# On the number of integers which can be represented by a binary form

Pál Erdős and Kurt Mahler

Summary. Let $F(x, y)$ be a binary form of degree $n \geqslant 3$ with integer coefficients and non-vanishing discriminant, and let $A(u)$ be the number of different positive integers $k \leqslant u$, for which $|F(x, y)| = k$ has at least one solution in integers $x, y$. In this paper, using Mahler's the $p$-adic generalisation of the Thue–Siegel theorem, Erdős and Mahler prove that

$$\liminf_{u \to \infty} A(u) u^{-2/n} > 0.$$

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Il est facile de prouver que

$$\lim_{n\to\infty} d_n = 0.$$

Ce fait entraîne que

$$\lim_{n\to\infty} \left| S_n(t) + \sum_{j=1}^{n} d_j r_j(t) \right| = 0$$

presque partout. En remarquant que

$$c_\kappa = d_\kappa - \tfrac{1}{2} d_{\kappa-1} \quad (\kappa > 1),$$

nous voyons que les séries

$$\sum_{\kappa=1}^{\infty} c_\kappa^2 \quad \text{et} \quad \sum_{\kappa=1}^{\infty} d_\kappa^2$$

sont simultanément convergentes ou divergentes. Ce fait, avec le théorème bien connu que la série

$$\sum_{\kappa=1}^{\infty} d_\kappa r_\kappa(t)$$

est convergente ou divergente presque partout suivant que la somme $\sum_{\kappa=1}^{\infty} d_\kappa^2$ est fini ou non, entraîne notre théorème.

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# ON THE NUMBER OF INTEGERS WHICH CAN BE REPRESENTED BY A BINARY FORM

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Let $F(x, y)$ be a binary form of degree $n \geqslant 3$ with integer coefficients and non-vanishing discriminant, and let $A(u)$ be the number of different positive integers $k \leqslant u$, for which $|F(x, y)| = k$ has at least one solution in integers $x$, $y$. We prove that

$$\liminf_{u\to\infty} A(u)\, u^{-2/n} > 0. \tag{a}$$

The proof is simple, but not elementary, since it depends on the $p$-adic generalization of the Thue-Siegel theorem. The result remains true when $x$ and $y$ are restricted by conditions

$$x \geqslant 0, \quad \alpha x \leqslant y \leqslant \beta x \quad (\alpha, \beta \text{ constants}).$$

* Received 15 December, 1937; read 16 December, 1937.

Thus, for instance, when $F(x, y)$ is not negative definite, and $A(u)$ is now the number of positive integers $k \leqslant u$, for which $F(x, y) = k$ has at least one solution, then again $(a)$ is true. In the special case $F(x, y) = x^n + y^n$, $n \geqslant 3$ and odd, one of us (Erdös) had already found an elementary proof for $(a)$ some weeks ago, but this proof could not be generalized.

1. The following notation will be used:

$F(x, y) = \sum_{h=0}^{n} a_h x^{n-h} y^h$ $(a_0 a_n \neq 0)$ is a binary form of degree $n \geqslant 3$ with integer coefficients and discriminant $d \neq 0$.

$x, y$ are two integers, for which $F(x, y) \neq 0$.

$|x, y| = \max(|x|, |y|)$.

$N$ is a sufficiently large positive integer.

$A$ is an integer not zero with sufficiently large modulus $|A|$.

$\vartheta$ is a number satisfying $0 < \vartheta \leqslant 1$, to be assigned later.

$c_0, c_1, \ldots$ are positive numbers, which depend only on the form $F$.

$\gamma = \max(|a_0|, |d|, n)$.

$p$ is a prime number satisfying $\gamma < p \leqslant N^3$.

$P$ is a prime number satisfying either $P \leqslant \gamma$ or $P > N^3$.

$p^a \| A$ denotes that $A$ is divisible by $p^a$, but not by $p^{a+1}$.

$g(A)$ is the arithmetical function defined by $g(A) = \prod_{\substack{\gamma < p \leqslant N^3 \\ p^a \| A \\ p^a \leqslant N^3}} p^a$.

2. LEMMA 1. *For sufficiently large $N$*

$$G(N) = \prod_{\substack{|x, y| \leqslant N \\ F(x, y) \neq 0}} g\Big(F(x, y)\Big) \leqslant N^{83n(2N+1)^2}.$$

*Proof.* By definition, $p > n$ and $p$ is prime to $a_0$ and $d$. Hence, for given $a$ and $y$, there are at most $n$ incongruent values of $x$ (mod $p^a$), for which $F(x, y) \equiv 0 \pmod{p^a}$. Therefore, for given $p$ and $a$ with

$$\gamma < p \leqslant p^a \leqslant N^3,$$

the conditions

$$|x, y| \leqslant N, \quad F(x, y) \neq 0, \quad F(x, y) \equiv 0 \pmod{p^a}$$

have at most

$$n(2N+1)\left\{\left[\frac{2N+1}{p^a}\right] + 1\right\} \leqslant \frac{2n(2N+1)^2}{p^a}$$

solutions $x, y$. It follows that the exponent $b$, with $p^b \| G(N)$, satisfies the inequality

$$b \leqslant \sum_{a=1}^{\infty} \frac{2n(2N+1)^2}{p^a} = \frac{2n(2N+1)^2}{p-1} \leqslant \frac{4n(2N+1)^2}{p}.$$

Hence, for sufficiently large $N$,

$$G(N) \leqslant \exp\left\{\sum_{\gamma < p \leqslant N^{\vartheta}} \frac{4n(2N+1)^2}{p} \log p\right\} \leqslant N^{2\vartheta \cdot 4n(2N+1)^2},$$

since

$$\sum_{p \leqslant u} \frac{\log p}{p} \leqslant 2 \log u$$

for sufficiently large $u$.

LEMMA 2. *If $\mu$ is the number of pairs $x, y$ with*

$$|F(x, y)| \leqslant N^{\frac{1}{2}}, \quad |x, y| \leqslant N,$$

*then $\mu \leqslant \frac{1}{3}N^2$ for sufficiently large $N$.*

*Proof.* For a given $m$ with $|m| \leqslant N^{\frac{1}{2}}$ and a given $y$ with $|y| \leqslant N$, the equation $F(x, y) = m$ has at most $n$ integer solutions $x$, and therefore

$$\mu \leqslant n(2\sqrt{N}+1)(2N+1) \leqslant \tfrac{1}{3}N^2.$$

LEMMA 3. *For sufficiently large $N$, there are at least $\frac{4}{3}N^2$ pairs of integers $x, y$ with $|x, y| \leqslant N$ and $(x, y) = 1$.*

*Proof.* Obviously, the number of these pairs is at least $4M$, where $M$ denotes the number of pairs with $1 \leqslant x \leqslant N$, $1 \leqslant y \leqslant N$, $(x, y) = 1$, so that

$$M \geqslant N^2 - \sum_p \frac{N^2}{p^2} \geqslant N^2\left(2 - \sum_{h=1}^{\infty} \frac{1}{h^2}\right) = N^2\left(2 - \frac{\pi^2}{6}\right) \geqslant \frac{N^2}{3}.$$

LEMMA 4. *For sufficiently large $N$, there are at least $\frac{1}{2}N^2$ pairs of integers $x, y$ with*

$$|x, y| \leqslant N, \quad F(x, y) \neq 0, \quad (x, y) = 1, \quad g\big(F(x, y)\big) \leqslant |F(x, y)|^{160\vartheta n}.$$

*Proof.* By Lemmas 2 and 3, there are at least $\frac{4}{3}N^2 - \frac{1}{3}N^2 = N^2$ pairs $x, y$ with

$$|x, y| \leqslant N, \quad |F(x, y)| \geqslant N^{\frac{1}{2}}, \quad (x, y) = 1.$$

Hence, if Lemma 4 were false, there would be more than $N^2 - \frac{1}{2}N^2 = \frac{1}{2}N^2$ pairs $x, y$ with

$$g\big(F(x, y)\big) \geqslant |F(x, y)|^{160\vartheta n} \geqslant N^{80\vartheta n},$$

and therefore $$G(N) \geqslant N^{80\vartheta \cdot n \frac{1}{2}N^2} > N^{8\vartheta n(2N+1)^2},$$

in contradiction to Lemma 1.

LEMMA 5. *For all $x$ and $y$,*

$$|F(x, y)| \leqslant c_1 |x, y|^n.$$

*Proof.* Obvious with $c_1 = |a_0| + \ldots + |a_n|$.

LEMMA 6. *For sufficiently large $N$, there are at least $\frac{1}{2}N^2$ pairs of integers $x$, $y$ with*

$$(1) \qquad |x, y| \leqslant N, \quad F(x, y) \neq 0, \quad (x, y) = 1,$$

*such that $|F(x, y)| = k_1 k_2$, where $k_1$ and $k_2$ are positive integers such that $k_1$ is divisible by at most $c_2$ different primes, and $k_2 \leqslant |F(x, y)|^{\frac{1}{7}}$.*

*Proof.* We apply Lemma 4 with $\vartheta = 1/(1120n)$ and

$$k_2 = g\big(F(x, y)\big), \quad k_1 = \frac{|F(x, y)|}{k_2}.$$

$k_1$ and $k_2$ are positive integers, since $g\big(F(x, y)\big)$ is a positive integer which divides $F(x, y)$. By Lemma 4, for at least $\frac{1}{2}N^2$ pairs $x$, $y$ satisfying (1),

$$k_2 \leqslant |F(x, y)|^{160\vartheta n} = |F(x, y)|^{\frac{1}{7}}.$$

The other factor $k_1$ is divisible only by prime numbers of the form $P$ with either $P \leqslant \gamma$ or $P > N^\vartheta$. But there are at most $\gamma$ primes of the first form, and, since by Lemma 5

$$|F(x, y)| \leqslant c_1 N^n,$$

there are at most $1200n^2$ different primes of the second form, which can divide $F(x, y)$, for sufficiently large $N$.

3. To conclude the proof we use the following generalization of the Thue-Siegel theorem*:

LEMMA 7. *Suppose that $x$ and $y$ are integers with*

$$F(x, y) \neq 0, \quad (x, y) = 1,$$

*that $P_1, P_2, \ldots, P_t$ are $t$ different prime numbers, and that*

$$Q(x, y) = P_1^{h_1} P_2^{h_2} \ldots P_t^{h_t}$$

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* See K. Mahler, *Math. Annalen*, 108 (1933), 51, Satz 6, from which Lemma 7 is a trivial consequence, if $F(x, y)$ is irreducible. But Satz 6 remains true when $F(x, y)$, though reducible, has a non-vanishing discriminant, if only the representations of $k = 0$ are excluded; a proof for this generalized theorem and so for the general case of Lemma 7 will be published in the near future.

*is the greatest product of powers of these primes which divides $F(x, y)$. Then the inequality*

$$\frac{|F(x, y)|}{Q(x, y)} \leqslant |x, y|^{\frac{1}{2}n-1-\frac{1}{28}}$$

*has at most $c_3^{t+1}$ solutions in different pairs $x$, $y$.*

Suppose now that $k$ is a positive integer, for which $|F(x, y)| = k$ has at least one solution. Then, by Lemma 5,

$$c_1 |x, y|^n \geqslant k, \quad \text{i.e.} \quad |x, y| \geqslant \left(\frac{k}{c_1}\right)^{1/n},$$

so that $|x, y|$ cannot be too small. The integer $k = k_1 k_2$ is a product of two positive integers $k_1$ and $k_2$, of which $k_1$ has no other prime factors than $P_1, \ldots, P_t$, while $k_2$ is prime to $P_1, \ldots, P_t$; hence, in particular,

$$k_1 = Q(x, y), \quad k_2 = \frac{|F(x, y)|}{Q(x, y)}.$$

Suppose that $$k_2 \leqslant k^{\frac{1}{7}},$$

so that $$k_2 \leqslant c_1^{\frac{1}{7}} |x, y|^{\frac{1}{7}n}.$$

Since $n \geqslant 3$, we have

$$\tfrac{1}{2}n - 1 - \tfrac{1}{28} \geqslant \tfrac{1}{2}n - \tfrac{1}{3}n - \tfrac{1}{28} = \tfrac{1}{6}n - \tfrac{1}{28} \geqslant \tfrac{1}{7}n + (\tfrac{3}{6} - \tfrac{3}{7} - \tfrac{1}{28}) = \tfrac{1}{7}n + \tfrac{1}{28}.$$

Thus, when

$$k \geqslant c_1^{4n+1} = c_4, \quad \text{i.e.} \quad |x, y| \geqslant \left(\frac{k}{c_1}\right)^{1/n} \geqslant c_1^{4},$$

we get

$$c_1^{\frac{1}{7}} |x, y|^{-\frac{1}{28}} \leqslant 1, \quad \text{and} \quad k_2 \leqslant c_1^{\frac{1}{7}} |x, y|^{-\frac{1}{28}} |x, y|^{\frac{1}{7}n+\frac{1}{28}} \leqslant |x, y|^{\frac{1}{2}n-1-\frac{1}{28}}.$$

Hence Lemma 7 leads to

LEMMA 8. *If the positive integer $k$ is larger than $c_4$, and if it can be written in the form $k = k_1 k_2$, where $k_1$ is divisible by only $t$ different prime numbers, and where $k_2 \leqslant k^{\frac{1}{7}}$, then the equation $|F(x, y)| = k$ has not more than $c_3^{t+1}$ different solutions $x$, $y$ in relatively prime integers $x$ and $y$.*

THEOREM 1. *For every sufficiently large positive $u$, there are at least $c_0 u^{2/n}$ different positive integers $k \leqslant u$, for which the equation $|F(x, y)| = k$ has at least one solution in relatively prime integers $x$ and $y$.*

*Proof.* Suppose, in Lemma 6, that

$$N = \left(\frac{u}{c_1}\right)^{1/n}, \quad \text{i.e.} \quad |F(x, y)| \leqslant u \quad \text{for} \quad |x, y| \leqslant N.$$

Then it follows that there are at least

$$\tfrac{1}{2}c_1^{-1/n}u^{2/n}$$

different pairs of relatively prime integers $x$, $y$ with $|x, y| \leqslant N$, for which

$$|F(x, y)| = k \neq 0$$

is a product of two positive integers $k = k_1 k_2$, such that $k_1$ is divisible by at most $c_2$ different primes, while $k_2 \leqslant k^{\frac{1}{4}}$. Hence, by Lemma 8, either $k \leqslant c_4$, or the number of different relatively prime solutions of $|F(x, y)| = k$ is not larger than $c_3^{c_2+1}$. Therefore, for sufficiently large $u$, there must be at least

$$\tfrac{1}{2}\, c_3^{-(c_2+1)} \cdot \tfrac{1}{2}c_1^{-1/n}u^{2/n}$$

different positive integers $k \leqslant u$, for which $|F(x, y)| = k$ has at least one solution in integers $x$, $y$ with $(x, y) = 1$.

4. By a theorem of Siegel*, the inequality

$$0 < |F(x, y)| \leqslant u$$

has only $O(u^{2/n})$ solutions in integers $x$, $y$. Hence the number of integers $k$, with $1 \leqslant k \leqslant u$, which can be represented by $|F(x, y)|$, say the number $A(u)$, must also be $O(u^{2/n})$, and so Theorem 1 gives the exact order of this function and shows that $\liminf A(u)/u^{2/n} > 0$, while $\limsup A(u)/u^{2/n} < \infty$.

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## ON THE PRODUCT OF THREE HOMOGENEOUS LINEAR FORMS

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1. Let $\xi$, $\eta$, $\zeta$ be three real homogeneous linear forms in $x$, $y$, $z$ with determinant 1. Let‡

$$M = \min|\xi\eta\zeta| \tag{1}$$

when $x$, $y$, $z$ assume integral values not all zero.

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* As Prof. Siegel's proof has not been published, see K. Mahler, *Acta Math.*, 62 (1934), 92 *ff*.

† Received and read 16 December, 1937.

‡ min means lower bound, whether attained or not.