On the approximation of π

Kurt Mahler

Summary. The aim of this paper is to determine an explicit lower bound, free of unknown constants, for the distance of π from a given rational or algebraic number. In particular, Mahler proves that, for all positive integers $p, q \geqslant 2$,

$$\left|\pi - \frac{p}{q}\right| > \frac{1}{q^{42}}.$$

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MATHEMATICS

ON THE APPROXIMATION OF π

BY

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The aim of this paper is to determine an explicit lower bound free of unknown constants for the distance of π from a given rational or algebraic number.

1. In my paper *"On the approximation of logarithms of algebraic numbers"*, which is to appear in the Transactions of the Royal Society, the following result was proved:

Lemma: Let x be a real or complex number different from 0 and 1; let $\log x$ denote the principal value of the natural logarithm of x; and let m and n be two positive integers such that

$$m + 1 \geqslant 2\,|\log x|. \tag{1}$$

There exist $(m+1)^2$ polynomials

$$A_{hk}(x) \qquad (h, k = 0,1, \dots, m)$$

in x with rational integral coefficients, of degrees not greater than n, and with the following further properties:

(a) *The determinant*

$$D(x) = \|A_{hk}(x)\|$$

does not vanish.

(b) $A_{hk}(x) << m!\, 2^{m-(3n/2)} (n+1)^{2m+1} (\sqrt{32})^{(m+1)n} (1 + x + \dots + x^n).$

(c) *The $m + 1$ functions*

$$R_h(x) = \sum_{k=0}^{m} A_{hk}(x) (\log x)^k \qquad (h = 0,1, \dots, m)$$

satisfy the inequalities

$$|R_h(x)| \leqslant m!\, 2^{-(3n/2)} (e\sqrt{n})^{m+1} e^{(2n+1)\,|\log x|} \left(\frac{\sqrt{8}\,|\log x|}{m+1}\right)^{(m+1)n}.$$

Denote by y a further real or complex number, and put

$$S_h(x,y) = \sum_{k=0}^{m} A_{hk}(x)\, y^k, \quad T_h(x,y) = \sum_{k=1}^{m} A_{hk}(x) \frac{(\log x)^k - y^k}{\log x - y} \quad (h = 0,1, \dots, m),$$

so that

$$R_h(x) - S_h(x,y) = T_h(x,y)(\log z - y), \tag{2}$$

identically in x and y. This identity will enable us to find a measure of irrationality for π.

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2. For this purpose, substitute in the last formulae the values

$$x = i, \ \log x = \pi \frac{i}{2}, \ y = \frac{p}{q}\frac{i}{2}$$

for x, $\log x$, and y; here p and q may be any two positive integers for which

(3) $$p < 4q.$$

Then

$$|\log x| < 2, \ |y| < 2,$$

so that

$$\left|\frac{(\log x)^k - y^k}{\log x - y}\right| = |(\log x)^{k-1} + (\log x)^{k-2} y + \ldots + (\log x) y^{k-2} + y^{k-1}| < 2^{k-1} k$$

and

$$\sum_{k=1}^{m}\left|\frac{(\log x)^k - y^k}{\log x - y}\right| < \sum_{k=1}^{m} 2^{k-1} k \leqslant \sum_{k=1}^{m} 2^{k-1} m < 2^m m.$$

Hence

(4) $$|T_h(x,y)| < 2^m m \cdot \max_{h,k=0,1,\ldots,m} |A_{hk}(x)|.$$

3. From now on assume that

$$m = 10 \text{ and } n \geqslant 50.$$

This choice of m satisfies the condition (1) of the lemma. The lemma may then be applied, and we find, first, that

$$\max_{h,k=0,1,\ldots,m} |A_{hk}(x)| \leqslant 10!\, 2^{10-(3n/2)} (n+1)^{21} 2^{(55/2)n} (1 + |x| + \ldots + |x|^n) =$$
$$= 10!\, 2^{10} (n+1)^{22} 2^{26n},$$

whence, by (4),

(5) $$|T_h(x,y)| < 10.10!\, 2^{20} (n+1)^{22} 2^{26n}.$$

Secondly,

(6) $$|R_h(x)| \leqslant 10!\, 2^{-(3n/2)} e^{11} n^{11/2} e^{n\pi + (\pi/2)} \left(\frac{\sqrt{2}\pi}{11}\right)^{11n} = 10!\, e^{11+(\pi/2)} n^{11/2} \left(\frac{16\pi^{11} e^{\pi}}{11^{11}}\right)^n.$$

Thirdly, $D(x) \neq 0$. Hence the index h, $= h_0$ say, can be chosen such that $S_{h_0}(x, y) \neq 0$. Now $(2q)^m S_{h_0}(x, y)$ evidently is an integer in the Gaussian field $K(i)$. Its absolute value is therefore not less than unity, whence, by the choice of m,

(7) $$|S_{h_0}(x,y)| \geqslant 2^{-10} q^{-10}.$$

4. Assume now that $n \geqslant 50$ can be selected so as to satisfy the inequality

(8) $$10!\, e^{11+(\pi/2)} n^{11/2} \left(\frac{16\pi^{11} e^{\pi}}{11^{11}}\right)^n \leqslant \tfrac{1}{2}\, 2^{-10} q^{-10}.$$

By (6) and (7), this inequality implies that

$$|R_{h_0}(x)| \leqslant \tfrac{1}{2} |S_{h_0}(x,y)|,$$

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and so, by (2),

$$\tfrac{1}{2}|S_{h_0}(x,y)| \leqslant |T_{h_0}(x,y)(\log x - y)|.$$

It follows then from (5) and (7) that

$$(9)\quad \left|\pi - \frac{p}{q}\right| = 2|\log x - y| \geqslant \left|\frac{S_{h_0}(x,y)}{T_{h_0}(x,y)}\right| \geqslant$$
$$\geqslant 2^{-10} q^{-10} \{10.10!\, 2^{20} (n+1)^{22}\, 2^{26n}\}^{-1}.$$

The two inequalities (8) and (9) are equivalent to

$$(10)\qquad \left(\frac{11^{11}}{16\pi^{11} e^{\pi}}\right)^n \geqslant 2^{11}\, 10!\, e^{11+(\pi/2)}\, n^{11/2}\, q^{10},$$

and

$$(11)\qquad \left|\pi - \frac{p}{q}\right| \geqslant \{10.10!\, 2^{30}(n+1)^{22}\, 2^{26n}\}^{-1} q^{-10},$$

respectively. Here

$$\frac{11^{11}}{16\pi^{11} e^{\pi}} > 10^{3.4181}, \qquad 2^{26} < 10^{7.8268},$$

and also, on account of $n \geqslant 50$,

$$2^{11}\, 10!\, e^{11+(\pi/2)} < 10^{15.3306} < 10^{0.3067n}, \quad 10.10!\, 2^{30} < 10^{16.5907} < 10^{0.3319n}.$$

Further, on denoting by Log N the decadic logarithm of N,

$$n^{11/2} = 10^{11/2\,(\mathrm{Log}\, n/n)n} \leqslant 10^{11/2\,(\mathrm{Log}\, 50/50)n} < 10^{0.1869n}$$

and

$$(n+1)^{22} = 10^{22\,(\mathrm{Log}\,(n+1)/n)n} \leqslant 10^{22\,(\mathrm{Log}\, 51/50)n} < 10^{0.7514n}.$$

These numerical formulae show that the inequality (10) certainly holds if

$$10^{3.4181n} > 10^{0.3067n+0.1869n}\, q^{10},$$

i.e., if

$$10^{2.9245n} > q^{10},$$

and they further give

$$10.10!\, 2^{30}(n+1)^{22}\, 2^{26n} < 10^{0.3319n+0.7514n+7.8268n} = 10^{8.9101n}.$$

We thus have proved the following result:

"*Let p and q be two positive integers such that $p < 4q$, and let n be an integer for which*

$$(12)\qquad n \geqslant 50, \qquad 10^{2.9245n} > q^{10}.$$

Then

$$(13)\qquad \left|\pi - \frac{p}{q}\right| > 10^{-8.9101n}\, q^{-10}."$$

5. This result be further simplified. Define n as function of q by the inequalities

$$10^{2.9245(n-1)} \leqslant q^{10} < 10^{2.9245n}.$$

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This choice of n is permissible provided q is so large that

$$q^{10} \geqslant 10^{2.9245\times 49} = 10^{143.3005}.$$

It suffices then to make the further assumption that

$$q \geqslant 2.14 \times 10^{14}, \tag{14}$$

because then

$$q^{10} > 10^{143.304}.$$

Since $n \geqslant 50$ and therefore $n - 1 \geqslant \frac{49}{50} n$, we have now

$$q^{10} \geqslant 10^{2.9245\times 0.98n} > 10^{2.8661n},$$

hence, by (13),

$$\left|\pi - \frac{p}{q}\right| > q^{-(8.9101/2.8661)\times 10-10} > q^{-41.09} > q^{-42}. \tag{15}$$

The proof assumed, as we saw, that $p < 4q$ and that (14) is satisfied. If (14) holds, but $p \geqslant 4q$, then trivially

$$\left|\pi - \frac{p}{q}\right| \geqslant 4 - \pi > q^{-42},$$

and (15) remains true.

6. It is now of greater interest that the remaining condition (14) can be replaced by a more natural one.

Theorem 1: *If p and $q \geqslant 2$ are positive integers, then*

$$\left|\pi - \frac{p}{q}\right| > q^{-42}.$$

Proof: By what has already been shown, it suffices to verify that there are no pairs of positive integers p, q for which

$$2 \leqslant q < 2.14 \times 10^{14}, \quad \left|\pi - \frac{p}{q}\right| \leqslant q^{-42}.$$

If such pairs of integers exist, they necessarily have the additional property that

$$\left|\pi - \frac{p}{q}\right| < \frac{1}{2q^2},$$

because otherwise

$$\frac{1}{2q^2} \leqslant \left|\pi - \frac{p}{q}\right| \leqslant q^{-42}, \qquad q^{40} \leqslant 2, \qquad q < 2,$$

which is false. It follows then, by the theory of continued fractions, that p/q must be one of the convergents p_n/q_n of the continued fraction

$$\pi = b_0 + \frac{1|}{|b_1} + \frac{1|}{|b_2} + \dots = [b_0; b_1, b_2, \dots]$$

for π; here the incomplete denominators $b_0, b_1, b_2, \dots$ are positive integers. According to J. WALLIS, the development begins as follows:

$$\pi = [3;\ 7,\ 15,\ 1,\ 292,\ 1,\ 1,\ 1,\ 2,\ 1,\ 3,\ 1,\ 14,\ 2,\ 1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 1,\ 84,\ 2,\ 1,\ 1,\ 15,\ 3,\ 13,\ 1,\ 4,\ 2,\ 6,\ 6,\ 1, \dots].$$

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A trivial computation shows that the convergent belonging to the incomplete denominator 13 is already greater than 2.14×10^{14}. The largest of the preceding incomplete denominators is 292. Hence, by the theory of continued fractions, we find that

$$\left|\pi - \frac{p_n}{q_n}\right| > \frac{1}{q_n(q_{n+1}+q_n)} =$$
$$= \frac{1}{q_n\{(b_{n+1}+1)q_n + q_{n-1}\}} > \frac{1}{(b_{n+1}+2)q_n^2} \geqslant \frac{1}{294 q_n^2} > q_n^{-42}$$

for every convergent the denominator of which lies in the range we are considering. There are therefore no pairs of integers p, q of the required kind. This completes the proof.

The theorem required that $q \geqslant 2$. If one is satisfied with an estimate for $|\pi - (p/q)|$ valid when q is greater than *some* large value q_0, then the exponent 42 can be replaced by 30. No new ideas being involved, the proof may be omitted.

7. As a second application of the lemma in § 1 we study now the approximation of π by arbitrary algebraic numbers.

Let ω be a real or complex algebraic number of degree ν over the Gaussian field $K(i)$, and let

$$f(z) = 0, \text{ where } f(z) = a_0 z^\nu + a_1 z^{\nu-1} + \ldots + a_\nu$$

and where further the coefficients $a_0 \neq 0, a_1, \ldots, a_\nu$ are integers in $K(i)$, be an irreducible equation for ω over this field. Denote by

$$a = \max(|a_0|, |a_1|, \ldots, |a_\nu|)$$

the height of this equation and by

$$\omega_0 = \omega, \omega_1, \ldots, \omega_{\nu-1}$$

its roots. These roots are all different, and it is well known that

$$|\omega_j| \leqslant a + 1 \qquad (j = 0, 1, \ldots, \nu - 1). \tag{16}$$

8. In the case when ω is a real algebraic number, the defining equation $f(z) = 0$ may be assumed to have *rational* integral coefficients. For let

$$F(z) = 0, \quad \text{where} \quad F(z) = A_0 z^N + A_1 z^{N-1} + \ldots + A_N,$$

and where $A_0 \neq 0, A_1, \ldots, A_N$ are rational integers, be an equation for ω irreducible over the rational field. It suffices to show that this equation is also irreducible over $K(i)$, hence that $F(z)$ differs from $f(z)$ only by a constant factor different from zero.

Let the assertion be false. Then $F(z)$ can be written as

$$F(z) = \{A(z) + iB(z)\}\{C(z) + iD(z)\}$$

where $A(z)$, $B(z)$, $C(z)$, and $D(z)$ are polynomials with rational coefficients such that neither $A(z) + iB(z)$ nor $C(z) + iD(z)$ is a constant. Since $F(z)$ is a real polynomial, also

$$F(z) = \{A(z) - iB(z)\}\{C(z) - iD(z)\}$$

and therefore, on multiplying the two equations,

$$F(z)^2 = \{A(z)^2 + B(z)^2\}\,\{C(z)^2 + D(z)^2\}.$$

Since unique factorization holds for polynomials in one variable over the rational field, this formula implies that

$$F(z) = c\,\{A(z)^2 + B(z)^2\}$$

where $c \neq 0$ is a rational constant.

Put now $z = \omega$. Then $F(z)$ and therefore $A(z)^2 + B(z)^2$ vanish, hence also both $A(z)$ and $B(z)$. This means that $A(z)$ and $B(z)$ are divisible by $z - \omega$, thus $F(z)$ by $(z-\omega)^2$. This is impossible because $F(z)$ is irreducible, so that it cannot have multiple linear factors.

9. Substitute now

$$x = i, \qquad \log x = \pi \frac{i}{2}, \qquad y = \omega \frac{i}{2}$$

for x, $\log x$, and y in the identity

$$R_h(z) - S_h(x, y) = T_h(x, y)\,(\log x - y) \tag{2}$$

of § 1, and assume further that

$$|\omega| < 4, \; m \geqslant 3.$$

One proves just as in § 2 and § 3 that

$$|R_h(x)| \leqslant m!\; 2^{-(3n/2)}\,(e\sqrt{n})^{m+1}\, e^{n\pi + (\pi/2)} \left(\frac{\sqrt{2}\,\pi}{m+1}\right)^{(m+1)n} \tag{17}$$

and

$$|T_h(x,y)| < 2^m\, m \cdot m!\; 2^{m-(3n/2)}\,(n+1)^{2m+2}\,(\sqrt{32})^{(m+1)n} \tag{18}$$

On the other hand, the then given lower bound for $S_{h_0}(x, y)$ is no longer valid and must be replaced by a more involved expression.

10. Since the determinent $D(x)$ does not vanish, there is again an index $h = h_0$ such that

$$S_{h_0}(x, y) = S_{h_0}\left(i, \omega\,\frac{i}{2}\right) \neq 0.$$

This means that also the $\nu - 1$ numbers

$$S_{h_0}\left(i, \omega_1 \frac{i}{2}\right), \quad S_{h_0}\left(i, \omega_2 \frac{i}{2}\right), \ldots, \quad S_{h_0}\left(i, \omega_{\nu-1} \frac{i}{2}\right)$$

obtained from $S_{h_0}(i, \omega i/2)$ on replacing ω by its conjugates $\omega_1, \omega_2, \ldots, \omega_{\nu-1}$ with respect to $K(i)$ do not vanish. For let z be a variable. The expression $S_{h_0}(i, zi/2)$ is a polynomial in z with coefficients in $K(i)$ which does not vanish at $z = \omega$. Therefore the polynomial cannot be divisible by the irreducible polynomial $f(z)$ of which ω is a root, and so it admits none of its other roots ω_j.

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It follows then that the product

$$\sigma = \prod_{j=0}^{\nu-1} S_{h_0}\left(i,\, \omega_j \frac{i}{2}\right)$$

does not vanish. This product is a symmetric polynomial in $\omega, \omega_1, \ldots, \omega_{\nu-1}$ which is in each ω_j of degree m; moreover, the coefficients of this polynomial are elements of $K(i)$, and their common denominator is a divisor of $2^{m\nu}$. Therefore σ itself lies in the Gaussian field, and its denominator is in absolute value not greater than

$$2^{m\nu}\, |a_0|^m \leqslant 2^{m\nu}\, a^m.$$

Since σ is not zero, the inequality

$$2^{m\nu}\, a^m\, |\sigma| \geqslant 1,$$

holds, and we find that

$$|S_{h_0}(x,y)| \geqslant \left\{2^{m\nu}\, a^m \prod_{j=1}^{\nu-1} \left| S_{h_0}\left(i,\, \omega_j \frac{i}{2}\right)\right|\right\}^{-1}. \tag{19}$$

11. By definition,

$$S_{h_0}\left(i, \omega_j \frac{i}{2}\right) = \sum_{k=0}^{m} A_{h_0 k}(i) \left(\omega_j \frac{i}{2}\right)^k.$$

Here, by (16),

$$|\omega_j| \leqslant a + 1,$$

so that

$$\sum_{k=0}^{m} \left|\omega_j \frac{i}{2}\right|^k \leqslant \sum_{k=0}^{m} \left(\frac{a+1}{2}\right)^k \leqslant (m+1)\left(\frac{a+1}{2}\right)^m \leqslant (m+1)\, a^m$$

since $a \geqslant 1$. Therefore

$$\left|S_{h_0}\left(i, \omega_j \frac{i}{2}\right)\right| \leqslant (m+1)\, a^m \max_{h,k=0,1,\ldots,m} |A_{hk}(i)|,$$

whence, by the lemma in 1.),

$$\left|S_{h_0}\left(i, \omega_j \frac{i}{2}\right)\right| \leqslant a^m\, (m+1)!\; 2^{m-(3n/2)}\, (n+1)^{2m+2}\, (\sqrt{32})^{(m+1)n}.$$

Therefore, from (19),

$$|S_{h_0}(x,y)| \geqslant \{2^{m\nu}\, a^m\, (a^m\, (m+1)!\, 2^{m-(3n/2)}\, (n+1)^{2m+2}\, (\sqrt{32})^{(m+1)n})^{\nu-1}\}^{-1}. \tag{20}$$

12. From now on we proceed in a similar way as in 4.). Let again $m \geqslant 3$ and n be chosen such that

$$|R_{h_0}(x)| \leqslant \tfrac{1}{2} |S_{h_0}(x, y)|; \tag{a}$$

then from the identity (2),

$$|S_{h_0}(x,y)| \leqslant 2|T_{h_0}(x,y)\, (\log x - y)|, \tag{b}$$

so that a lower bound for

$$2|\log x - y| = |\pi - \omega|$$

is obtained.

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By (17) and (20), the condition (a) is certainly satisfied if

$$m!\, 2^{-(3n/2)} (e\sqrt{n})^{m+1} e^{n\pi+(\pi/2)} \left(\frac{\sqrt{2}\,\pi}{m+1}\right)^{(m+1)n} \leqslant$$
$$\leqslant \tfrac{1}{2}\{2^{m\nu} a^m (a^m (m+1)!\, 2^{m-(3n/2)} (n+1)^{2m+2} (\sqrt{32})^{(m+1)n})^{\nu-1}\}^{-1},$$

or, what is the same, if

$$(21)\quad \left(\frac{4\,(m+1)}{2^{5\nu/2}\,\pi}\right)^{(m+1)m} \geqslant$$
$$\geqslant \frac{(m+1)!^{\nu}}{m+1}\, 2^{(2\nu-1)m-(3n\nu/2)+1}\, e^{m+n\pi+(\pi/2)+1} (\sqrt{n}\,(n+1)^{2(\nu-1)})^{m+1} a^{m\nu}.$$

Under this hypothesis, we find from (b), by (18) and (20), that

$$|\pi-\omega| > \{2^{m\nu} a^m (a^m (m+1)!\, 2^{m-(3n/2)} (n+1)^{2m+2} (\sqrt{32})^{(m+1)n})^{\nu-1}\}^{-1} \times$$
$$\times \{2^m m\cdot m!\, 2^{m-(3n/2)} (n+1)^{2m+2} (\sqrt{32})^{(m+1)n}\}^{-1}.$$

whence, after some trivial simplification,

$$(22)\quad |\pi-\omega| > \left\{\frac{m}{m+1} (m+1)!^{\nu}\, 2^{(2\nu+1)m-(3n\nu/2)} (n+1)^{2(m+1)\nu} (\sqrt{32})^{(m+1)n\nu} a^{m\nu}\right\}^{-1}.$$

In order to put (21) and (22) into a more convenient form, we now apply the well-known inequality

$$(m+1)! \leqslant e\sqrt{m+1}\,(m+1)^{m+1} e^{-(m+1)}.$$

It follows that (21) is satisfied if

$$\left(\frac{4(m+1)}{2^{5\nu/2}\,\pi}\right)^{(m+1)n} \geqslant e^{\nu} (m+1)^{(\nu/2)-1} (m+1)^{(m+1)\nu} e^{-(m+1)\nu}\, 2^{2(m+1)\nu-2\nu-(m+1)-(3n\nu/2)+2} \times$$
$$\times e^{(m+1)+n\pi+(\pi/2)} \left(\frac{\sqrt{n}}{(n+1)^2}(n+1)^{2\nu}\right)^{m+1} a^{m\nu},$$

and so even more if

$$(23)\quad \left\{\begin{aligned} &\left(\frac{4(m+1)}{2^{5\nu/2}\,\pi}\right)^{(m+1)n} \geqslant \frac{4e^{\pi/2}(m+1)^{(\nu/2)-1}(e/4)^{\nu}}{(n+1)^{(m+1)/2}} \cdot \frac{(e/2)^{m+1}(4/e)^{(m+1)\nu}}{(n+1)^{m+1}} \cdot (e^{\pi}\cdot 2^{-(3\nu)/2})^n \times \\ &\qquad \times (m+1)^{(m+1)\nu} (n+1)^{2(m+1)\nu} a^{m\nu}. \end{aligned}\right.$$

Therefore, assuming that (23) holds, by (22)

$$|\pi-\omega|^{-1} < \frac{m}{m+1} e^{\nu} (m+1)^{\nu/2} (m+1)^{(m+1)\nu} e^{-(m+1)\nu}\, 2^{2\nu(m+1)-2\nu+(m+1)-(3n\nu/2)-1} \times$$
$$\times (n+1)^{2(m+1)\nu} (\sqrt{32})^{(m+1)n\nu} a^{m\nu},$$

whence

$$(24)\quad \left\{\begin{aligned} &|\pi-\omega|^{-1} < \left(\frac{e}{4}\right)^{\nu} (m+1)^{\nu/2} \left(\frac{4}{e}\right)^{(m+1)\nu} 2^{m+1}\, 2^{-\frac{3}{2}n\nu-1} \cdot (m+1)^{(m+1)\nu} (n+1)^{2(m+1)\nu} \times \\ &\qquad \times (\sqrt{32})^{(m+1)n\nu} a^{m\nu}. \end{aligned}\right.$$

13. So far $m \geqslant 3$ and n are restricted solely by the condition (23). In order further to simplify (23) and (24), assume from now on that

$$(25)\qquad m+1 \geqslant 20\cdot 2^{\frac{5}{2}(\nu-1)}, \qquad n \geqslant (m+1)\log(m+1).$$

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Since $\frac{3}{2} \log 2 > 1$, by the first of these conditions,

$$m + 1 \geqslant 20\, e^{\nu-1} \geqslant 20\,(1 + (\nu - 1)) = 20\,\nu > 3.$$

The second condition implies then that

$$n \geqslant 20\nu \log (20\nu).$$

Now $20 \log 20 > 59$, $20 \log 40 > 73$, and so

$$n \geqq 60\nu,$$

both when $\nu = 1$ and when $\nu \geqslant 2$.

As a first application of (25), we determine an upper estimate for the expression

$$A_0 = (m + 1)^{\nu/n}\,(n + 1)^{2\nu/n}.$$

Since $n \geqslant 60\nu \geqq 60$,

$$n + 1 \leqslant \frac{61}{60}\,n,\ \left(\frac{61}{60}\right)^{2\nu/n} \leqslant \left(\frac{61}{60}\right)^{1/30},\ A_0 \leqslant \left(\frac{61}{60}\right)^{1/30} (m+1)^{\nu/n}\, n^{2\nu/n},\ = B_0 \text{ say.}$$

Next

$$\frac{\partial \log B_0}{\partial n} = -\frac{\nu}{n^2} \log (m + 1) - \frac{2\nu}{n^2} (\log n - 1)$$

is negative because $\log n \geqslant \log 60 > 1$. Therefore B_0 is not decreased on replacing n by $(m + 1) \log (m + 1)$, and we find that

$$A_0 \leqslant \left(\frac{61}{60}\right)^{1/30} \exp\left\{\frac{\nu \log (m+1) + 2\nu\,(\log (m+1) + \log\log (m+1))}{(m+1) \log (m+1)}\right\}$$

or

$$A_0 \leqslant \left(\frac{61}{60}\right)^{1/30} \exp\left\{\frac{3\,\nu}{m+1} + \frac{2\,\nu}{m+1}\,\frac{\log\log (m+1)}{\log (m+1)}\right\}.$$

Here $\frac{\log\log (m+1)}{\log (m+1)}$ decreases with increasing m because $\log (m + 1) \geqslant$ $\geqslant \log\ 20 > e$; hence

$$\frac{\log\log (m+1)}{\log (m+1)} \leqslant \frac{\log\log 20}{\log 20} < \tfrac{1}{2},$$

whence finally,

$$A_0 \leqslant \left(\frac{61}{60}\right)^{1/30} \exp\left(\frac{3\,\nu+\nu}{20\,\nu}\right) = \left(\frac{61}{60}\right)^{1/30} e^{1/5} < \tfrac{5}{4}.$$

We next discuss certain factors that occur on the right-hand sides of (23) and (24).

In

$$A_1 = \frac{4\,e^{\pi/2}\,(m+1)^{(\nu/2)-1}\,(e/4)^{\nu}}{(n+1)^{(m+1)/2}},$$

evidently

$$\log (m+1) > e,\ \ n+1 > (m+1)\ \log (m+1) > e(m+1),\ \ m+1 \geqslant 20\nu,\ \ (e/4)^{\nu} < 1,$$

whence

$$A_1 < \frac{4\,e^{\pi/2}\,(m+1)^{(\nu/2)-1}\cdot 1}{\{e(m+1)\}^{10\nu}} < 4\,e^{(\pi/2)-10}\,(m + 1)^{-9\nu} < 1.$$

Next let

$$A_2 = \frac{(e/2)^{m+1}\,(4/e)^{(m+1)\nu}}{(n+1)^{m+1}}.$$

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Then by the last inequalities and by (25),

$$A_2 < \left\{\frac{(e/2)^1 (4/e)^\nu}{e(m+1)}\right\}^{m+1} \leqslant \left\{\frac{(e/2)^\nu (4/e)^\nu}{e\cdot 20\cdot 2^{5(\nu-1)/2}}\right\}^{(m+1)} = \left(\frac{2^{5/2}}{20\, e\cdot 2^{3\nu/2}}\right)^{m+1} < 1.$$

Let further

$$A_3 = (e^\pi\cdot 2^{-(3\nu/2)})^{1/(m+1)}.$$

Since $\nu \geqslant 1$ and $m+1 \geqslant 20$,

$$A_3 \leqslant (e^\pi\cdot 2^{-(3/2)})^{1/20} < \tfrac{6}{5}.$$

Consider finally the expression

$$A_4 = \left(\frac{e}{4}\right)^\nu (m+1)^{\nu/2} \left(\frac{4}{e}\right)^{(m+1)\nu} 2^{(m+1)-(3n\nu/2)-1}.$$

Here

$$\nu \geqslant 1,\ \left(\frac{e}{4}\right)^\nu 2^{-1} < 1,\ m+1 < e^{m+1},\ n \geqslant (m+1)\log(m+1),$$

so that

$$A_4 < e^{(m+1)\nu/2}\left(\frac{4}{e}\right)^{(m+1)\nu} 2^{(m+1)\nu-(3/2)(m+1)\nu\log(m+1)} = \left(\frac{8\,e^{-1/2}}{(m+1)^{(3/2)\log 2}}\right)^{(m+1)\nu}.$$

Since now $\frac{3}{2}\log 2 > 1$ and $m+1 \geqslant 20$, we find that

$$A_4 < \left(\frac{2\,e^{-1/2}}{5}\right)^{(m+1)\nu} < 1.$$

14. The inequalities for the A's lead easily to a great simplification of the result in 12.).

The right-hand side of (23) can be written as

$$A_1\, A_2\, A_3^{(m+1)n}\, A_0^{(m+1)n}\, a^{m\nu}$$

and so, by what has just been proved, is less than

$$1\cdot 1\cdot\left(\frac{6}{5}\right)^{(m+1)n}\left(\frac{5}{4}\right)^{(m+1)n} a^{(m+1)\nu} = \left(\frac{3}{2}\right)^{(m+1)n} a^{(m+1)\nu}.$$

Similarly the right-hand side of (24) has the value

$$A_4\, A_0^{(m+1)n}\, 2^{(5/2)(m+1)n\nu}\, a^{m\nu}$$

and is therefore smaller than

$$\left(\frac{5}{4}\cdot 2^{(5/2)\nu}\right)^{(m+1)n} a^{(m+1)\nu}.$$

We have therefore the following result:

"*Let m and n satisfy the inequalities* (25) *and let further*

(26) $$\left(\frac{4(m+1)}{2^{5\nu/2}\pi}\right)^n \geqslant \left(\frac{3}{2}\right)^n a^\nu.$$

Then

(27) $$|\pi-\omega| > \left\{\left(\frac{5}{4}\cdot 2^{(5/2)\nu}\right)^n a^\nu\right\}^{-(m+1)}."$$

The proof assumed that $|\omega| < 4$, but we may now dispense with this condition. For if $|\omega| \geqslant 4$, then trivially,

$$|\pi-\omega| \geqslant 4-\pi > \frac{1}{5} > \left\{\left(\frac{5}{4}\cdot 2^{(5/2)\nu}\right)^n a^\nu\right\}^{-(m+1)}.$$

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15. The first inequality (25) is satisfied if

$$m = [20 \cdot 2^{(5/2)(\nu-1)}], \tag{28}$$

for then

$$20 \times 2^{(5/2)(\nu-1)} < m + 1 \leqslant 20 \cdot 2^{(5/2)(\nu-1)} + 1.$$

This choice of m means that

$$\frac{2}{3} \times \frac{4(m+1)}{2^{5\nu/2}\pi} \geqslant \frac{2}{3} \times \frac{4 \times 20}{2^{5/2}\pi} = \frac{20\sqrt{2}}{3\pi} > e.$$

The condition (26) is therefore certainly fulfilled if

$$e^n \geqslant a^\nu, \text{ i.e., } n \geqslant \nu \log a.$$

Let then from now on n be defined by the formula,

$$n = [\max\,((m+1)\log(m+1), \nu \log a)] + 1, \tag{29}$$

so that both inequalities (25) and (26) hold, hence also the inquality (27) for $|\pi - \omega|$.

It is now convenient to distinguish two cases.

If, firstly,

$$a < (m+1)^{(m+1)/\nu},$$

then

$$(m+1)\log(m+1) > \nu \log a,$$

and therefore, by (29),

$$n = [(m+1)\log(m+1)] + 1 \leqslant (m+1)\log(m+1) + 1.$$

Further

$$\frac{5}{2}\, 2^{(5/2)\nu} = \frac{1}{\sqrt{8}}\, 20 \cdot 2^{(5/2)(\nu-1)} < \frac{m+1}{\sqrt{8}} < \frac{m+1}{e},$$

whence

$$\left(\frac{5}{4}\, 2^{(5/2)\nu}\right)^n a^\nu < \left(\frac{m+1}{e}\right)^{(m+1)\log(m+1)+1} (m+1)^{m+1} = \frac{m+1}{e}\, e^{(m+1)\{\log(m+1)\}^2}.$$

Let, secondly,

$$a \geqslant (m+1)^{(m+1)/\nu},$$

so that

$$(m+1)\log(m+1) \leqslant \nu \log a.$$

Now

$$n = [\nu \log a] + 1 \leqslant \nu \log a + 1,$$

hence

$$\left(\frac{5}{4}\, 2^{(5/2)\nu}\right)^n a^\nu < \left(\frac{m+1}{e}\right)^{\nu \log a + 1} a^\nu = \frac{m+1}{e}\, a^{\nu \log(m+1)}.$$

The following result has therefore been obtained:

Theorem 2: *Let ω be a real or complex algebraic number. Denote by R the rational field K if ω is real, and the Gaussian imaginary field $K(i)$ if ω is non-real. Further denote by ν the degree of ω over R, by*

$$a_0 z^\nu + a_1 z^{\nu-1} + \ldots + a_\nu = 0 \qquad (a_0 \neq 0)$$

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an equation for ω with integral coefficients in R which is irreducible over this field, and by

$$a = \max(|a_0|,\ |a_1|, \ldots, |a_\nu|)$$

the height of this equation. Put

$$m = [20 \cdot 2^{(5/2)(\nu-1)}], \qquad \bar{a} = \max(a, (m+1)^{(m+1)/\nu}).$$

Then

$$|\pi - \omega| > \left(\frac{m+1}{e}\right)^{-(m+1)} \bar{a}^{-(m+1)\nu \log(m+1)}. \tag{30}$$

Remarks: 1) We note that the theorem remains true if $\bar{a}$ is replaced by any larger number.

2) When

$$a < (m+1)^{(m+1)/\nu},$$

the estimate (30) is not as good as that by N. I. FEL'DMAN (Izvestiya Akad. Nauk SSSR, ser. mat. 15, 1951, 53–74), viz.

$$|\pi - \omega| > \exp\{-\gamma_1 \nu (1 + \nu \log \nu + \log a) \log(2 + \nu \log \nu + \log a)\},$$

where γ_1, just as γ_2 in the next line, is a positive absolute constant. Fel'dman's inequality implies that

$$\pi^n - [\pi^n] > \exp\{-\gamma_2 n^2 (\log n)^2\}$$

for all sufficiently large positive integers n, while my result yields a much less good lower estimate.

If, however,

$$a \geqslant (m+1)^{(m+1)/\nu},$$

then Theorem 2 is much stronger, and it furthermore gives a lower bound for $|\pi - \omega|$ free of unknown constants. The exponent of $1/a$,

$$(m+1)\,\nu \log(m+1),$$

is not greater than

$$(20 \cdot 2^{(5/2)(\nu-1)} + 1)\,\nu\, \log\,(20 \cdot 2^{(5/2)(\nu-1)} + 1)$$

and therefore, for large n, is of the order

$$O(2^{(5/2)\nu} \nu^2)$$

16. As an application of Theorem 2, let us determine a lower bound for $|\sin u\alpha|$ when α is a fixed positive algebraic number and u is a positive integral variable such that $u \geqslant \pi/\alpha$.

Define a second positive integer v by

$$-\frac{\pi}{2} < u\alpha - v\pi \leqslant \frac{\pi}{2}.$$

Then

$$\frac{\alpha}{2\pi} u \leqslant \frac{\alpha}{\pi} u - \tfrac{1}{2} \leqslant v < \frac{\alpha}{\pi} u + \tfrac{1}{2} < \frac{2\alpha}{\pi} u,$$

and therefore

$$\max(u, v) \leqslant \max\left(u, \frac{2\alpha}{\pi} u\right) < \left(\frac{2\alpha}{\pi} + 1\right) u.$$

Let, say, α have the degree ν over the rational field, and let it satisfy the irreducible equation

$$A_0 z^\nu + A_1 z^{\nu-1} + \ldots + A_\nu = 0 \qquad (A_0 \neq 0)$$

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with rational integral coefficients of height

$$A = \max(|A_0|, |A_1|, \ldots, |A_\nu|) \geqslant 1.$$

Then the rational multiple of α,

$$\omega = \frac{u}{v}\alpha,$$

is a root of the equation

$$A_0 v^\nu z^\nu + A_1 u v^{\nu-1} z^{\nu-1} + \ldots + A_\nu u^\nu = 0$$

of height

$$a = \max(|A_0 v^\nu|, |A_1 u v^{\nu-1}|, \ldots, |A_\nu u^\nu|) \leqslant A (\max(u, v))^\nu < \left(\frac{2\alpha}{\pi} + 1\right)^\nu A u^\nu.$$

Let again

$$m = [20 \cdot 2^{(5/2)(\nu-1)}], \quad \bar{a} = \max(a, (m+1)^{(m+1)/\nu}),$$

so that

$$\bar{a} \leqslant \max\left(\left(\frac{2\alpha}{\pi} + 1\right)^\nu A u^\nu, (m+1)^{(m+1)/\nu}\right), = a^* \text{ say},$$

whence, by Theorem 2,

$$|\pi - \omega| > \left(\frac{m+1}{e}\right)^{-(m+1)} a^{*-(m+1)\nu \log(m+1)}.$$

On the other hand,

$$|\sin t| \geqslant \frac{2}{\pi}|t| \quad \text{if } |t| \leqslant \frac{\pi}{2},$$

hence

$$|\sin u\alpha| = |\sin(u\alpha - v\pi)| \geqslant \frac{2}{\pi} v |\pi - \omega|,$$

and we find, finally, that

$$|\sin u\alpha| > \frac{\alpha}{\pi^2} u \left(\frac{m+1}{e}\right)^{-(m+1)} a^{*-(m+1)\nu \log(m+1)}.$$

In the special case when $\alpha = 1$, Theorem 1 gives a stronger result, viz.

$$|\sin u| > \frac{1}{\pi^2} u^{-41}.$$

This inequality has been proved for $u \geqslant \pi$, i.e. for $u \geqslant 4$, but it is easily verified that it holds also for $1 \leqslant u \leqslant 3$.

By way of example, the power series

$$\sum_{u=1}^{\infty} \frac{z^u}{\sin u\alpha}$$

has the radius of convergence 1, and the Dirichlet series

$$\sum_{u=1}^{\infty} \frac{u^{-s}}{\sin u\alpha}$$

converges when the real part of s is greater than $(m+1)\nu \log(m+1)$.

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