Motivic Equivalence and Similarity of Quadratic Forms

To Sasha Merkurjev on the occasion of his 60th birthday

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Abstract. A result by Vishik states that given two anisotropic quadratic forms of the same dimension over a field of characteristic not 2, the Chow motives of the two associated projective quadrics are isomorphic iff both forms have the same Witt indices over all field extensions, in which case the two forms are called motivically equivalent. Izhboldin has shown that if the dimension is odd, then motivic equivalence implies similarity of the forms. This also holds for even dimension ≤ 6, but Izhboldin also showed that this generally fails in all even dimensions ≥ 8 except possibly in dimension 12. The aim of this paper is to show that motivic equivalence does imply similarity for fields over which quadratic forms can be classified by their classical invariants provided that in the case of formally real such fields the space of orderings has some nice properties. Examples show that some of the required properties for the field cannot be weakened.

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1. Introduction

Throughout this note, we will consider only fields of characteristic not 2. By a form over F we will mean a finite dimensional nondegenerate quadratic form over F, and by a quadric over F a smooth projective quadric $X_\varphi = \{\varphi = 0\}$ for some form φ over F.

An important theme in the theory of quadratic forms is the study of forms in terms of geometric properties of their associated quadrics. Suppose, for example, that for two given forms φ and ψ over F one has that the motives $M(X_\varphi)$ and $M(X_\psi)$ are isomorphic in the category of Chow motives, in which case we call φ and ψ *motivically equivalent* and we write $\varphi \overset{\text{mot}}{\sim} \psi$. Does this already imply that the quadrics are isomorphic as projective varieties? The converse is of course trivially true. It is well known that the quadrics X_φ and X_ψ are isomorphic iff φ and ψ are similar (see, e.g. [18, Th. 2.2]), i.e. there exists $c \in F^\times = F\setminus\{0\}$ with $\varphi \cong c\psi$ in which case we write $\varphi \overset{\text{sim}}{\sim} \psi$. The above question then reads as follows: Let φ and ψ be forms of the same dimension over F. Does $\varphi \overset{\text{mot}}{\sim} \psi$ imply $\varphi \overset{\text{sim}}{\sim} \psi$?

In fact, Izhboldin has shown that the answer is yes if $\dim\varphi$ is odd ([14, Cor. 2.9]) or even and at most 6 ([14, Prop. 3.1]), and that there are counterexamples in every even dimension ≥ 8 except possibly 12 over suitably chosen fields ([15, Th. 0.1]). To our knowledge, it seems to be still open if such counterexamples exist in dimension 12.

The purpose of the present note is to give criteria for fields that guarantee that motivic equivalence implies similarity in all dimensions. We show that it holds for fields over which forms of a given dimension can be classified by their classical invariants determinant, Clifford invariant and signatures provided that in the case of formally real fields the space of orderings satisfies a certain property called *effective diagonalization* ED (which will be defined below). We show furthermore that there are counterexamples once the condition ED is only slightly weakened.

Rather than working with motives of quadrics, we will use an alternative criterion for motivic equivalence due to Vishik [24, Th. 1.4.1] (see also Vishik [25, Th. 4.18] or Karpenko [16, § 5]). If we denote the Witt index of a form φ by $i_W(\varphi)$, this important criterion reads as follows.

Vishik's Criterion 1.1. *Let φ and ψ be forms over F with $\dim\varphi = \dim\psi$. Then $\varphi \overset{\text{mot}}{\sim} \psi$ if and only if $i_W(\varphi_E) = i_W(\psi_E)$ for every field extension E/F.*

Let us remark that while Vishik formulated his criterion in terms of integral Chow motives, it still holds for Chow motives with $\mathbb{Z}/2\mathbb{Z}$ coefficients, see [8].

The proofs of our results will concern mainly formally real fields (in the sequel we will call such fields *real* for short). For nonreal fields, the results are still valid but can often be shown in a much quicker and simpler fashion. The real case will involve various arguments concerning the space of orderings X_F of a real field and the signatures $\text{sgn}_P(\varphi)$ of a form φ over F with respect to an ordering $P \in X_F$.

Consider the Witt ring WF and the torsion ideal W_tF (we have $WF = W_tF$ iff F in nonreal). By Pfister's local-global principle (see, e.g., [20, Ch. VIII, Th. 3.2]), a form φ is torsion iff $\operatorname{sgn}_P(\varphi) = 0$ for all $P \in X_F$. We call a form totally indefinite if $|\operatorname{sgn}_P(\varphi)| < \dim \varphi$ for all $P \in X_F$. Also, we will use the fact that the Witt ring only contains 2-primary torsion.

Let IF be the fundamental ideal in WF generated by even-dimensional forms in F and let $I^nF = (IF)^n$. We define $I_t^nF = I^nF \cap W_tF$. A real field F is said to satisfy *effective diagonalization* (ED) if any form φ over F has a diagonalization $\langle a_1, \ldots, a_n\rangle$ such that for all $1 \leq i < n$ and for all $P \in X_F$ one has $a_i <_P 0 \Longrightarrow a_{i+1} <_P 0$ (see [26] or [23]). Recall that the u-invariant and the Hasse number $\tilde{u}$ are defined as follows:

$$\begin{aligned} u(F) &= \sup\{\dim \varphi \mid \varphi \text{ is anisotropic and } \varphi \in W_tF\} \\ \tilde{u}(F) &= \sup\{\dim \varphi \mid \varphi \text{ is anisotropic and totally indefinite}\} \end{aligned}$$

For nonreal F, we thus have $u(F) = \tilde{u}(F)$. It is also well known that these invariants cannot take the values $3, 5, 7$ (see [5, Ths. F–G] for the more involved case $\tilde{u}$ for real fields).

Our main result reads as follows.

Main Theorem 1.2. *Let F be an ED-field and let φ, ψ be anisotropic forms over F of the same dimension. If $\varphi \overset{\text{mot}}{\sim} \psi$ then there exists $x \in F^\times$ such that $\varphi \perp -x\psi \in I_t^3F$.*

Corollary 1.3. *Let F be an ED-field with $I_t^3F = 0$ and let φ, ψ be anisotropic forms over F of the same dimension. Then $\varphi \overset{\text{mot}}{\sim} \psi$ if and only if $\varphi \overset{\text{sim}}{\sim} \psi$.*

Recall that fields with $I_t^3F = 0$ are exactly those fields over which quadratic forms can be classified by their classical invariants dimension, (signed) determinant, Clifford invariant and signatures, see [4].

Now fields with finite $\tilde{u}$ are always ED (see, e.g., [7, Th. 2.5]). By the Arason-Pfister Hauptsatz (see, e.g., [20, Ch. X, 5.1]) we thus get

Corollary 1.4. *Let F be a field with $\tilde{u}(F) \leq 6$ and let φ, ψ be anisotropic forms over F of the same dimension. Then $\varphi \overset{\text{mot}}{\sim} \psi$ if and only if $\varphi \overset{\text{sim}}{\sim} \psi$.*

This corollary applies to global fields for which $\tilde{u} = 4$ (this follows from the well known Hasse-Minkowski theorem) and fields of transcendence degree one over a real closed field for which $\tilde{u} = 2$ (see, e.g., [5, Th. I]). However, for each $k \in \{2n \mid n \in \mathbb{N}\} \cup \{\infty\}$ there exist ED-fields F (in fact, fields F with a unique ordering) with $\tilde{u}(F) = k$ and $I_t^3F = 0$ (see [13, Th. 2.7] or [11, Th. 3.1]) to which Corollary 1.3 can still be applied.

In § 2, we investigate how determinants and Clifford invariants behave under motivic equivalence. The third section does the same for signatures and there we also prove the main theorem by putting all this together. In § 4, we give a few examples that show that under weakening some of the imposed conditions, one cannot expect any longer that motivic equivalence implies similarity.

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2. Comparing determinants and Clifford invariants

We will freely use without reference various basic facts from the algebraic theory of quadratic forms in characteristic $\neq 2$. All such facts and any unexplained terminology can be found in the books [20] or [3]. If φ is a form defined on an F-vector space V, we put $D_F(\varphi) = \{\varphi(x) \mid x \in V\} \cap F^\times$. We use the convention $\langle\langle a_1, \dots, a_n\rangle\rangle$ to denote the n-fold Pfister form $\langle 1, -a_1\rangle \otimes \dots \otimes \langle 1, -a_n\rangle$. A form φ over a field F is called a Pfister neighbor if there exists a Pfister form π over F and some $a \in F^\times$ such that $a\varphi$ is a subform of π (i.e. there exists another form ψ over F with $a\varphi \perp \psi \cong \pi$) and $2 \dim \varphi > \dim \pi$. Since such a Pfister form π is known to be either anisotropic or hyperbolic, it follows that a Pfister neighbor φ of π is anisotropic iff π is anisotropic. We call two forms φ and ψ over F half-neighbors if there exist an integer $n \geq 0$, $a, b \in F^\times$ and an $(n+1)$-fold Pfister form π such that $\dim \varphi = \dim \psi = 2^n$ and $a\varphi \perp -b\psi \cong \pi$. Now in this situation, if E is any field extension of F over which φ or ψ is isotropic then π_E is hyperbolic and thus $a\varphi_E \cong b\psi_E$ and it readily follows that $\varphi \overset{\text{mot}}{\sim} \psi$. Thus, a good way to construct examples of nonsimilar motivically equivalent forms is to find nonsimilar half-neighbors, see § 4. The function field $F(\varphi)$ of a form φ is defined to be the function field of the associated quadric $F(X_\varphi)$ (we put $F(\varphi) = F$ if $\dim \varphi = 1$ or φ a hyperbolic plane).

In the sequel, we state some definitions and facts concerning generic splitting of quadratic forms. We refer to Knebusch's original paper [17] on that topic for details.

Let φ be a form over F. The generic splitting tower of φ is constructed inductively as follows. Let $F = F_0$ and $\varphi_0 = \varphi_{\text{an}}$ be its anisotropic part over F. Suppose that for $i \geq 0$ we have constructed the field extension F_i/F. Consider the anisotropic form $\varphi_i \cong (\varphi_{F_i})_{\text{an}}$. If $\dim \varphi_i \geq 2$ we put $F_{i+1} = F_i(\varphi_i)$ and $\varphi_{i+1} \cong (\varphi_{F_{i+1}})_{\text{an}}$. Note that if $\dim \varphi_i \geq 2$, we have $2i_W(\varphi_{F_i}) = \dim \varphi - \dim \varphi_i < 2i_W(\varphi_{F_{i+1}})$ or, equivalently, $\dim \varphi_i > \dim \varphi_{i+1}$. The smallest h such that $\dim \varphi_h \leq 1$ is called the height of φ. The generic splitting tower of φ is then given by

$$F = F_0 \subset F_1 \subset \dots \subset F_{h-1} \subset F_h .$$

F_{h-1} is called the leading field of φ. It is known that

$$\mathfrak{S}_a(\varphi) := \{i_W(\varphi_E) \mid E/F \text{ field extension}\} = \{i_W(\varphi_{F_i}) \mid 0 \leq i \leq h\} .$$

We call $\mathfrak{S}_a(\varphi)$ the absolute splitting pattern of φ. In the literature, it has often proved to be of advantage to consider instead the relative splitting pattern $\mathfrak{S}_r(\varphi)$ defined as follows. If $\mathfrak{S}_a(\varphi) = \{i_\ell = i_W(\varphi_{F_\ell}) \mid 0 \leq \ell \leq h\}$, then put $j_m = i_m - i_{m-1}$, $1 \leq m \leq h$, the increase of the Witt index at the m-th step in the splitting tower. Then $\mathfrak{S}_r(\varphi) = (j_1, \dots, j_h)$ as an ordered sequence, but we won't need this here.

The degree $\deg(\varphi)$ is defined as follows. If the dimension of φ is odd, then $\deg(\varphi) = 0$. If φ is hyperbolic one defines $\deg(\varphi) = \infty$. So suppose φ is not hyperbolic and $\dim\varphi$ is even. Then the anisotropic form φ_{h-1} over F_{h-1} becomes hyperbolic over its own function field $F_h = F_{h-1}(\varphi_{h-1})$ and is thus similar to an n-fold Pfister form for some $n \geq 1$. We then define $\deg(\varphi) = n$. Now the above implies that if φ is not hyperbolic then

$$2^{\deg(\varphi)} = \min\{\dim(\varphi_E)_{\mathrm{an}} \mid E/F \text{ is a field extension with } \varphi_E \text{ not hyperbolic}\}\ ,$$

and it follows that if $\dim(\varphi_E)_{\mathrm{an}} = 2^{\deg(\varphi)}$, then $(\varphi_E)_{\mathrm{an}}$ is similar to an n-fold Pfister form over E. An important and deep theorem which we will also use states that $I^nF = \{\varphi \in WF \mid \deg\varphi \geq n\}$, see [22, Th. 4.3].
While part (i) of the following lemma is rather trivial, part (ii) is a bit less so and seems to be due to Izhboldin (see [16, Remark 2.7]) but to our knowledge a proof was not yet in the literature, so we included one for the reader's convenience.

Lemma 2.1. *Let φ and ψ be anisotropic forms over F with $\varphi \overset{\mathrm{mot}}{\sim} \psi$. Then*
- (i) $\deg(\varphi) = \deg(\psi)$;
- (ii) *For every $a \in F^\times$ we have* $\deg(\varphi \perp -a\psi) > \deg(\varphi)$.

Proof. Part (i) follows immediately from the definition of degree and Vishik's criterion for motivic equivalence.
Let now $\deg(\varphi) = \deg(\psi) = n$. Part (ii) is trivial for $n = 0$, so assume $n \geq 1$. If $\varphi \perp -a\psi$ is hyperbolic there is nothing to show. So assume $\tau \cong (\varphi \perp -a\psi)_{\mathrm{an}} \neq 0$. By the degree characterization of I^nF, we have $\tau \in I^nF$ and hence $\deg(\tau) \geq n$. Suppose $\deg(\tau) = n$. Let E/F be the leading field of φ. By what was said preceding the lemma, $(\varphi_E)_{\mathrm{an}}$ and $(\psi_E)_{\mathrm{an}}$ are anisotropic n-fold Pfister forms which are clearly motivically equivalent and thus similar (this follows readily from, e.g., [20, Ch. X, Cor. 4.9]). Hence, there exist an n-fold Pfister form π over E and $x, y \in E^\times$ such that in WE, $\varphi_E = x\pi$, $\psi_E = y\pi$. Thus, $\tau_E = \langle x, -ay\rangle \otimes \pi \in I^{n+1}E$ and therefore $\deg(\tau) = n < n+1 \leq \deg(\tau_E)$. But this implies $\deg(\varphi) \leq n - 2$ by [1, Satz 19], a contradiction. □

The signed determinant of a form φ over F will be denoted by $d(\varphi)$. For a diagonalization $\varphi \cong \langle a_1, \ldots, a_n\rangle$ we have $d(\varphi) = (-1)^{n(n-1)/2}\prod_{i=1}^n a_i \in F^\times/F^{\times 2}$ and the map $\varphi \mapsto d(\varphi)$ induces an isomorphism $IF/I^2F \to F^\times/F^{\times 2}$. The Clifford invariant $c(\varphi)$ of φ is defined as follows. The Clifford algebra $C(\varphi)$ is a central simple algebra over F if $\dim\varphi$ is even, and its even part $C_0(\varphi)$ is central simple if $\dim\varphi$ is odd. In both cases, these algebras are Brauer-equivalent to a tensor product of quaternion algebras and thus their classes lie in the 2-torsion part $\mathrm{Br}_2(F)$ of the Brauer group of F. One defines

$$c(\varphi) = \begin{cases} [C(\varphi)] \in \mathrm{Br}_2(F) & \text{if } \dim\varphi \text{ even} \\ [C_0(\varphi)] \in \mathrm{Br}_2(F) & \text{if } \dim\varphi \text{ odd} \end{cases}$$

By Merkurjev's theorem [21], c induces an isomorphism $I^2F/I^3F \to \mathrm{Br}_2(F)$.

Corollary 2.2. *Let φ and ψ be forms over F of even dimension* $\dim\varphi = \dim\psi$. *Let $d = d(\varphi) \in F^\times/F^{\times 2}$ and $K = F$ if $d = 1$ and $K = F(\sqrt{d})$ if $d \neq 1$. If $\varphi \overset{\text{mot}}{\sim} \psi$ then $d = d(\varphi) = d(\psi)$ and $c(\varphi_K) = c(\psi_K)$.*

Proof. We have $\varphi, \psi \in IF$ and also $\varphi \perp -\psi \in I^2F$ and thus $\varphi \equiv \psi \bmod I^2F$ since $\varphi \overset{\text{mot}}{\sim} \psi$ and by Lemma 2.1. The above isomorphism $IF/I^2F \cong F^\times/F^{\times 2}$ immediately implies $d(\varphi) = d(\psi)$.
Now over K we then have $\varphi_K, \psi_K \in I^2K$ since $d(\varphi_K) = d(\psi_K) = 1$. This time, Lemma 2.1 yields $\varphi_K \equiv \psi_K \bmod I^3K$ and by invoking Merkurjev's theorem we readily get $c(\varphi_K) = c(\psi_K)$. □

Corollary 2.3. *Let φ and ψ be forms over F of even dimension* $\dim\varphi = \dim\psi$. *Let $d = d(\varphi) \in F^\times/F^{\times 2}$ and suppose that $\varphi \overset{\text{mot}}{\sim} \psi$.*

(i) *There exists $a \in F^\times$ such that $\varphi \perp -\psi \equiv \langle\langle a, d\rangle\rangle \bmod I^3F$.*
(ii) *With a as in (i), if $b \in F^\times$, then $\varphi \perp -b\psi \equiv \langle\langle ab, d\rangle\rangle \bmod I^3F$.*

In particular, with a as before, we have $\varphi \perp -a\psi \in I^3F$.

Proof. (i) If $d = 1$ then Corollary 2.2 together with Merkurjev's theorem implies $\varphi, \psi \in I^2F$ and $\varphi \perp -\psi \equiv 0 \bmod I^3F$. The result follows since $\langle\langle a, d\rangle\rangle = \langle\langle a, 1\rangle\rangle = 0$ in WF for any $a \in F^\times$.
If $d \neq 1$, we still have $\varphi \perp -\psi \in I^2F$ since $d(\psi) = d$ and this time for $K = F(\sqrt{d})$ that $(\varphi \perp -\psi)_K \in I^3K$. Hence, the central simple F-algebra $C(\varphi \perp -\psi)$ splits over the quadratic extension K, so its index is at most 2 and it is well known that then there exists a quaternion algebra $(a, d)_F$ for some $a \in F^\times$ such that $C(\varphi \perp -\psi) \sim (a, d)_F$ in $\mathrm{Br}_2(F)$. Hence, it follows again readily from Merkurjev's theorem and the fact that $c(\langle\langle a, d\rangle\rangle) = [(a, d)_F]$ that we have $\varphi \perp -\psi \equiv \langle\langle a, d\rangle\rangle \bmod I^3F$.
(ii) We have $\varphi \perp -\psi, \psi \perp -b\psi \in I^2F$ and $-\psi \perp \psi = 0 \in WF$. Furthermore, by denoting the class of a quaternion algebra by its own symbol and using well known rules for manipulating Clifford invariants (see, e.g., [20, p. 118]), we get

$$\begin{aligned} c(\varphi \perp -b\psi) &= c(\varphi \perp -\psi \perp \psi \perp -b\psi) \\ &= c(\varphi \perp -\psi)c(\psi \perp -b\psi) \\ &= (a, d)_F c(\psi) c(-db\psi) \\ &= (a, d)_F c(\psi) c(\psi) (-db, d)_F \\ &= (ab, d)_F \ . \end{aligned}$$

We conclude as in (i) that now $\varphi \perp -b\psi \equiv \langle\langle ab, d\rangle\rangle \bmod I^3F$. □

3. Comparing signatures and proof of the Main Theorem

The following lemma compares signatures of motivically equivalent forms.

Lemma 3.1. *Let φ and ψ be forms of the same dimension over a real field F. If $\varphi \overset{\text{mot}}{\sim} \psi$ then $|\operatorname{sgn}_P(\varphi)| = |\operatorname{sgn}_P(\psi)|$ for all $P \in X_F$.*

Proof. We first note that if γ is any form of dimension ≥ 2 over any real field K and if $Q \in X_K$, then for $L = K(\gamma)$ we have that Q extends to an

ordering $Q' \in X_L$ iff γ is indefinite at Q, i.e. $\dim\gamma > |\operatorname{sgn}_Q(\gamma)|$ (see, e.g. [6, Th. 3.5]). In this case, we clearly have $\operatorname{sgn}_Q(\gamma) = \operatorname{sgn}_{Q'}(\gamma_L)$ which implies $\dim(\gamma_L)_{\mathrm{an}} \geq |\operatorname{sgn}_{Q'}(\gamma_L)| = |\operatorname{sgn}_Q(\gamma)|$.
Applied to φ, ψ and $P \in X_F$, it now follows readily that there exists an extension E/F with E in the generic splitting tower of φ such that P extends to $P' \in X_E$ and

$$\dim(\varphi_E)_{\mathrm{an}} = |\operatorname{sgn}_{P'}(\varphi_E)| = |\operatorname{sgn}_P(\varphi)|\ .$$

By motivic equivalence, we have $\dim(\varphi_E)_{\mathrm{an}} = \dim(\psi_E)_{\mathrm{an}}$ and hence

$$|\operatorname{sgn}_P(\varphi)| = \dim(\psi_E)_{\mathrm{an}} \geq |\operatorname{sgn}_{P'}\psi_E| = |\operatorname{sgn}_P\psi|\ .$$

By symmetry, we also have $|\operatorname{sgn}_P\psi| \geq |\operatorname{sgn}_P(\varphi)|$. □

Remark 3.2. The above proof also shows that $\frac{1}{2}(\dim\varphi - |\operatorname{sgn}_P(\varphi)|) \in \mathfrak{S}_a(\varphi)$, a fact that was already noticed in [9, Prop. 2.2].

We need a few properties regarding spaces of orderings of real fields. For more details regarding the following, we refer to [19], [7], [23]. Recall that the space of orderings X_F is a topological space whose topology has as sub-basis the so-called Harrison sets $H(a) = \{P \in X_F \mid a >_P 0\}$ for $a \in F^\times$. These are clopen sets, and F has the strong approximation property SAP if each clopen set is a Harrison set. F has the property S_1 if every binary torsion form represents a totally positive element. SAP and S_1 together are equivalent to ED, see [23, Th. 2].

Lemma 3.3. *Let F be a real SAP field and let φ and ψ be forms over F of the same dimension with $\varphi \overset{\mathrm{mot}}{\sim} \psi$. Then there exist $a, b \in F^\times$ such that $\operatorname{sgn}_P(a\varphi) = \operatorname{sgn}_P(b\psi) \geq 0$ for all $P \in X_F$.*

Proof. Let $U = \{P \in X_F \mid \operatorname{sgn}_P(\varphi) < 0\}$. Then $U \subset X_F$ is clopen and SAP implies that there exists $a \in F^\times$ with $U = H(-a)$. Then $\operatorname{sgn}_P(a\varphi) \geq 0$ for all $P \in X_F$. Similarly, there exists $b \in F^\times$ with $\operatorname{sgn}_P(b\psi) \geq 0$ for all $P \in X_F$. Since $a\varphi \overset{\mathrm{mot}}{\sim} \varphi \overset{\mathrm{mot}}{\sim} \psi \overset{\mathrm{mot}}{\sim} b\psi$, we have $\operatorname{sgn}_P(a\varphi) = \operatorname{sgn}_P(b\psi)$ for all $P \in X_F$ by Lemma 3.1. □

Let $\sum^\times F^2$ denote the set of nonzero sums of squares in F. If F is nonreal, then it is well known that $F^\times = \sum^\times F^2$.

Lemma 3.4. *Let F be a real S_1 field and let φ and ψ be forms over F of the same dimension with $\varphi \overset{\mathrm{mot}}{\sim} \psi$ and $\operatorname{sgn}_P(\varphi) = \operatorname{sgn}_P(\psi)$ for all $P \in X_F$. Then there exists $s \in \sum^\times F^2$ with $\varphi \perp -s\psi \in I_t^3 F$.*

Proof. Note first that the signatures don't change by scaling with an $s \in \sum^\times F^2$. Hence $\varphi \perp -s\psi$ has total signature zero for any such s and thus $\varphi \perp -s\psi \in W_t F$.
On the other hand, by Corollary 2.3, there exists $a \in F^\times$ with $\varphi \perp -\psi \equiv \langle\langle a, d\rangle\rangle \bmod I^3F$ where $d = d(\varphi) = d(\psi) \in F^\times/F^{\times 2}$. Now if $P \in X_F$ and if π is an n-fold Pfister form over F, then $\operatorname{sgn}_P(\pi) \in \{0, 2^n\}$, hence, for $\tau \in I^nF$

we have $\mathrm{sgn}_P(\tau) \equiv 0 \bmod 2^n$. Now comparing signatures mod 8 immediately yields that $\langle\langle a,d\rangle\rangle \cong \langle 1,-a,-d,ad\rangle$ has total signature zero and is therefore torsion.

Consider the n-fold Pfister form $\sigma_n \cong 2^n \times \langle 1\rangle$. For n large enough, the $(n+2)$-fold Pfister form $\sigma_n \otimes \langle 1,-a,-d,ad\rangle$ will now be hyperbolic, so its Pfister neighbor $\sigma_n \otimes \langle 1,-d\rangle \perp \langle -a\rangle$ will be isotropic. It follows readily that there exist $u,v \in D_F(\sigma_n) \subseteq \sum^\times F^2$ with $\langle u,-a,-dv\rangle$ isotropic, so in particular, $au \in D_F(\langle 1,-duv\rangle)$. Since $uv \in \sum^\times F^2$, we can apply the characterization of S_1 in [12, Lemma 2.2(iii)] to find $t \in \sum^\times F^2$ such that $aut \in D_F(\langle 1,-d\rangle)$. But then $s := ut \in \sum^\times F^2$ and $\langle 1,-as,-d\rangle$ is isotropic. Therefore the Pfister form $\langle\langle as,d\rangle\rangle$ is hyperbolic, i.e. $\langle\langle as,d\rangle\rangle = 0$ in WF.

By the above and Corollary 2.3, we now have $\varphi \perp -s\psi \in W_tF \cap I^3F = I_t^3F$ as desired. □

Proof of Main Theorem 1.2. Let F be an ED-field and let φ, ψ be anisotropic forms over F of the same dimension n with $\varphi \overset{\mathrm{mot}}{\sim} \psi$. We have to show that there exists $x \in F^\times$ such that $\varphi \perp -x\psi \in I_t^3F$.

The theorem is trivial for odd n by Izhboldin's result because it implies $\varphi \overset{\mathrm{sim}}{\sim} \psi$. So we may assume that n is even.

If F is nonreal (in which case $I_t^3F = I^3F$ and ED is an empty condition), the result follows already from Corollary 2.3 with $x = b = a$.

So suppose that F is real. Now ED is equivalent to SAP plus S_1. Because of SAP, we may assume by Lemma 3.3 that, possibly after scaling, $\mathrm{sgn}_P(\varphi) = \mathrm{sgn}_P(\psi)$ for all $P \in X_F$. Since we also have S_1, we can apply Lemma 3.4 to conclude. □

4. Examples

The following two examples show that in Corollary 1.3 the condition $I_t^3F = 0$ does not suffice for motivic equivalence to imply similarity once the condition ED is only slightly weakened.

Example 4.1. Let $F = \mathbb{R}((x))((y))$ be the iterated power series field in two variables x,y over the reals. It is well known that $S = \{\pm 1, \pm x, \pm y, \pm xy\}$ is a set of representatives of $F^\times/F^{\times 2}$. Let $\tau_n \cong n \times \langle 1\rangle$ (where we allow the 0-dimensional form τ_0). Then Springer's theorem implies that up to isometry the anisotropic forms over F are exactly the forms of type

$$\epsilon_1\tau_{n_1} \perp \epsilon_2 x\tau_{n_2} \perp \epsilon_3 y\tau_{n_3} \perp \epsilon_4 xy\tau_{n_4}$$

with $\epsilon_i \in \{\pm 1\}$ and $n_i \geq 0$, and that the isometry type is uniquely determined by the four pairs (ϵ_i, n_i) (see, e.g., [20, Ch. VI, Cor. 1.6, Prop 1.9]).

Since $u(\mathbb{R}) = 0$, it also follows from the above that $u(F) = 0$, in particular $W_tF = I_t^3F = 0$. Now consider the anisotropic forms

$$\varphi \cong \langle 1,1,1,x,x,x,y,y\rangle \quad \text{and} \quad \psi \cong \langle 1,x,y,y,xy,xy,xy,xy\rangle\ .$$

We have $\varphi \perp \psi \cong \langle\langle -1, -1, -x, -y\rangle\rangle$, so φ and ψ are half-neighbors and thus $\varphi \overset{\text{mot}}{\sim} \psi$. However, one also readily sees that there is no $s \in S$ with $s\varphi \cong \psi$, hence $\varphi \overset{\text{sim}}{\not\sim} \psi$.
Of course, it is also well known that F lacks the property SAP and thus ED as, for example, the totally indefinite form $\langle 1, x, y, -xy\rangle$ is not weakly isotropic.
We can be more precise. Recall that the reduced stability index $\text{st}(F)$ of a field F can be characterized as the least n such that $I^{n+1}F = 2I^nF \bmod W_tF$, and that SAP is equivalent to $\text{st}(F) \leq 1$ (see [2]).
For $F = \mathbb{R}((x))((y))$, we trivially have property S_1 since $W_tF = 0$, and one also readily sees that $\text{st}(F) = 2$.
Now Corollary 1.3 applies to fields with $I_t^3F = 0$, S_1 and $\text{st}(F) \leq 1$, but the above shows that generally, it cannot be extended to fields satisfying $I_t^3F = 0$, S_1 and $\text{st}(F) = 2$. □

In [7], the property S_1 has been generalized as follows. A field F is said to have property S_n for $n \geq 1$ if for every n-fold Pfister form $\pi \cong \langle 1\rangle \perp \pi'$ over F and every $a \in \sum^\times F^2$ there exists an $m \geq 1$ with

$$D_F(\langle 1, -a\rangle) \cap D_F(\underbrace{\langle 1, \ldots, 1\rangle}_{m} \otimes \pi') \neq \emptyset\ .$$

Example 4.2. It is not difficult to construct real fields K with $|K^\times/K^{\times 2}| = 4$ and where the square classes are represented by $\{\pm 1, \pm 2\}$ (see, e.g., [20, Remark II.5.3]). Clearly, K is uniquely ordered and $u(K) = \tilde{u}(K) = 2$. Consider $F = K((t))$. Then $u(F) = 4$, so in particular $I_t^3F = 0$, F has two orderings (see, e.g., [20, Prop. VIII.4.11]) and thus is SAP. Furthermore, one readily checks that F has property S_2.
Now consider the anisotropic forms

$$\varphi \cong \langle 1, 1, 1, 1, 1, 1\rangle \perp t\langle 1, 2\rangle \quad \text{and} \quad \psi \cong \langle 1, 1\rangle \perp t\langle 1, 1, 1, 1, 1, 2\rangle\ .$$

Since $\langle 1, 1\rangle \cong \langle 2, 2\rangle$ we have $\varphi \perp \psi \cong \langle\langle -1, -1, -1, -t\rangle\rangle$. So φ and ψ are half-neighbors and hence $\varphi \overset{\text{mot}}{\sim} \psi$. On the other hand, since $2 \notin F^{\times 2}$, it follows readily that $\varphi \overset{\text{sim}}{\not\sim} \psi$.
Hence, in general, Corollary 1.3 cannot be extended to fields satisfying $I_t^3F = 0$, S_2 and SAP (i.e. $\text{st}(F) \leq 1$). □

Note that the two forms in the previous example also provide motivically equivalent nonsimilar forms over $\mathbb{Q}((t))$, a field that also satisfies S_2 and SAP. However, this would give a weaker counterexample in the sense that $I_t^4\mathbb{Q}((t)) = 0$ but $I_t^3\mathbb{Q}((t)) \neq 0$ as can be readily seen.

Example 4.3. If F is nonreal and $u(F) < 2^{n+1}$, then $(n+1)$-fold Pfister forms will always be hyperbolic over F and thus half-neighbors of dimension 2^n will always be similar. However, in [10, Cor. 3.6], it was shown that for any $n \geq 3$ there exist nonreal fields F with $u(F) = 2^{n+1}$ over which one can find nonsimilar half-neighbors of dimension 2^n. In fact, one can take any nonreal field E with $u(E) = 4$ and take $F = E((x_1))\ldots((x_{n-1}))$. As a consequence, there exist nonreal fields F with $u(F) = 16$ and motivically equivalent nonsimilar forms of dimension 8. $\square$

It should be noted that to our knowledge, all constructions of nonsimilar motivically equivalent forms over nonreal fields (e.g. in [15]) require the existence of anisotropic 4-fold Pfister forms, so for these fields one would have $I^4F \neq 0$ and in particular $u(F) \geq 16$. Thus, also in view of the above examples, we ask the following.

Question 4.4. Are there fields F with $u(F) < 16$ which in the real case also satisfy ED, such that there exist nonsimilar motivically equivalent forms over F?

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