9. A Mahler Miscellany

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1 Introduction

Kurt Mahler's first paper [M1] was published in 1927. Like most first papers from students, it came about for many reasons—certainly interest was one of them—but the setting was an important one. It was the mid-late 1920s in Germany, which was the place (and time) to be for mathematics and physics. The mathematics and physics culture in Germany was booming and this boom was nowhere more pronounced than in Göttingen in 1926. In that year, Mahler found himself working in an illustrious group of applied mathematicians. Indeed, in 1926, the famous American applied mathematician Norbert Wiener received a Guggenheim fellowship to work with Max Born in Göttingen and then to travel on to work with Niels Bohr in Copenhagen. In that year, Born's assistant was Werner Heisenberg, who would follow Wiener to Copenhagen and there develop what would later become his famous uncertainty principle. It is in this setting that, while in Göttingen, Wiener was given an (unpaid) assistant—a (barely) 23-year-old Kurt Mahler! Collectively, Wiener [24] and Mahler [M1] produced a two-part series of papers entitled, "The spectrum of an array and its application to the study of the translational properties of a simple class of arithmetical functions." Wiener describes the purpose of the series in the first paragraphs of his part. "The purpose of the present paper is to extend the spectrum theory already developed by the author in a series of papers to the harmonic analysis of functions only defined for a denumerable set of arguments—*arrays*, as we shall call them—and the application of this theory to the study of certain power series admitting the unit circle as an essential boundary."

Concerning the actual contribution, given a sequence A, Wiener describes a method to construct a monotone non-decreasing function $A(x)$, which he calls the *spectral function of* A. By a result of Fréchet, $A(x)$ may contain three possible additive parts: a monotone step function, a function which is the integral of its derivative, and a continuous function which has almost everywhere a zero derivative. In more modern terminology, what Wiener is describing is how one creates the diffraction measure associated to the sequence A. The three possible parts of the measure are then described by the

Lebesgue Decompositon Theorem: *Any regular Borel measure* μ *on* $\mathbb{R}^d$ *has a unique decomposition* $\mu = \mu_{\mathrm{pp}} + \mu_{\mathrm{ac}} + \mu_{\mathrm{sc}}$ *where* μ_{pp}, μ_{ac} *and* μ_{sc} *are mutually orthogonal and also* $|\mu| = |\mu_{\mathrm{pp}}| + |\mu_{\mathrm{ac}}| + |\mu_{\mathrm{sc}}|$. Here, μ_{pp} is a pure point measure corresponding to the monotone step function, μ_{ac} is an absolutely continuous measure corresponding to the function that is the integral of its derivative, and μ_{sc} is a singular continuous measure corresponding to the continuous function which has almost everywhere a zero derivative. Wiener provided examples giving pure point measures and absolutely continuous measures. As one might expect, periodic sequences have pure point diffraction measures. Wiener's example of a sequence with an absolutely continuous diffraction measure is reminiscent of the sequence of digits of Champernowne's number, which will be described in a later section of this chapter. Mahler's contribution was an example of a sequence whose associated diffraction measure is purely singular continuous; it remains a central result in diffraction theory in the context of aperiodic order (see Baake and Grimm [4, Section 10.1] for details and further advances and impact). His example—the Thue–Morse sequence, sometimes called the Thue–Morse–Mahler sequence—is paradigmatic and a fundamental example in an area of transcendence theory now called Mahler's method; see Boris Adamczewski's chapter in this volume for more details about Mahler's method. The Thue–Morse sequence $\mathbf{t} = \{t(n)\}_{n\geqslant 0}$ is defined by $t(0) = 1$, $t(1) = -1$, $t(2n) = t(n)$ and $t(2n+1) = -t(n)$. The sequence starts

$$\mathbf{t} := 1, -1, -1, 1, -1, 1, 1, -1, -1, 1, 1, -1, 1, -1, -1, 1, \ldots;$$

it is now ubiquitous in the areas of theoretical computer science and symbolic dynamics.

While it is considered a work of dynamical systems, Mahler's first work touches on many ideas and structures related to number theory. Even its purpose was the study of certain transcendental power series—"power series admitting the unit circle as an essential boundary." Special sequences with interesting arithmetical structure took a leading role, including Champernowne's sequence and the Thue–Morse sequence, and in Mahler's arguments we see the importance of finite characteristic. All of these concepts played important roles in Mahler's further research and by considering the details in these areas, Mahler created an influential body of work, much of which has been described in the previous chapters of this volume.

In the remainder of this short chapter, we briefly describe several works of Kurt Mahler not yet considered in the previous chapters, together with some of their further developments. We use [Mx] to quote Mahler's paper numbered x in the Kurt Mahler Archive—also contained in this volume as Bibliography of Kurt Mahler.

Remark 1.1. *It is amusing to note that only a handful of papers have been written in collaboration. In total, Mahler had fourteen collaborators and, with the exception of Erdős—with whom he wrote two papers—wrote only one paper with each of them.*

2 The Cugiani–Mahler theorem

Roth's celebrated theorem (1955) in Diophantine approximation [19] has a long history. Liouville established in 1844 that a non-zero real algebraic number cannot be approximated by rational numbers at an order greater than its degree; this result was subsequently improved by Thue (1909), Siegel (1921) and later by Dyson (1947) and Gelfond (1948). In the meantime, in 1936, Schneider [21] proved an important, and almost forgotten, result; see also [M52].

Theorem 2.1 (Schneider, 1936). *Let ξ be an irrational, algebraic real number. Let ε be a positive real number. Let $\{p_j/q_j\}_{j\geqslant 1}$ be the sequence of reduced rational solutions of*

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}},$$

ordered such that $3 \leqslant q_1 < q_2 < \ldots$ Then either the sequence $\{p_j/q_j\}_{j\geqslant 1}$ is finite, or

$$\limsup_{j\to+\infty} \frac{\log q_{j+1}}{\log q_j} = +\infty.$$

Mahler [M41, M42] established a p-adic analogue and generalisation of Schneider's theorem. He noted [M45, M46] the interest of constructing b-ary expansions, whose transcendence can be proved without being trivial, and he used the (p-adic extension of the) Schneider theorem to derive several transcendence statements, including the transcendence of the Champernowne number

$$0.1234567891011121314\ldots$$

See also [M209, Section 18]. Let us highlight the main result in [M46].

Theorem 2.2 (Mahler, 1937). *Let $f(x)$ be a non-constant polynomial with integer coefficients such that $f(k) \geqslant 0$ for positive integers k. Then, the decimal number*

$$0.f(1)f(2)f(3)\cdots,$$

formed by concatenating the decimal expansions of the $f(k)$, is both transcendental and not a Liouville number.

In 1958, Cugiani [12], by means of a subtle modification of the proof of Roth's theorem, proved the following improvement of Schneider's result.

Theorem 2.3 (Cugiani, 1958). *Let ξ be a real algebraic number of degree d. For an integer $q \geqslant 16$, set*

$$\varepsilon(q) = \frac{9d}{(\log\log\log q)^{1/2}}.$$

Let $\{p_j/q_j\}_{j\geqslant 1}$ be the sequence of reduced rational solutions of

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon(q)}},$$

ordered such that $16 \leqslant q_1 < q_2 < \dots$. *Then, either the sequence* $\{p_j/q_j\}_{j\geqslant 1}$ *is finite, or*

$$\limsup_{j\to+\infty} \frac{\log q_{j+1}}{\log q_j} = +\infty.$$

Cugiani's theorem was extended by Mahler in his monograph [M146] to include p-adic valuations. The corresponding statement is referred to as the Cugiani–Mahler theorem. For further extensions together with a much simpler proof, see [7] and the survey [9].

3 Fields of power series

Mahler investigated Diophantine approximation and the geometry of numbers [M72] in fields of power series. In [M106], he proved the analogue of Liouville's inequality in fields of power series and established that, unlike for real algebraic numbers, Liouville's inequality is best possible in fields of power series over a finite field; that is, he showed that an analogue of Roth's theorem does not hold in fields of power series over a finite field. To do this, he observed that, for any prime number p, the power series

$$\Theta_p = X^{-1} + X^{-p} + X^{-p^2} + X^{-p^3} + \cdots$$

is a root in $\mathbb{F}_p((X^{-1}))$ of the algebraic equation $Z^p - Z + X^{-1} = 0$, and for $n \geqslant 1$, its n-th partial sum, a_n/b_n, where

$$a_n := X^{p^{n-1}}(X^{-1} + X^{-p} + X^{-p^2} + \cdots + X^{-p^{n-1}}) \quad \text{and} \quad b_n := X^{p^{n-1}}$$

are polynomials in $\mathbb{F}_p[X]$, satisfies

$$\left|\Theta_p - \frac{a_n}{b_n}\right| = \frac{1}{|b_n|^p}.$$

Here, the norm $|\Theta|$ of the power series $\Theta = a_{-n}X^n + \ldots + a_1X^{-1} + \ldots$ in $\mathbb{F}_p((X^{-1}))$ with $a_{-n} \neq 0$ is defined by $|\Theta| = p^n$.

Remark 3.1. *We note that Baum and Sweet* [5] *gave an explicit example of an algebraic power series over the field of two elements, whose continued fraction expansion has bounded partial quotients. An interested reader may consult the surveys* [17, 20].

4 Fractional parts of powers of rational numbers

In [M135], Mahler improved his previous result of [M51] by showing that for any $\varepsilon > 0$, there exists an integer $n_0(\varepsilon)$ such that

$$\|(3/2)^n\| > 2^{-\varepsilon n}, \tag{1}$$

for every $n > n_0(\varepsilon)$, where $\|\cdot\|$ denotes the distance to the nearest integer. This result is deeply connected to Waring's problem. The proof depends on Ridout's p-adic extension of Roth's theorem and so, the number $n_0(\varepsilon)$ cannot be given explicitly. Weaker lower bounds can be obtained using estimates for p-adic linear forms in logarithms or by the hypergeometric method; see [10, Section 3.7] for references.

Corvaja and Zannier [11] extended (1) by replacing $3/2$ by general algebraic numbers greater than one. Let $\alpha > 1$ be a real algebraic number and ℓ a real number in $(0,1)$. They established that, if $\|\alpha^n\| < \ell^n$ for infinitely many positive integers n, there is a positive integer d such that α^d is a Pisot number (recall that a Pisot number is a real algebraic integer greater than one, all of whose Galois conjugates are in the open unit disc). Their conclusion is best possible. The proof rests on a skilful application of the Schmidt Subspace Theorem; see [16] for an extension.

In [M167], Mahler introduced the notion of Z-numbers—the positive real numbers ξ such that

$$0 \leqslant \{\xi(3/2)^n\} < 1/2$$

holds for every $n \geqslant 0$, where $\{\cdot\}$ denotes the fractional part. Mahler proved that, for any non-negative integer m, the real interval $(m, m+1)$ contains at most one Z-number. The existence of Z-numbers remains an open problem.

In 1995, Flatto, Lagarias and Pollington [15] proved that

$$\limsup_{n\to+\infty}\{\xi(3/2)^n\} - \liminf_{n\to+\infty}\{\xi(3/2)^n\} \geqslant 1/3,$$

for every real number $\xi > 0$. An alternative (and much simpler) proof, together with an extension to fractional parts of powers of algebraic numbers, was given by Dubickas [13], who established the following result. Recall that a Salem number is a real algebraic integer greater than one, all of whose Galois conjugates are in the closed unit disc, with only one of them in the open unit disc.

Theorem 4.1 (Dubickas, 2006). *Let $\xi > 0$ and $\alpha > 1$ be real numbers, with α algebraic. Let $P(X)$ denote the minimal defining polynomial of α over $\mathbb{Z}$. Further, suppose that ξ lies outside the field $\mathbb{Q}(\alpha)$ if α is a Pisot number or a Salem number. Then,*

$$\limsup_{n\to+\infty}\ \{\xi\alpha^n\} - \liminf_{n\to+\infty}\ \{\xi\alpha^n\} \geqslant \frac{1}{\ell(\alpha)},$$

where

$$\ell(\alpha) := \inf\{L(PG) : G(X) = b_0 + b_1X + \cdots + b_mX^m \in \mathbb{R}[X], b_0 = 1 \text{ or } b_m = 1\}$$

and $L(PG)$ is the sum of the absolute values of the coefficients of the polynomial $PG(X)$.

At present, it is unknown whether the sequence of fractional parts of $(3/2)^n$ is dense in $[0,1]$; see [10, Chapter 3] for more references and further results.

5 Algebraic number theory

In [M157], extending results from [M43, M53], Mahler used an inequality from the reduction theory of quadratic forms to establish new results for ideal bases in algebraic number fields. In particular, let

$$\Phi(x) := \Phi(x_1, \dots, x_n) = \sum_{h=1}^{n} \sum_{k=1}^{n} \varphi_{hk} x_h x_k$$

be a symmetric positive definite quadratic form of positive discriminant D_Φ, and let $F(x) := |\sqrt{\Phi(x)}|$. The convex body $K = K(\Phi)$ is the ellipsoid

$$K := \{x : \Phi(x) < 1\}$$

of volume

$$V := \pi^{n/2} \Gamma\left(\frac{n}{2} + 1\right)^{-1} D_\Phi^{-1/2}.$$

Then, there exist n lattice points $g_k := (g_{1k}, g_{2k}, \dots, g_{nk})$, $k = 1, \dots, n$, of determinant 1, such that

$$\prod_{k=1}^{n} \Phi(g_{1k}, g_{2k}, \dots, g_{nk}) \leqslant (n!)^3 D_\Phi.$$

Using this inequality, Mahler [M157] proved the following theorem.

Theorem 5.1 (Mahler, 1964). *Let ϑ be an algebraic number of degree $n \geqslant 2$ and $\lambda(\mathfrak{p})$ be an arbitrary ceiling of $\mathbb{Q}(\vartheta)$ with corresponding divisor $\mathfrak{a}_\lambda$. Then, there exists a basis $\alpha_1, \dots, \alpha_n$ of the (fractional) ideal $[\mathfrak{a}_\lambda]$ such that, for each $k = 1, 2, \dots, n$,*

$$C^{-(n-1)} \lambda(\mathfrak{q}) \leqslant |\alpha_k|_\mathfrak{q} \leqslant C\lambda(\mathfrak{q})$$
$$C^{-n} \lambda(\mathfrak{r}) \leqslant |\alpha_k|_\mathfrak{r} \leqslant \lambda(\mathfrak{r})$$

holds for all infinite prime divisors $\mathfrak{q}$ and finite prime divisors $\mathfrak{r}$, where, here, $C := C(\vartheta) \geqslant 1$ is an explicitly computable constant.

The benefit of this result is that one could then construct ideal bases rather than just a system of independent ideal elements.

6 p-adic analysis

Weierstraß' famous approximation theorem states that, if $\varepsilon > 0$ and f is a continuous real-valued function defined on an interval $[a, b]$, there is a polynomial p such that $|f(x) - p(x)| < \varepsilon$ on $[a, b]$, that is, continuous real-valued functions can be uniformly approximated on closed intervals by polynomials.

In [M139], on the suggestion of J. F. Koksma, Mahler proved a p-adic analogue of Weierstraß' approximation theorem for a continuous function defined on the ring of p-adic integers and taking p-adic values. In particular, he proved the following theorem.

Theorem 6.1 (Mahler, 1958). *Let f be a continuous function on $\mathbb{Z}_p$, the p-adic integers. For integers $n \geqslant 0$, define*

$$a_n := \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(n-k).$$

Then, $\{a_n\}_{n \geqslant 0}$ is a p-adic null sequence and

$$f(x) = \sum_{n \geqslant 0} a_n \binom{x}{n}$$

holds for all x in $\mathbb{Z}_p$.

As de Shalit [22] points out, this is closely related to the fact that the functions $\binom{x}{n}$, for $n \geqslant 0$, form an algebraic basis for the $\mathbb{Z}_p$-module of polynomials over $\mathbb{Q}_p$ which take integral values on $\mathbb{Z}_p$. See [22] for a further extension.

7 On the digits of the multiples of an irrational number

Let $b \geqslant 2$ be an integer. Weyl's Theorem asserts that, for any irrational number ξ, the sequence $\{m\xi\}_{m \geqslant 1}$ is uniformly distributed modulo one. This implies that, for any finite block D of digits from $\{0, 1, \ldots, b-1\}$, there exist arbitrarily large integers m such that D occurs at least once in the b-ary expansion of $m\xi$. This does not, however, provide any information regarding the number of occurrences of D in the b-ary expansion of $m\xi$. The question whether there is a positive integer m such that D occurs infinitely often in the b-ary expansion of $m\xi$ was addressed by Mahler in [M185].

Theorem 7.1 (Mahler, 1973). *Let ξ be an irrational number, $b \geqslant 2$ be an integer and k be any positive integer. Then, there exists a positive integer $B = B(b, k)$, independent of ξ, with the following property. There is an integer m in $\{1, \ldots, B\}$ such that the b-ary representation of $m\xi$ contains infinitely many occurrences of every possible sequence D of k digits* $0, 1, 2, \ldots, b-1$.

In fact, Mahler gave an explicit value for $B(b, k)$; see [10, Section 8.6] for more references and a proof that one can take $B(b, k) = b^k(b+1)$.

Mahler considered the p-adic analogue in [M191] and proved the following.

Theorem 7.2 (Mahler, 1974). *Let*

$$\xi = \sum_{h \geqslant 0} a_h p^h$$

be any irrational p-adic integer and r any positive integer. Then, there exists a positive integer m less than p^{2p^r} such that every possible sequence of r digits occurs infinitely often among the digits of $m\xi$.

8 Linear differential equations

Before moving on to the final section, we note that Mahler's lecture notes [M200] provide us with a very good reference on the works of Siegel and Šhidlovskiĭ on E-functions. On the same topic, see [M166] and [M171], among other papers, as well as the chapter in this volume by Nesterenko.

9 Open problems

In his lecture notes [M200], Mahler asked whether there exists an entire transcendental function

$$f(z) = \sum_{n \geqslant 0} a_n z^n \in \mathbb{Q}[[z]]$$

such that the image and the pre-image of any algebraic number are algebraic numbers. This was recently confirmed in [18].

The paper "Some suggestions for further research" [M216] was very influential.

It starts with a result of Maillet asserting that, if λ is a Liouville number and $f(z)$ any non-constant rational function with rational coefficients, then $f(\lambda)$ is also a Liouville number. Mahler asked whether there are entire transcendental functions with the same property.

Mahler also formulated explicitly the following question:

How close can irrational elements of Cantor's set be approximated by rational numbers

(i) in Cantor's set, and

(ii) by rational numbers not in Cantor's set?

This and analogous questions on intrinsic/extrinsic approximation have motivated several recent works, including [6, 8, 14].

Mahler further asked [M216] whether Cantor's set contains no irrational algebraic elements and suggested as a possible approach the use of "a p-adic form of Schmidt's theorem on the rational approximations of algebraic numbers." It should be pointed out that this very approach has been used by Adamczewski, Bugeaud and Luca [1, 2] to prove the transcendence of irrational numbers whose expansion in some integer base is stammering (see also [23] for a special case).

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