An inequality for the discriminant of a polynomial

Kurt Mahler

Summary. In this paper, Mahler provides inequalities relating length (the sum of the absolute values of the coefficients), Mahler measure, discriminant and minimal distance between zeros of a polynomial. In particular, he gives an upper bound on the absolute value of the discriminant and a lower bound for the minimal distance between zeros.

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AN INEQUALITY FOR THE DISCRIMINANT OF A POLYNOMIAL

K. Mahler

Let

$$f(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_m = a_0 \prod_{h=1}^{m} (x - \alpha_h) \qquad (m \geq 2)$$

be an arbitrary polynomial with real or complex coefficients; put

$$L(f) = |a_0| + |a_1| + \cdots + |a_m|, \qquad M(f) = |a_0| \prod_{h=1}^{m} \max(1, |\alpha_h|).$$

Then, as I proved in [2],

$$2^{-m} L(f) \leq M(f) \leq L(f). \tag{1}$$

Here I shall establish and apply an upper estimate for the discriminant D(f) of f(x) in terms of either L(f) or M(f). This estimate is best-possible, and slightly better than one by R. Güting [1].

1. The main tool in the proof of the inequality is Hadamard's theorem on determinants, which may be stated as follows.

LEMMA 1. *If the elements of the determinant*

$$d = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

are arbitrary complex numbers, then

$$|d|^2 \leq \prod_{j=1}^{n} \left(\sum_{h=1}^{n} |a_{hj}|^2 \right),$$

and equality holds if and only if

$$\sum_{h=1}^{n} a_{hj} \bar{a}_{hk} = 0 \quad \textit{for } 1 \leq j < k \leq n.$$

Here $\bar{a}_{hk}$ denotes the complex conjugate of a_{hk}.

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2. Let $\alpha_1, \cdots, \alpha_m$, the zeros of f, be numbered so that

$$|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_M| > 1 \geq |\alpha_{M+1}| \geq |\alpha_{M+2}| \geq \cdots \geq |\alpha_m|. \tag{2}$$

Here M may have any one of the values 0, 1, $\cdots$, m. Further, put

$$P = \prod_{1 \leq h < k \leq m} (\alpha_h - \alpha_k),$$

with the convention that

$$P = 1 \text{ in the excluded case where } m = 1.$$

Written as a Vandermonde determinant,

$$P = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_m \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_m^2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \alpha_1^{m-1} & \alpha_2^{m-1} & \cdots & \alpha_m^{m-1} \end{vmatrix}.$$

We denote by r and s any two suffices satisfying the conditions

$$1 \leq r < s \leq m, \quad \alpha_r \neq \alpha_s,$$

and we use the notation

$$Q = (\alpha_1 \alpha_2 \cdots \alpha_M)^{-(m-1)} P.$$

Thus, in particular, $Q = P$ if $M = 0$.

By its definition, Q may be written as the determinant

$$Q = \begin{vmatrix} \alpha_1^{-(m-1)} & \cdots & \alpha_M^{-(m-1)} & 1 & \cdots & 1 \\ \alpha_1^{-(m-2)} & \cdots & \alpha_M^{-(m-2)} & \alpha_{M+1} & \cdots & \alpha_m \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \alpha_1^{-1} & \cdots & \alpha_M^{-1} & \alpha_{M+1}^{m-2} & \cdots & \alpha_m^{m-2} \\ 1 & \cdots & 1 & \alpha_{M+1}^{m-1} & \cdots & \alpha_m^{m-1} \end{vmatrix}.$$

Since the absolute value of no element of this new determinant exceeds 1, it follows from Lemma 1 that

$$|Q| \leq m^{m/2}. \tag{3}$$

Here equality can only hold if both

$$|\alpha_1| = |\alpha_2| = \cdots = |\alpha_m| = 1$$

and

$$\sum_{k=0}^{m-1} \alpha_h^k \bar{\alpha}_j^k = 0 \quad \text{for } 1 \leq h < j \leq m.$$

It follows then that the m quotients

$$\frac{\alpha_1}{\alpha_1} = 1, \frac{\alpha_2}{\alpha_1}, \cdots, \frac{\alpha_m}{\alpha_1}$$

are equal to the m distinct mth roots of unity, and f is of the form

$$f(x) = a_0 x^m + a_m, \quad \text{where } |a_0| = |a_m| > 0.$$

3. An upper bound for $|Q/(\alpha_r - \alpha_s)|$ is obtained by a method very similar to that just applied to $|Q|$.

In the Vandermonde determinant for P, subtract the sth column from the rth column, so that the new rth column consists of the elements

$$0, \alpha_r - \alpha_s, \alpha_r^2 - \alpha_s^2, \cdots, \alpha_r^{m-2} - \alpha_s^{m-2}, \alpha_r^{m-1} - \alpha_s^{m-1},$$

all of which are multiples of $\alpha_r - \alpha_s$. For brevity, write

$$q_0 = 0, \qquad q_h = \frac{\alpha_r^h - \alpha_s^h}{\alpha_r - \alpha_s} = \alpha_r^{h-1} + \alpha_r^{h-2}\alpha_s + \cdots + \alpha_r \alpha_s^{h-2} + \alpha_s^{h-1} \quad \text{for } h \geq 1.$$

The quotient $P/(\alpha_r - \alpha_s)$ can now be written as a determinant in which the rth column consists of the elements

$$q_0, q_1, \cdots, q_{m-2}, q_{m-1},$$

while the other $m - 1$ columns are the same as in the original determinant for P. On dividing the 1st, 2nd, $\cdots$, Mth column of the new determinant again by the factors

$$\alpha_1^{m-1}, \alpha_2^{m-1}, \cdots, \alpha_M^{m-1},$$

respectively, we obtain a determinant with the value $Q/(\alpha_r - \alpha_s)$. Except for its rth column, this determinant is identical with that for Q; but its rth column consists of the elements

$$q_0 \alpha_r^{-(m-1)}, q_1 \alpha_r^{-(m-1)}, \cdots, q_{m-2} \alpha_r^{-(m-1)}, q_{m-1} \alpha_r^{-(m-1)} \qquad \text{if } r \leq M,$$

and of the elements

$$q_0, q_1, \cdots, q_{m-2}, q_{m-1} \qquad \text{if } r > M.$$

Since

$$|\alpha_r| \geq |\alpha_s| \quad \text{and} \quad |\alpha_r| \begin{cases} > 1 & \text{for } r \leq M, \\ \leq 1 & \text{for } r > M, \end{cases}$$

the absolute values of the consecutive elements of the rth column of the determinant do not exceed the values

$$0, 1, \cdots, m-2, m-1,$$

respectively. Therefore, by Lemma 1,

$$|Q/(\alpha_r - \alpha_s)|^2 \leq m^{m-1}\{0^2 + 1^2 + \cdots + (m-2)^2 + (m-1)^2\}.$$

Since

$$0^2 + 1^2 + \cdots + (m-2)^2 + (m-1)^2 = \frac{m(m-1)(2m-1)}{6} < \frac{m^3}{3},$$

the final result takes the form

$$\left|\frac{Q}{\alpha_r - \alpha_s}\right| < \frac{1}{\sqrt{3}}\, m^{(m+2)/2}. \tag{4}$$

This inequality is nearly best-possible. For choose for $\alpha_1, \cdots, \alpha_m$ all the distinct mth roots of unity. The minimum of $|\alpha_r - \alpha_s|$ is then attained, for example, if

$$\alpha_r = 1 \quad \text{and} \quad \alpha_s = e^{2\pi i/m},$$

and so it has the value

$$|\alpha_r - \alpha_s| = 2\sin\frac{\pi}{m}.$$

In this special case we further have

$$|P| = |Q| = m^{m/2}.$$

It follows then that

$$\left|\frac{Q}{\alpha_r - \alpha_s}\right| = \frac{m^{m/2}}{2\sin\frac{\pi}{m}} \sim \frac{m^{(m+2)/2}}{2\pi} \quad \text{as } m \to \infty.$$

This shows that the inequality (4) cannot be improved except perhaps that the constant factor $1/\sqrt{3}$ may be replaced by a smaller number. It would be of some interest to determine the least possible constant factor.

4. The discriminant $D(f)$ of f is defined by the formula

$$D(f) = a_0^{2m-2} P^2.$$

On the other hand, by (2),

$$M(f) = |a_0 \alpha_1 \alpha_2 \cdots \alpha_M|,$$

so that evidently

$$|D(f)|\,M(f)^{-(2m-2)} = |Q|^2. \tag{5}$$

Hence, from (3) and its corollary we immediately obtain the following result.

THEOREM 1. *For all polynomials* f *of degree* $m \geq 2$,

$$|D(f)| \leq m^m\, M(f)^{2m-2},$$

with equality if and only if f *has the form*

$$f(x) = a_0 x^m + a_m, \qquad \textit{where } |a_0| = |a_m| > 0.$$

COROLLARY. *The inequality* (1) *therefore implies that*

$$|D(f)| < m^m\, L(f)^{2m-2},$$

because $L(f) = 2M(f)$ *for the extremal polynomial.*

5. Next, denote by

$$\Delta(f) = \min_{1 \leq h < j \leq m} |\alpha_h - \alpha_j|$$

the shortest distance between any two zeros of f. We assume that

$$D(f) \neq 0,$$

so that also

$$\Delta(f) > 0.$$

Choose for r and s a pair of suffices such that

$$\Delta(f) = |\alpha_r - \alpha_s|, \qquad 1 \leq r < s \leq m.$$

On combining the inequality (4) with the identity (5) and applying Theorem 1, we obtain the following result.

THEOREM 2. *For all polynomials* f *of degree* $m \geq 2$,

$$\Delta(f) > \sqrt{3}\, m^{-(m+2)/2}\, |D(f)|^{1/2}\, M(f)^{-(m-1)}.$$

COROLLARY. *If follows therefore from* (1) *that*

$$\Delta(f) > \sqrt{3}\, m^{-(m+2)/2}\, |D(f)|^{1/2}\, L(f)^{-(m-1)}.$$

This is slightly better than the corresponding formula by Güting.

Assume in particular that f has rational integral coefficients and that therefore, since $D(f) \neq 0$,

$$|D(f)| \geq 1.$$

It follows then at once that

$$\Delta(f) > \sqrt{3}\, m^{-(m+2)/2} L(f)^{-(m-1)},$$

hence that every nonreal zero of f has an imaginary part of absolute value greater than

$$\sqrt{3/4}\, m^{-(m+2)/2}\, L(f)^{-(m-1)}.$$

For another application, put

$$g_r(x) = \frac{f(x)}{x - \alpha_r} \qquad (1 \leq r \leq m),$$

so that $f'(\alpha_r) = g_r(\alpha_r)$. Then

$$D(f) = D(g_r)\, f'(\alpha_r)^2, \qquad M(f) = M(g_r) \max(1, |\alpha_r|).$$

Hence, by Theorem 1,

$$|D(g_r)| \leq (m-1)^{m-1}\, M(f)^{2m-4} \max(1, |\alpha_r|)^{-(2m-4)}.$$

It follows then easily that

$$|f'(\alpha_r)| \geq (m-1)^{-(m-1)/2}\, |D(f)|^{1/2}\, M(f)^{-(m-2)} \max(1, |\alpha_r|)^{m-2},$$

hence also that

$$|f'(\alpha_r)| \geq (m-1)^{-(m-1)/2}\, |D(f)|^{1/2}\, L(f)^{-(m-2)}.$$

REFERENCES

1. R. Güting, *Approximation of algebraic numbers by algebraic numbers*, Michigan Math. J. 8 (1961), 149-159.

2. K. Mahler, *An application of Jensen's formula to polynomials*, Mathematika 7 (1960), 98-100.

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