n, we have

$$\omega_n^*(\xi) \geq (\omega_n(\xi) + 1)/2 \quad (\text{resp. } \omega_n^*(\xi) \geq \omega_n(\xi)/2).$$

This shows that $\omega(\xi) \neq 0$ implies $\omega^*(\xi) \neq 0$. Therefore, Koksma's classification is indeed equivalent to Mahler's classification.

It was also proved by Wirsing [45] that, for a real (resp., complex) transcendental number ξ and for all n,

$$\omega_n^*(\xi) \geq \omega_n(\xi)/(\omega_n(\xi) - n + 1) \quad (\text{resp. } \omega_n^*(\xi) \geq \omega_n(\xi)/(2\omega_n(\xi) - n + 2)),$$

which, combined with $\omega_n^*(\xi) \leq \omega_n(\xi)$, implies that

$$\omega_n^*(\xi) = n \text{ if } \omega_n(\xi) = n \quad (\text{resp. } \omega_n^*(\xi) = (n-1)/2 \text{ if } \omega_n(\xi) = (n-1)/2).$$

Therefore, Sprindžuk's solution to Mahler's conjecture (2nd form) also confirms the analogous conjecture where ω_m in (8) is replaced by ω_m^*. Moreover, Wirsing [45] conjectured that $\omega_n^*(\xi) \geq n$ (resp., $\omega_n^*(\xi) \geq (n-1)/2$) holds for every transcendental real (resp., complex) number ξ and every positive integer n. This conjecture is still open.

6 EXISTENCE OF NUMBERS IN MAHLER'S AND KOKSMA'S CLASSIFICATIONS

We know from the previous section that almost all real numbers ξ are S-numbers satisfying $\omega_n(\xi) = n$ for every n. On the other hand, in 1970, A. Baker and W. Schmidt [9] proved that there exist S-numbers ξ for which $\omega(\xi)$ is arbitrarily large; see also [16, Chs. 5.6 and 5.7]. In 1953, LeVeque [27] proved the existence of U-numbers ξ with $\mu(\xi)$ being any given positive integer; see also [16, Ch. 7.6]. The existence of T-numbers, which remained an open question for many years, was confirmed in 1968 by Schmidt [36]; see also [37]. We state here a refinement of Schmidt's result in [37] given by R. C. Baker [10] as follows (see also [16, Chs. 7.1-7.5]): let $(\omega_n)_{n\in\mathbb{N}}$ and $(\omega_n^*)_{n\in\mathbb{N}}$ be two nondecreasing sequences of elements of $\mathbb{R}_{\geq 0} \cup \{\infty\}$ such that, for all n,

$$\omega_n^* \leq \omega_n \leq \omega_n^* + (n-1)/n, \quad \omega_n > n^3 + 3n^2 + 5n + 1.$$

Then there exists a real transcendental number ξ such that, for all n,

$$\omega_n^*(\xi) = \omega_n^* \quad \text{and} \quad \omega_n(\xi) = \omega_n. \tag{9}$$

The following problem was posed in connection with this result [16, Problem 1]: let $(\omega_n)_{n\in\mathbb{N}}$ and $(\omega_n^*)_{n\in\mathbb{N}}$ be two nondecreasing sequences of elements of $R_{\geq 0} \cup \{\infty\}$ such that, for all n,

$$n \leq \omega_n^* \leq \omega_n \leq \omega_n^* + n - 1.$$

Then prove (or disprove) the existence of a real transcendental number ξ such that the equalities (9) hold for all n.

Since, for a real transcendental number ξ, we have $\omega_n(\xi) \leq \omega_n^*(\xi) + n - 1$ (see [38, Hilfssatz 19]), assuming the validity of Wirsing's conjecture, the setting of this problem is as general as possible. Defining Ω_n to be the set consisting of the values $\omega_n(\xi) - \omega_n^*(\xi)$ for the real numbers ξ, we have by R. C. Baker's result above that $[0, (n-1)/n]$ is contained in Ω_n for all n. For this part of the problem, we have better results, namely

$$[0,1] \subset \Omega_2, \quad [0,2) \subset \Omega_3, \quad \text{and} \quad \left[0, \tfrac{n}{2} + \tfrac{n-2}{4(n-1)}\right) \subset \Omega_n \text{ for } n \geq 4$$

(see [17, 18] for the first two results and [19] for the last one). Refining the lower bound $O(n^3)$ for ω_n in R. C. Baker's result seems to be a difficult problem.

7 Sprindžuk's classification

In 1962, Sprindžuk [41] (see also [43, pp. 140-142]) introduced a classification of numbers, which is also based on the behaviour of the quantity $\omega_n(\xi|H)$ defined in Section 3 for a given number ξ. However, in his classification, the role of the variable n in Mahler's classification (corresponding to the degree) is replaced by that of the variable H (corresponding to the height).

Let ξ be a real or complex number. For positive integers n and H, we rewrite the quantity $\omega_n(\xi|H)$ as $\omega_n(\xi, H)$. Then we define

$$\tilde{\omega}(\xi, H) := \limsup_{n\to\infty} \frac{\log\log(1/\omega_n(\xi, H))}{\log n}, \quad \tilde{\omega} = \tilde{\omega}(\xi) := \sup_{H\in\mathbb{N}} \tilde{\omega}(\xi, H).$$

The quantity $\tilde{\omega}$ is called the *order* of ξ. If it is finite, we also define

$$\tilde{\mu}(\xi, H) := \limsup_{n\to\infty} \frac{\log(1/\omega_n(\xi, H))}{n^{\tilde{\omega}(\xi)}}, \quad \tilde{\mu} = \tilde{\mu}(\xi) := \limsup_{H\to\infty} \frac{\tilde{\mu}(\xi, H)}{\log H}.$$

In the case where $\tilde{\mu}(\xi) = \infty$, we set $H_0 = H_0(\xi) = \infty$ if there exists no H such that $\tilde{\mu}(\xi, H) = \infty$, and, otherwise, $H_0 = H_0(\xi)$ denotes the smallest H for which $\tilde{\mu}(\xi, H) = \infty$.

Definition 7.1 (Sprindžuk's classification). *A number ξ is called an*

$\tilde{A}$-number, if $0 \leq \tilde{\omega} < 1$ or if $\tilde{\omega} = 1$ and $\tilde{\mu} = 0$;
$\tilde{S}$-number, if $1 < \tilde{\omega} < \infty$ or if $\tilde{\omega} = 1$ and $\tilde{\mu} > 0$;
$\tilde{T}$-number, if $\tilde{\omega} = \infty$ and $H_0 = \infty$;
$\tilde{U}$-number, if $\tilde{\omega} = \infty$ and $H_0 < \infty$.

Sprindžuk proved that the class $\tilde{A}$ is the set of algebraic numbers, and that almost all real and almost all complex numbers are $\tilde{S}$-numbers of order at most 2. For the second assertion, Chudnovsky claimed (with a sketch of the proof) that the upper bound 2 can be replaced by the best possible value 1; see [5, Appendix] for a complete proof. Indeed, the following more precise statement was proved later (see [6]): let ε be a positive real number; then, for almost all real numbers ξ, there exists a positive constant $c(\xi, \varepsilon)$, depending only on ξ and ε, such that

$$|P(\xi)| > \exp\{-(2+\varepsilon)n \log H - (2.5+\varepsilon)n \log n\}$$

holds for all nonzero polynomials $P(X)$ in $\mathbb{Z}[X]$ of degree n and height H provided that $\max(n, H) \geq c(\xi, \varepsilon)$.

We may expect that a stronger assertion holds, namely that, for every positive real number ε, for almost all real numbers ξ, there exist a positive constant $c(\xi, \varepsilon)$, depending only on ξ and ε, and a constant $C(n)$, depending only on n, such that

$$|P(\xi)| > \exp\{-(1+\varepsilon)n \log H - C(n)\}$$

holds for all nonzero polynomials $P(X)$ in $\mathbb{Z}[X]$ of degree n and height H provided that $\max(n, H) \geq c(\xi, \varepsilon)$ [16, Problem 51]. This problem can be viewed as a refinement of Mahler's conjecture on S-numbers, in a different direction from the refinements mentioned in Section 4.

We now turn to a problem on Sprindžuk's classification analogous to the one on Mahler's and Koksma's classifications discussed in Section 6. Let $(\tilde{\omega}_H)_{H\in\mathbb{N}}$ be a sequence of elements of $\mathbb{R}_{\geq 0} \cup \{\infty\}$ satisfying

$$\tilde{\omega}_H = 0 \quad \text{for } H < H_0, \quad 1 \leq \tilde{\omega}_H \leq \tilde{\omega}_H \leq \infty \quad \text{for } H \geq H_0, \tag{10}$$

for some positive integer H_0. It is easily seen that, for any real number ξ, the sequence $(\tilde{\omega}_H)_{H\in\mathbb{N}}$ with $\tilde{\omega}_H = \tilde{\omega}(\xi, H)$ satisfies (10) for some H_0. Conversely, it was proved in [5] that, for any sequence $(\tilde{\omega}_H)_{H\in\mathbb{N}}$ satisfying (10), there exist uncountably many real numbers ξ which satisfy $\tilde{\omega}_H = \tilde{\omega}(\xi, H)$ for all H; see also [16, Ch. 8.1]. This result gives a full answer to the existence of real numbers in a given subclass defined according to Sprindžuk's classification.

8 Values of certain functions in Mahler's classification

It is very likely that values taken by 'classical' functions at (reasonable) nonzero algebraic numbers are S-numbers if they are transcendental. We quote below several results supporting this guess.

(a) Let $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ be nonzero algebraic numbers, and set

$$\Lambda = \sum_{j=1}^{n} \beta_j \log \alpha_j.$$

If $\Lambda \neq 0$, then it is a consequence of Baker's results (see [8, Chs. 2 and 3]) that Λ is either an S- or a T-number.

(b) Let $\wp(z)$ be a Weierstrass elliptic function with algebraic invariants, and α a nonzero algebraic number. Then the value $\wp(\alpha)$ is either an S- or a T-number [34]. Moreover, if $\wp(z)$ has complex multiplication, then $\wp(\alpha)$ is an S-number [20].

(c) Let $E(z)$ be an E-function over an algebraic number field $\mathbb{K}$ (see [26, Ch. II §1] and [39, Ch. 3 §1] for the definition of E-functions) satisfying a homogeneous linear differential equation

$$\sum_{j=0}^{m} Q_j(z)E^{(m-j)}(z) \;=\; 0, \quad Q_j(z) \in \mathbb{K}[z].$$

Assume that the functions $E(z), E'(z), \ldots, E^{(m-1)}(z)$ are algebraically independent over $\mathbb{K}(z)$. Then, for any nonzero algebraic number α distinct from the zeros of $Q_0(z)$, the value $E(\alpha)$ is either an S- or a T-number; see [25], [26, Ch. VII §5] and [39, Chs. 11 and 12]. This result includes the case where $E(z)$ is the Bessel function $J_0(z)$ and α is an arbitrary nonzero algebraic number [40]. If $m = 1$, then the value $E(\alpha)$ is an S-number.

(d) Let $R(z)$ be a power series over an algebraic number field $\mathbb{K}$ which satisfies a k-Mahler equation, namely, a functional equation of the form

$$\sum_{j=0}^{m} Q_j(z)R(z^{k^{m-j}}) \;=\; 0, \quad Q_j(z) \in \mathbb{K}[z].$$

Assume that the functions $R(z), R(z^k), \ldots, R(z^{k^{m-1}})$ are algebraically independent over $\mathbb{K}(z)$. Then, for any algebraic number α with $0 < |\alpha| < 1$ in the disk of convergence of $R(z)$ centred at the origin such that α^{k^ν} is distinct from the zeros of $Q_0(z)$ for every nonnegative integer ν, the value $R(\alpha)$ is either an S- or a T-number; see [30, Ch. 4.4]. If $m = 1$, then the value $R(\alpha)$ is an S-number [22]. If $m = 1$ and $\alpha = 1/b$ for a nonzero integer b, then we further have $\omega(R(1/b)) \leq k + 2\sqrt{k} + 2$; see [4].

(e) Let $R(z)$ be the generating series of a k-automatic sequence, or more generally a k-regular sequence over $\mathbb{Z}$; see [2, Chs. 5.1 and 16.1] for the definitions. It is known that $R(z)$ satisfies a k-Mahler equation; see [11] and [30, Ch. 5.1]. Let b be an integer with $|b| \geq 2$. Then, if $R(1/b)$ is irrational, it is either an S- or a T-number; see [1] or [12] for the automatic case or the regular case, respectively.

The above results, except very few of them, ensure only that each value under consideration is either an S- or a T-number. To exclude the latter possibility in each case is an important, but presumably very difficult, problem.

9 Other works

Let p be a prime number. In 1935, Mahler [M27] proposed a classification of p-adic numbers, in analogy with his classification of complex numbers. He proved several fundamental results corresponding to those given in [M11]. The analog of Mahler's conjecture for p-adic S-numbers was established by Sprindžuk [43, Part II, Ch. II], and the existence of p-adic T-numbers was proved by Schlickewei [35]; see also [31].

In 1971, Mahler [M179] introduced a new classification of numbers. For a given complex number ξ, he defined the 'order function' $O(u|\xi)$ of ξ in the integer variable u, and classified complex numbers by a certain equivalence relation between their order functions. He proved fundamental results on the classification including the fact that two algebraically dependent numbers belong to the same class. Answering the problems posed by Mahler [M179], Durand [21] proved, in particular, the existence of uncountably many classes and that of a particular class to which almost all real and almost all complex numbers belong; the latter result was obtained independently by Nesterenko [29]. For related works, see [3] and [32]; see also [16, Ch. 8.2].

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