On the fractional parts of the powers of a rational number. II

Kurt Mahler

Summary. Let $\|x\|$ denote the distance of the real number x to the nearest integer. In this paper, Mahler proves that, if u and v are coprime integers satisfying $u > v \geqslant 2$ and $\varepsilon > 0$ is an arbitrarily small positive number, the inequality

$$\left\| \left(\frac{u}{v}\right)^n \right\| < e^{\varepsilon n}$$

is satisfied by at most a finite number of positive integer solutions n. He uses this result to show that, except for a finite number of values k,

$$g(k) = 2^k - \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor - 2,$$

where $g(k)$ is the function in Waring's problem.

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ON THE FRACTIONAL PARTS OF THE POWERS OF A RATIONAL NUMBER (II)

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1. About twenty years ago, in a note of the same title [2], I obtained the following result.

THEOREM 1. *Let u and v be relatively prime integers satisfying $u > v \geqslant 2$ and let ϵ be an arbitrarily small positive number. Suppose the inequality*

$$\left| \left(\frac{u}{v}\right)^n - (\text{nearest integer}) \right| < e^{-\epsilon n} \tag{1}$$

is satisfied by an infinite sequence of positive integers $n_1, n_2, \ldots$. Then

$$\limsup_{r \to \infty} \frac{n_{r+1}}{n_r} = \infty.$$

The proof of this theorem was based on a method of Th. Schneider [6] as extended by myself [3]; see also a recent paper of Schneider [7].

It may be of interest to note that the new method of K. F. Roth [5] for studying the rational approximations to algebraic numbers enables one to replace Theorem 1 by the following much stronger result.

THEOREM 2. *Let u, v and ϵ be as in Theorem* 1. *Then the inequality* (1) *is satisfied by at most a finite number of positive integers n.*

This result has a curious application in connection with the value of the number $g(k)$ in Waring's Problem. This number is now known for $k \geqslant 6$, as a result of the work of several mathematicians (see Hardy and Wright [1], 337), but the formula for $g(k)$ depends on whether B is less than or greater than $2^k - A$, where

$$A = \left[\left(\frac{3}{2}\right)^k\right], \quad B = 3^k - 2^k A.$$

In the former case, we have $g(k) = 2^k + A - 2$, in the latter case there is a different result. It follows from Theorem 2 that the latter case can occur for at most a finite number of values of k; for if $B > 2^k - A$ we have

$$0 < (A+1) - \left(\frac{3}{2}\right)^k < \frac{A}{2^k} < \left(\frac{3}{4}\right)^k,$$

and thus (1) holds with $u = 3$, $v = 2$, $\epsilon = \log \frac{4}{3}$, $n = k$.

It follows that, *except possibly for a finite number of values of k, we have*

$$g(k) = 2^k + \left[\left(\frac{3}{2}\right)^k\right] - 2.$$

[MATHEMATIKA 4 (1957), 122–124]

2. Roth's theorem states that if ϑ is an irrational algebraic number, and if $\gamma > 2$, there are at most finitely many rational numbers p/q $(q > 0)$ satisfying the inequality

$$\left|\vartheta - \frac{p}{q}\right| < \frac{1}{q^\gamma}.$$

The proof actually remains valid if ϑ is rational, provided only rational numbers p/q distinct from ϑ are considered, though of course the result is then trivial.

The method of my paper [3], by which I formerly generalized Schneider's result, can be used to prove an analogous extension of Roth's result, and this has been carried through by Ridout [4]. He proves:

THEOREM 3. *Let ϑ be any algebraic number other than* 0; *let $P_1, \ldots, P_s, Q_1, \ldots, Q_t$ be finite sets of distinct primes; and let α, β, γ, c be real numbers satisfying*

$$0 \leqslant \alpha \leqslant 1, \quad 0 \leqslant \beta \leqslant 1, \quad \gamma > \alpha + \beta, \quad c > 0. \tag{2}$$

Let p, q be restricted to be integers of the form

$$p = p^* P_1^{h_1} \ldots P_s^{h_s}, \quad q = q^* Q_1^{k_1} \ldots Q_t^{k_t},$$

where $h_1, \ldots, h_s, k_1, \ldots, k_t$ are non-negative integers and p^*, q^* are integers satisfying

$$0 < |p^*| \leqslant c p^\alpha, \quad 0 < q^* \leqslant c q^\beta. \tag{3}$$

There exists a positive number C depending on $\vartheta, \alpha, \beta, \gamma, c$ and the primes $P_1, \ldots, Q_1, \ldots,$ such that, for all p and q of the above form, we have

$$\left|\vartheta - \frac{p}{q}\right| > \frac{C}{q^\gamma} \quad \text{provided} \quad \vartheta - \frac{p}{q} \neq 0. \tag{4}$$

3. We can now easily deduce Theorem 2 from Theorem 3, and even obtain a slightly more general result.

Let ϑ be any positive algebraic number, and let u, v, ϵ be as in Theorem 1. Put

$$\lambda = \frac{\log v}{\log u},$$

so that $v = u^\lambda$ and $0 < \lambda < 1$. Let $P_1, \ldots, P_s$ be the distinct prime factors of v and $Q_1, \ldots, Q_t$ those of u. Take

$$\alpha = 1 - \lambda, \quad \beta = 0, \quad c = (2\vartheta)^\lambda + 1,$$

$$\gamma = 1 - \lambda + \tfrac{1}{2}\epsilon(\log u)^{-1} > \alpha + \beta.$$

Apply Theorem 3 with

$$p = p^* v^n, \quad q = u^n \quad (q^* = 1),$$

where p^* denotes the integer nearest to $\vartheta(u/v)^n$. This is permissible because v^n is a product of powers of $P_1, \ldots, P_s$ and u^n is a product of powers of $Q_1, \ldots, Q_t$. If n is sufficiently large, we have

$$0 < p^* < 2\vartheta(u/v)^n = 2\vartheta\, v^{n(1-\lambda)/\lambda},$$

whence

$$0 < p^* < cp^{1-\lambda},$$

so that (3) is satisfied. Further, $\vartheta(u/v)^n$ obviously cannot be an integer if n is sufficiently large. Hence (4) implies that

$$|\vartheta(u/v)^n - p^*| > (u/v)^n\, Cu^{-\gamma n} = C \exp\left(-\tfrac{1}{2}\epsilon n\right).$$

Thus *for all but a finite number of values of n we have*

$$|\vartheta(u/v)^n - p^*| > e^{-\epsilon n}. \tag{5}$$

Theorem 2 is the case $\vartheta = 1$.

The conclusion would no longer hold if u/v were replaced by a suitable algebraic number, *e.g.* by $\frac{1}{2}(1+\sqrt{5})$, and ϑ were again taken to be 1. It would be of some interest to know which algebraic numbers have the same property as u/v in Theorem 2.

References.

1. G. H. Hardy and E. M. Wright, *Introduction to the Theory of Numbers* (3rd ed., Oxford, 1954).
2. K. Mahler, *Acta Arithmetica*, 3 (1938), 89–93.
3. ———, *Proc. K. Akad. Wet. Amsterdam*, 39 (1936), 633–640 and 729–737.
4. D. Ridout, *Mathematika*, 4 (1957), 125–131.
5. K. F. Roth, *Mathematika*, 2 (1955), 1–20.
6. Th. Schneider, *J. für die reine und angew. Math.*, 175 (1936), 182–192.
7. ———, *J. für die reine und angew. Math.*, 188 (1950), 115–128.

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