On a theorem of Liouville in fields of positive characteristic

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Summary. In this paper, Mahler gives two interesting results related to Liouville's rational approximation theorem for algebraic numbers. First, he proves the analogue of Liouville's result for function fields with coefficients in an arbitrary field. Second, he shows that, in contrast to the situation for fields of characteristic zero, Liouville's theorem for algebraic functions cannot be improved if the ground field is of finite characteristic.

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ON A THEOREM OF LIOUVILLE IN FIELDS OF POSITIVE CHARACTERISTIC

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A classical theorem of J. Liouville[1] states that if z is a real algebraic number of degree $n \geq 2$, then there exists a constant $c > 0$ such that

$$\left| z - \frac{a}{b} \right| \geq \frac{c}{|b|^n}$$

for every pair of integers a, b with $b \neq 0$.

This theorem has an analogue in function fields. Let k be an arbitrary field, x an indeterminate, $k[x]$ the ring of all polynomials in x with coefficients in k, $k(x)$ the field of all rational functions in x with coefficients in k, and $k<x>$ the field of all formal series

$$z = a_f x^f + a_{f-1} x^{f-1} + a_{f-2} x^{f-2} + \dots$$

in x where the coefficients $a_f, a_{f-1}, a_{f-2}, \dots$ are in k. Thus $k(x)$ is the quotient field of $k[x]$ and a subfield of $k<x>$.

A valuation $|z|$ in $k<x>$ is now defined by putting $|0| = 0$; but $|z| = e^f$ if $z = a_f x^f + a_{f-1} x^{f-1} + a_{f-2} x^{f-2} + \dots$ and $a_f \neq 0$. If z lies in $k[x]$, then $\log |z|$ is simply the degree of z.

With this notation, the analogue to Liouville's theorem states:

Theorem 1. *If the element z of $k<x>$ is algebraic of degree $n \geq 2$ over $k(x)$, then there exists a constant $c > 0$ such that*

$$\left| z - \frac{a}{b} \right| \geq \frac{c}{|b|^n}$$

for all pairs of elements a and $b \neq 0$ of $k[x]$.

Proof. Denote by

$$f(y) = a_0 y^n + a_1 y^{n-1} + \dots + a_n, \qquad \text{where } a_0 \neq 0,$$

a polynomial in y with coefficients in $k[x]$ which is irreducible over $k(x)$ and vanishes for $y = z$; further put

$$g(y) = a_0 y^{n-1} + (a_0 z + a_1) y^{n-2} + (a_0 z^2 + a_1 z + a_2) y^{n-3} + \dots + (a_0 z^{n-1} + a_1 z^{n-2} + \dots + a_{n-1}).$$

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[1] *C.R. Acad. Sci. Paris*, vol. 18 (1844), 883-885, 910-911.

Then

$$\frac{f(y)}{y-z} = \frac{f(y)-f(z)}{y-z} = g(y)$$

identically in y, and therefore

$$y - z = \frac{f(y)}{g(y)}.$$

Put

$$\max(|a_0|, |a_1|, \ldots, |a_n|) = c_1, \ \max(1, |z|) = c_2$$

and take

$$y = \frac{a}{b}$$

where a and $b \neq 0$ are in $k[x]$.

If

$$\left|\frac{a}{b}\right| > c_2 = |z|,$$

then

(1) $$\left|z - \frac{a}{b}\right| = \left|\frac{a}{b}\right| > c_2 \geq \frac{c_2}{|b|^n}, \qquad \text{since } |b| \geq 1.$$

Next let

$$\left|\frac{a}{b}\right| \leq c_2,$$

so that

$$\left|g\left(\frac{a}{b}\right)\right| \leq c_1 c_2^{n-1}.$$

The expression

$$b^n f\left(\frac{a}{b}\right) = a_0 a^n + a_1 a^{n-1} b + \ldots + a_n b^n$$

lies in $k[x]$ and does not vanish since $f(y)$ is irreducible and at least of the second degree. Therefore

$$\left|b^n f\left(\frac{a}{b}\right)\right| \geq 1, \ \left|f\left(\frac{a}{b}\right)\right| \geq |b|^{-n},$$

whence

(2) $$\left|z - \frac{a}{b}\right| = \left|\frac{f\left(\frac{a}{b}\right)}{g\left(\frac{a}{b}\right)}\right| \geq \frac{1}{c_1 c_2^{n-1} |b|^n}.$$

If we now put

$$c = \min\left(c_2, \frac{1}{c_1 c_2^{n-1}}\right),$$

then the assertion of the theorem is contained in (1) and (2).

In the case of a real algebraic number of degree $n \geq 3$, Liouville's theorem is not the best-possible, and it was first improved by A. Thue,[2] who showed that, for every $\epsilon > 0$, there is a constant $c(\epsilon) > 0$ such that

$$\left| z - \frac{a}{b} \right| \geq \frac{c(\epsilon)}{|b|^{\frac{n}{2}+1+\epsilon}}$$

for all pairs of integers a and $b \neq 0$. Still better inequalities were given by C. L. Siegel[3] and F. J. Dyson.[4] A similar improvement is possible in the case of the analogue of Liouville's theorem for algebraic functions, if the constant field k is the field of all complex numbers, or, more generally, any field of characteristic 0, as was proved by B. P. Gill.[5]

It is then of some interest to note that *the analogue of Liouville's theorem for algebraic functions cannot be improved if the ground field k is of characteristic p where p is a positive prime number.* Indeed, the following result holds.

THEOREM 2. *Let k be any field of characteristic p, x an indeterminate, and z the element*

$$z = x^{-1} + x^{-p} + x^{-p^2} + x^{-p^3} + \dots$$

of $k\langle x \rangle$. Then z is of exact degree p over $k(x)$, and there exists an infinite sequence of pairs of elements a_n and $b_n \neq 0$ of $k[x]$ such that

$$\left| z - \frac{a_n}{b_n} \right| = |b_n|^{-p}, \quad \lim_{n\to\infty} |b_n| = \infty.$$

Proof. If $a, b, c, \dots$ are elements of $k\langle x \rangle$, then

$$(a + b + c + \dots)^p = a^p + b^p + c^p + \dots,$$

by a well-known property of fields of characteristic p. Hence, in particular,

$$z = x^{-1} + (x^{-p} + x^{-p^2} + x^{-p^3} + \dots) = x^{-1} + (x^{-1} + x^{-p} + x^{-p^2} + \dots)^p$$

and so z is a root of the algebraic equation[6]

$$z^p - z + x^{-1} = 0 \tag{3}$$

of degree p over $k(x)$.

Put, for $n = 1, 2, 3, \dots,$

$$a_n = x^{p^{n-1}}(x^{-1} + x^{-p} + \dots + x^{-p^{n-1}}), \quad b_n = x^{p^{n-1}}$$

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[2] *Norske Vid. Selsk. Scr.* (1908), Nr. 7.
[3] *Math. Zeit.*, vol. 10 (1921), 173-213.
[4] *Acta Math.*, vol. 79 (1947), 225-240.
[5] *Ann. of Math.* (2) 31 (1930), 207-218.
[6] I am indebted to E. Artin for the remark that z is algebraic if k is of characteristic p. If k is of characteristic 0, then z is, of course, transcendental over $k(x)$.

so that

$$|b_n| = e^{p^{n-1}}, \text{ and } \left|z - \frac{a_n}{b_n}\right| = |x^{-p^n} + x^{-p^{n+1}} + \ldots| = e^{-p^n} = |b_n|^{-p}.$$

The assertion will therefore be proved if we can show that z is of exact degree p. But, by Theorem 1, z cannot be of lower degree than p, unless it is of degree 1 and lies in $k(x)$. Suppose then that

$$z = \frac{A}{B},$$

where A and $B \neq 0$ are elements of $k[x]$. Since the fractions a_n/b_n are all different,

$$\frac{a_n}{b_n} \neq z, \ Ab_n - a_nB \neq 0, \ |Ab_n - a_nB| \geq 1,$$

for all sufficiently large n. But then

$$|b_n|^{-p} = \left|z - \frac{a_n}{b_n}\right| = \left|\frac{A}{B} - \frac{a_n}{b_n}\right| = \left|\frac{Ab_n - a_nB}{Bb_n}\right| \geq \frac{1}{|B|\,|b_n|},$$

whence

$$|B| \geq |b_n|^{p-1},$$

contrary to the fact that

$$\lim_{n\to\infty} |b_n| = \infty.$$

It would be of interest to investigate whether the analogue of Liouville's theorem remains still the best-possible for elements $k\langle x\rangle$ not in $k(x)$ which are of a degree *less than p* over $k(x)$.

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