An application of Jensen's formula to polynomials

Kurt Mahler

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AN APPLICATION OF JENSEN'S FORMULA TO POLYNOMIALS

K. MAHLER

In this note new proofs will be given for two inequalities on polynomials due to N. I. Feldman [1] and A. O. Gelfond [2], respectively; these inequalities are of importance in the theory of transcendental numbers. While the original proofs by the two authors are quite unconnected, we shall deduce both results from the same source, viz. from Jensen's integral formula in the theory of analytic functions.

1. Let $F(x)$ be a function of the complex variable x, which is regular inside and on the circle $|x| = \rho$; let $F(0) \neq 0$, and let $\xi_1, \xi_2, \ldots, \xi_N$ be all the zeros of $F(x)$ satisfying $|\xi_\nu| \leqslant \rho$. Here multiple zeros are counted as many times as their multiplicity.

Jensen's integral formula states now that

$$\frac{1}{2\pi}\int_0^{2\pi} \log|F(\rho e^{\theta i})|\,d\theta = \log|F(0)| + \sum_{\nu=1}^{N} \log\frac{\rho}{|\xi_\nu|}. \tag{1}$$

2. Denote by

$$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$$

a polynomial with arbitrary real or complex coefficients. We shall assume that $f(x)$ has the exact degree $d(f) = n$, and does not vanish for $x = 0$, so that

$$a_0 \neq 0, \quad a_n \neq 0.$$

Let $\xi_1, \xi_2, \ldots, \xi_n$ be all the zeros of $f(x)$; these are then all distinct from 0. There is no restriction in assuming that the zeros have been numbered so that

$$|\xi_1| \leqslant |\xi_2| \leqslant \ldots \leqslant |\xi_N| \leqslant 1 < |\xi_{N+1}| \leqslant |\xi_{N+2}| \leqslant \ldots \leqslant |\xi_n|.$$

Put $\rho = 1$ in Jensen's formula (1) and identify $F(x)$ with $f(x)$; it follows then that

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{\theta i})|\,d\theta = \log|a_n| + \sum_{\nu=1}^{N} \log\frac{1}{|\xi_\nu|} = \log\left|\frac{a_n}{\xi_1 \xi_2 \ldots \xi_N}\right|.$$

Since

$$a_0 \xi_1 \xi_2 \ldots \xi_n = \mp a_n,$$

this formula may also be written as

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{\theta i})|\,d\theta = \log|a_0 \xi_{N+1} \xi_{N+2} \ldots \xi_n|. \tag{2}$$

3. Next let $i_1, i_2, \ldots, i_m$ be an arbitrary set of not more than n distinct suffixes $1, 2, \ldots, n$; neither of the cases when $m = 0$ and $m = n$ is excluded.

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From the numbering of the zeros, it is obvious that

$$|a_0 \xi_{i_1} \xi_{i_2} \cdots \xi_{i_m}| \leqslant |a_0 \xi_{N+1} \xi_{N+2} \cdots \xi_n|.$$

Hence the identity (2) implies the inequality

$$\log|a_0 \xi_{i_1} \xi_{i_2} \cdots \xi_{i_m}| \leqslant \frac{1}{2\pi} \int_0^{2\pi} \log|f(e^{\theta i})|\, d\theta. \tag{3}$$

Apart from a factor ∓ 1, each coefficient a_m of $f(x)$ is a sum of $\binom{n}{m}$ terms of the same form

$$a_0 \xi_{i_1} \xi_{i_2} \cdots \xi_{i_m}$$

as occur in (3). In particular,

$$|a_0| + |a_1| + \dots + |a_n| \leqslant \Sigma |a_0 \xi_{i_1} \xi_{i_2} \cdots \xi_{i_m}|,$$

where the summation extends over all possible sets of suffixes $i_1, i_2, \ldots, i_m$. Since m can have the values $0, 1, \ldots, n$, the sum has

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$$

terms. Moreover, each term satisfies the inequality (3). We find then that

$$\log(|a_0| + |a_1| + \dots + |a_n|) \leqslant d(f) \log 2 + \frac{1}{2\pi} \int_0^{2\pi} \log|f(e^{\theta i})|\, d\theta. \tag{4}$$

There is a similar upper estimate for the integral. For it is obvious that

$$|f(e^{\theta i})| \leqslant |a_0| + |a_1| + \dots + |a_n| \quad \text{for } 0 \leqslant \theta \leqslant 2\pi,$$

and so also

$$\frac{1}{2\pi} \int_0^{2\pi} \log|f(e^{\theta i})|\, d\theta \leqslant \log(|a_0| + |a_1| + \dots + |a_n|). \tag{5}$$

4. The two inequalities of Feldman and Gelfond follow now immediately.

First, by combining (3) and (5), we find that

$$|a_0 \xi_{i_1} \xi_{i_2} \cdots \xi_{i_m}| \leqslant |a_0| + |a_1| + \dots + |a_n|. \tag{6}$$

This is Feldman's inequality, although in a rather strengthened form.

Secondly, let $f(x)$ be written as a product of polynomials

$$f(x) = \prod_{\sigma=1}^{s} f_\sigma(x).$$

We use the abbreviation

$$S(f) = |a_0| + |a_1| + \dots + |a_n|.$$

The formula (4) implies therefore the s relations

$$\log S(f_\sigma) \leqslant d(f_\sigma) \log 2 + \frac{1}{2\pi} \int_0^{2\pi} \log|f_\sigma(e^{\theta i})|\, d\theta \quad (\sigma = 1, 2, \ldots, s),$$

where

$$\sum_{\sigma=1}^{s} d(f_\sigma) = d(f).$$

Hence, on adding these inequalities,

$$\log \prod_{\sigma=1}^{s} S(f_\sigma) \leqslant d(f) \log 2 + \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{\theta i})| \, d\theta,$$

and it follows from (5) that

$$\prod_{\sigma=1}^{s} S(f_\sigma) \leqslant 2^{d(f)} S(f). \tag{7}$$

This is essentially Gelfond's inequality*.

References.

1. N. I. Feldman, "Approksimatsiya nekotorych transtsendentnych tchisel, I," *Izvestiya Akad. Nauk SSSR.*, ser. mat. 15 (1951), 53–74; Lemma 2, p. 54.

2. A. O. Gelfond, *Transtsendentnye i algebraitcheskie tchisla* (*Moskva* 1952), pp. 22–24 (Lemma 4) and pp. 168–173 (Lemma II).

Mathematics Department,
Manchester University.

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* The inequalities (6) and (7), although proved only for polynomials that do not vanish for $x=0$, evidently remain valid when this restriction is removed.