Triality and algebraic groups of type ${}^3\mathsf{D}_4$

Dedicated to Sasha Merkurjev on the occasion of his 60th birthday, in fond memory of the time spent writing the Book of Involutions

Max-Albert Knus and Jean-Pierre Tignol[1]

Received: September 8, 2014
Revised: April 12, 2014

Abstract. We determine which simple algebraic groups of type ${}^3\mathsf{D}_4$ over arbitrary fields of characteristic different from 2 admit outer automorphisms of order 3, and classify these automorphisms up to conjugation. The criterion is formulated in terms of a representation of the group by automorphisms of a trialitarian algebra: outer automorphisms of order 3 exist if and only if the algebra is the endomorphism algebra of an induced cyclic composition; their conjugacy classes are in one-to-one correspondence with isomorphism classes of symmetric compositions from which the induced cyclic composition stems.

2010 Mathematics Subject Classification: 20G15, 11E72, 17A75.
Keywords and Phrases: Algebraic group of outer type ${}^3\mathsf{D}_4$, triality, outer automorphism of order 3, composition algebra, symmetric composition, cyclic composition, octonions, Okubo algebra.

[1]The second author is supported in part by the Fonds de la Recherche Scientifique–FNRS under grant n° 1.5054.12. Both authors acknowledge the hospitality of the Fields Institute, Toronto, where this research was initiated. The first author also acknowledges the hospitality of the Université catholique de Louvain and enlightening conversations with Michel Racine.

1. Introduction

Let G_0 be an adjoint Chevalley group of type D_4 over a field F. Since the automorphism group of the Dynkin diagram of type D_4 is isomorphic to the symmetric group $\mathfrak{S}_3$, there is a split exact sequence of algebraic groups

$$(1) \qquad 1 \longrightarrow G_0 \xrightarrow{\text{Int}} \mathbf{Aut}(G_0) \xrightarrow{\pi} \mathfrak{S}_3 \longrightarrow 1.$$

Thus, $\mathbf{Aut}(G_0) \cong G_0 \rtimes \mathfrak{S}_3$; in particular G_0 admits outer automorphisms of order 3, which we call *trialitarian automorphisms*. Adjoint algebraic groups of type D_4 over F are classified by the Galois cohomology set $H^1(F, G_0 \rtimes \mathfrak{S}_3)$ and the map induced by π in cohomology

$$\pi_*\colon H^1(F, G_0 \rtimes \mathfrak{S}_3) \to H^1(F, \mathfrak{S}_3)$$

associates to any group G of type D_4 the isomorphism class of a cubic étale F-algebra L. The group G is said to be of type ${}^1\mathsf{D}_4$ if L is split, of type ${}^2\mathsf{D}_4$ if $L \cong F \times \Delta$ for some quadratic separable field extension Δ/F, of type ${}^3\mathsf{D}_4$ if L is a cyclic field extension of F and of type ${}^6\mathsf{D}_4$ if L is a non-cyclic field extension. An easy argument given in Proposition 4.2 below shows that groups of type ${}^2\mathsf{D}_4$ and ${}^6\mathsf{D}_4$ do not admit trialitarian automorphisms defined over the base field. Trialitarian automorphisms of groups of type ${}^1\mathsf{D}_4$ were classified in [3], and by a different method in [2]: the adjoint groups of type ${}^1\mathsf{D}_4$ that admit trialitarian automorphisms are the groups of proper projective similitudes of 3-fold Pfister quadratic spaces; their trialitarian automorphisms are shown in [3, Th. 5.8] to be in one-to-one correspondence with the symmetric composition structures on the quadratic space. In the present paper, we determine the simple groups of type ${}^3\mathsf{D}_4$ that admit trialitarian automorphisms, and we classify those automorphisms up to conjugation.

Our main tool is the notion of a trialitarian algebra, as introduced in [9, Ch. X]. Since these algebras are only defined in characteristic different from 2, we assume throughout (unless specifically mentioned) that the characteristic of the base field F is different from 2. In view of [9, Th. (44.8)], every adjoint simple group G of type D_4 can be represented as the automorphism group of a trialitarian algebra $T = (E, L, \sigma, \alpha)$. In the datum defining T, L is the cubic étale F-algebra given by the map π_* above, E is a central simple L-algebra with orthogonal involution σ, known as the *Allen invariant* of G (see [1]), and α is an isomorphism relating (E, σ) with its Clifford algebra $C(E, \sigma)$ (we refer to [9, §43] for details). We show in Proposition 4.2 that if G admits an outer automorphism of order 3 modulo inner automorphisms, then L is either split (i.e., isomorphic to $F \times F \times F$), or it is a cyclic field extension of F (so G is of type ${}^1\mathsf{D}_4$ or ${}^3\mathsf{D}_4$), and the Allen invariant E of G is a split central simple L-algebra. This implies that T has the special form $T = \operatorname{End}\Gamma$ for some cyclic composition Γ. We further show in Theorem 4.3 that if G carries a trialitarian automorphism, then the cyclic composition Γ is *induced*, which means that it is built from some symmetric composition over F, and we establish a one-to-one

correspondence between trialitarian automorphisms of G up to conjugation and isomorphism classes of symmetric compositions over F from which Γ is built.
Note that we only consider outer automorphisms of order 3, hence we do not investigate the weaker property considered by Garibaldi in [6], about the existence of outer automorphisms whose third power is inner. Nevertheless, our Theorem 4.3 has bearing on it, in view of a result recently announced by Garibaldi and Petersson [7], establishing the existence of outer automorphisms whose third power is inner for any group of type $^3\mathsf{D}_4$ with trivial Allen invariant. If Γ is a cyclic composition that is not induced (examples are given in Remark 2.1), the group of automorphisms of $\operatorname{End}\Gamma$ does not admit trialitarian automorphisms, but the Garibaldi–Petersson result shows that it has outer automorphisms whose third power is inner.
The notions of symmetric and cyclic compositions are recalled in §2. Trialitarian algebras are discussed in §3, which contains the most substantial part of the argument: we determine the trialitarian algebras that have semilinear automorphisms of order 3 (Theorem 3.1) and we classify these automorphisms up to conjugation (Theorem 3.5). The group-theoretic results follow easily in §4 by using the correspondence between groups of type D_4 and trialitarian algebras.
Notation is generally as in the Book of Involutions [9], which is our main reference. For an algebraic structure S defined over a field F, we let $\operatorname{Aut}(S)$ denote the group of automorphisms of S, and write $\mathbf{Aut}(S)$ for the corresponding group scheme over F.
We gratefully thank Vladimir Chernousov, Alberto Elduque, and Sasha Merkurjev for their help during the preparation of this paper. We are also grateful to Skip Garibaldi for his comments on a preliminary version of the paper, and to an anonymous referee for their careful reading.

2. Cyclic and symmetric compositions

Cyclic compositions were introduced by Springer in his 1963 Göttingen lecture notes ([11], [12]) to get new descriptions of Albert algebras. We recall their definition from [12][1] and [9, §36.B], restricting to the case of dimension 8.
Let F be an arbitrary field (of any characteristic). A *cyclic composition* (of dimension 8) over F is a 5-tuple $\Gamma = (V, L, Q, \rho, *)$ consisting of

- a cubic étale F-algebra L;
- a free L-module V of rank 8;
- a quadratic form $Q\colon V \to L$ with nondegenerate polar bilinear form b_Q;
- an F-automorphism ρ of L of order 3;
- an F-bilinear map $*\colon V \times V \to V$ with the following properties: for all x, y, $z \in V$ and $\lambda \in L$,

$$(x\lambda) * y = (x * y)\rho(\lambda), \qquad x * (y\lambda) = (x * y)\rho^2(\lambda),$$

[1]A cyclic composition is called a *normal twisted composition* in [11] and [12].

$$Q(x * y) = \rho\big(Q(x)\big) \cdot \rho^2\big(Q(y)\big),$$
$$b_Q(x * y, z) = \rho\big(b_Q(y * z, x)\big) = \rho^2\big(b_Q(z * x, y)\big).$$

These properties imply the following (see [9, §36.B] or [12, Lemma 4.1.3]): for all x, $y \in V$,

$$(2) \qquad (x * y) * x = y\rho^2\big(Q(x)\big) \quad \text{and} \quad x * (y * x) = y\rho\big(Q(x)\big).$$

Since the cubic étale F-algebra L has an automorphism of order 3, L is either a cyclic cubic field extension of F, and ρ is a generator of the Galois group, or we may identify L with $F \times F \times F$ and assume ρ permutes the components cyclically. We will almost exclusively restrict to the case where L is a field; see however Remark 2.3 below.

Let $\Gamma' = (V', L', Q', \rho', *')$ be also a cyclic composition over F. An *isotopy*[2] $\Gamma \to \Gamma'$ is defined to be a pair (ν, f) where $\nu\colon (L,\rho) \xrightarrow{\sim} (L',\rho')$ is an isomorphism of F-algebras with automorphisms (i.e., $\nu \circ \rho = \rho' \circ \nu$) and $f\colon V \xrightarrow{\sim} V'$ is a ν-semilinear isomorphism for which there exists $\mu \in L^\times$ such that

$$Q'\big(f(x)\big) = \nu\big(\rho(\mu)\rho^2(\mu) \cdot Q(x)\big) \quad \text{and} \quad f(x) *' f(y) = f(x * y)\nu(\mu)$$

for x, $y \in V$. The scalar μ is called the *multiplier* of the isotopy. Isotopies with multiplier 1 are *isomorphisms*. When the map ν is clear from the context, we write simply f for the pair (ν, f), and refer to f as a *ν-semilinear isotopy*.

Examples of cyclic compositions can be obtained by scalar extension from symmetric compositions over F, as we now show. Recall from [9, §34] that a *symmetric composition* (of dimension 8) over F is a triple $\Sigma = (S, n, \star)$ where (S, n) is an 8-dimensional F-quadratic space (with nondegenerate polar bilinear form b_n) and $\star\colon S \times S \to S$ is a bilinear map such that for all x, y, $z \in S$

$$n(x \star y) = n(x)n(y) \quad \text{and} \quad b_n(x \star y, z) = b_n(x, y \star z).$$

If $\Sigma' = (S', n', \star')$ is also a symmetric composition over F, an *isotopy* $\Sigma \to \Sigma'$ is a linear map $f\colon S \to S'$ for which there exists $\lambda \in F^\times$ (called the *multiplier*) such that

$$n'\big(f(x)\big) = \lambda^2 n(x) \quad \text{and} \quad f(x) \star' f(y) = f(x \star y)\lambda \quad \text{for } x,\ y \in S.$$

Note that if $f\colon \Sigma \to \Sigma'$ is an isotopy with multiplier λ, then $\lambda^{-1}f\colon \Sigma \to \Sigma'$ is an isomorphism. Thus, symmetric compositions are isotopic if and only if they are isomorphic. For an explicit example of a symmetric composition, take a Cayley (octonion) algebra $(C, \cdot)$ with norm n and conjugation map $\overline{}$. Letting $x \star y = \overline{x} \cdot \overline{y}$ for x, $y \in C$ yields a symmetric composition $\widetilde{C} = (C, n, \star)$, which is called a *para-Cayley composition* (see [9, §34.A]).
Given a symmetric composition $\Sigma = (S, n, \star)$ and a cubic étale F-algebra L with an automorphism ρ of order 3, we define a cyclic composition $\Sigma \otimes (L, \rho)$ as follows:

$$\Sigma \otimes (L, \rho) = (S \otimes_F L, L, n_L, \rho, *)$$

[2] The term used in [9, p. 490] is *similarity*.

where n_L is the scalar extension of n to L and $*$ is defined by extending $\star$ linearly to $S \otimes_F L$ and then setting

$$x * y = (\mathrm{Id}_S \otimes \rho)(x) \star (\mathrm{Id}_S \otimes \rho^2)(y) \qquad \text{for } x,\, y \in S \otimes_F L.$$

(See [9, (36.11)].) Clearly, every isotopy $f \colon \Sigma \to \Sigma'$ of symmetric compositions extends to an isotopy of cyclic compositions $(\mathrm{Id}_L, f) \colon \Sigma \otimes (L,\rho) \to \Sigma' \otimes (L,\rho)$. Observe for later use that the map $\widehat{\rho} = \mathrm{Id}_S \otimes \rho \in \mathrm{End}_F(S \otimes_F L)$ defines a ρ-semilinear automorphism

$$\widehat{\rho} \colon \Sigma \otimes (L,\rho) \xrightarrow{\sim} \Sigma \otimes (L,\rho) \tag{3}$$

such that $\widehat{\rho}^3 = \mathrm{Id}$.

We call a cyclic composition that is isotopic to $\Sigma \otimes (L,\rho)$ for some symmetric composition Σ *induced*. Cyclic compositions induced from para-Cayley symmetric compositions are called *reduced* in [12].

Remark 2.1. Induced cyclic compositions are not necessarily reduced. This can be shown by using the following cohomological argument. We assume for simplicity that the field F contains a primitive cube root of unity ω. There is a cohomological invariant $g_3(\Gamma) \in H^3(F, \mathbb{Z}/3\mathbb{Z})$ attached to any cyclic composition Γ. The cyclic composition Γ is reduced if and only if $g_3(\Gamma) = 0$ (we refer to [12, §8.3] or [9, §40] for details). We construct an induced cyclic composition Γ with $g_3(\Gamma) \neq 0$. Let $a, b \in F^\times$ and let $A(a,b)$ be the F-algebra with generators α, β and relations $\alpha^3 = a$, $\beta^3 = b$, $\beta\alpha = \omega\alpha\beta$. The algebra $A(a,b)$ is central simple of dimension 9 and the space A^0 of elements of $A(a,b)$ of reduced trace zero admits the structure of a symmetric composition $\Sigma(a,b) = (A^0, n, \star)$ (see [9, (34.19)]). Such symmetric compositions are called *Okubo symmetric compositions*. From the Elduque–Myung classification of symmetric compositions [5, p. 2487] (see also [9, (34.37)]), it follows that symmetric compositions are either para-Cayley or Okubo. Let $L = F(\gamma)$ with $\gamma^3 = c \in F^\times$ be a cubic cyclic field extension of F, and let ρ be the F-automorphism of L such that $\gamma \mapsto \omega\gamma$. We may then consider the induced cyclic composition $\Gamma(a,b,c) = \Sigma(a,b) \otimes (L,\rho)$. Its cohomological invariant $g_3\big(\Gamma(a,b,c)\big)$ can be computed by the construction in [12, §8.3]: Using ω, we identify the group μ_3 of cube roots of unity in F with $\mathbb{Z}/3\mathbb{Z}$, and for any $u \in F^\times$ we write $[u]$ for the cohomology class in $H^1(F, \mathbb{Z}/3\mathbb{Z})$ corresponding to the cube class $uF^{\times 3}$ under the isomorphism $F^\times/F^{\times 3} \cong H^1(F, \mu_3)$ arising from the Kummer exact sequence (see [9, p. 413]). Then $g_3\big(\Gamma(a,b,c)\big)$ is the cup-product $[a] \cup [b] \cup [c] \in H^3(F, \mathbb{Z}/3\mathbb{Z})$. Thus any cyclic composition $\Gamma(a,b,c)$ with $[a] \cup [b] \cup [c] \neq 0$ is induced but not reduced. Another cohomological argument can be used to show that there exist cyclic compositions that are not induced. We still assume that F contains a primitive cube root of unity ω. There is a further cohomological invariant of cyclic compositions $f_3(\Gamma) \in H^3(F, \mathbb{Z}/2\mathbb{Z})$ which is zero for any cyclic composition induced by an Okubo symmetric composition[3] and is given by the class in $H^3(F, \mathbb{Z}/2\mathbb{Z})$ of the 3-fold Pfister form which is the norm of $\widetilde{C}$ if Γ is induced

[3]The fact that F contains a primitive cubic root of unity is relevant for this claim.

from the para-Cayley $\widetilde{C}$ (see for example [9, §40]). Thus a cyclic composition Γ with $f_3(\Gamma) \neq 0$ and $g_3(\Gamma) \neq 0$ is not induced. Such examples can be given with the help of the Tits process used for constructing Albert algebras (see [9, §39 and §40]). However, for example, cyclic compositions over finite fields, p-adic fields or algebraic number fields are reduced, see [12, p. 108].

Examples 2.2. (i) Let $F = \mathbb{F}_q$ be the field with q elements, where q is odd and $q \equiv 1 \bmod 3$. Thus F contains a primitive cube root of unity and we are in the situation of Remark 2.1. Let $L = \mathbb{F}_{q^3}$ be the (unique, cyclic) cubic field extension of F, and let ρ be the Frobenius automorphism of L/F. Because $H^3(F, \mathbb{Z}/3\mathbb{Z}) = 0$, every cyclic composition over F is reduced; moreover every 3-fold Pfister form is hyperbolic, hence every Cayley algebra is split. Therefore, up to isomorphism there is a unique cyclic composition over F with cubic algebra (L, ρ), namely $\Gamma = \widetilde{C} \otimes (L, \rho)$ where $\widetilde{C}$ is the split para-Cayley symmetric composition. If Σ denotes the Okubo symmetric composition on 3×3 matrices of trace zero with entries in F, we thus have $\Gamma \cong \Sigma \otimes (L, \rho)$, which means that Γ is also induced by Σ. By the Elduque–Myung classification of symmetric compositions, every symmetric composition over F is isomorphic either to the Okubo composition Σ or to the split para-Cayley composition $\widetilde{C}$. Therefore, Γ is induced by exactly two symmetric compositions over F up to isomorphism.

(ii) Assume that F contains a primitive cube root of unity and that F carries an anisotropic 3-fold Pfister form n. Let C be the non-split Cayley algebra with norm n and let $\widetilde{C}$ be the associated para-Cayley algebra. For any cubic cyclic field extension (L, ρ) the norm n_L of the cyclic composition $\widetilde{C} \otimes (L, \rho)$ is anisotropic. Thus it follows from the Elduque–Myung classification that any symmetric composition Σ such that $\Sigma \otimes (L, \rho)$ is isotopic to $\widetilde{C} \otimes (L, \rho)$ must be isomorphic to $\widetilde{C}$.

(iii) Finally, we observe that the cyclic compositions of type $\Gamma(a, b, c)$, described in Remark 2.1, have invariant g_3 equal to zero if $c = a$. Since the f_3-invariant is also zero, they are all isotopic to the cyclic composition induced by the split para-Cayley algebra. Thus we can get (over suitable fields) examples of many mutually non-isomorphic symmetric compositions $\Sigma(a, b)$ that induce isomorphic cyclic compositions $\Gamma(a, b, c)$.

Of course, besides this construction of cyclic compositions by induction from symmetric compositions, we can also extend scalars of a cyclic composition: if $\Gamma = (V, L, Q, \rho, *)$ is a cyclic composition over F and K is any field extension of F, then $\Gamma_K = (V \otimes_F K, L \otimes_F K, Q_K, \rho \otimes \mathrm{Id}_K, *_K)$ is a cyclic composition over K.

Remark 2.3. Let $\Gamma = (V, L, Q, \rho, *)$ be an arbitrary cyclic composition over F with L a field. Write θ for ρ^2. We have an isomorphism of L-algebras

$$\nu\colon L \otimes_F L \xrightarrow{\sim} L \times L \times L \quad \text{given by} \quad \ell_1 \otimes \ell_2 \mapsto (\ell_1 \ell_2, \rho(\ell_1)\ell_2, \theta(\ell_1)\ell_2).$$

Therefore, the extended cyclic composition Γ_L over L has a split cubic étale algebra. To give an explicit description of Γ_L, note first that under the isomorphism ν the automorphism $\rho\otimes\mathrm{Id}_L$ is identified with the map $\widetilde{\rho}$ defined by $\widetilde{\rho}(\ell_1,\ell_2,\ell_3)=(\ell_2,\ell_3,\ell_1)$. Consider the twisted L-vector spaces ${}^\rho V$, ${}^\theta V$ defined by

$$ {}^\rho V=\{{}^\rho x\mid x\in V\},\qquad {}^\theta V=\{{}^\theta x\mid x\in V\} $$

with the operations

$$ {}^\rho(x+y)={}^\rho x+{}^\rho y,\ {}^\theta(x+y)={}^\theta x+{}^\theta y,\ \text{and } {}^\rho(x\lambda)=({}^\rho x)\rho(\lambda),\ {}^\theta(x\lambda)=({}^\theta x)\theta(\lambda) $$

for x, $y\in V$ and $\lambda\in L$. Define quadratic forms ${}^\rho Q\colon {}^\rho V\to L$ and ${}^\theta Q\colon {}^\theta V\to L$ by

$$ {}^\rho Q({}^\rho x)=\rho\bigl(Q(x)\bigr)\quad\text{and}\quad {}^\theta Q({}^\theta x)=\theta\bigl(Q(x)\bigr)\quad\text{for } x\in V, $$

and L-bilinear maps

$$ *_{\mathrm{Id}}\colon {}^\rho V\times{}^\theta V\to V,\quad *_\rho\colon {}^\theta V\times V\to{}^\rho V,\quad *_\theta\colon V\times{}^\rho V\to{}^\theta V $$

by

$$ {}^\rho x *_{\mathrm{Id}} {}^\theta y = x*y,\quad {}^\theta x *_\rho y={}^\rho(x*y),\quad x*_\theta{}^\rho y={}^\theta(x*y)\quad\text{for } x,\,y\in V. $$

We may then consider the quadratic form

$$ Q\times{}^\rho Q\times{}^\theta Q\colon V\times{}^\rho V\times{}^\theta V\to L\times L\times L $$

and the product $\diamond\colon (V\times{}^\rho V\times{}^\theta V)\times(V\times{}^\rho V\times{}^\theta V)\to(V\times{}^\rho V\times{}^\theta V)$ defined by

$$ (x,{}^\rho x,{}^\theta x)\diamond(y,{}^\rho y,{}^\theta y)=({}^\rho x*_{\mathrm{Id}}{}^\theta y,{}^\theta x*_\rho y,\ x*_\theta{}^\rho y). $$

Straightforward calculations show that the F-vector space isomorphism $f\colon V\otimes_F L\to V\times{}^\rho V\times{}^\theta V$ given by

$$ f(x\otimes\ell)=(x\ell,({}^\rho x)\ell,({}^\theta x)\ell)\qquad\text{for } x\in V \text{ and } \ell\in L $$

defines with ν an isomorphism of cyclic compositions

$$ \Gamma_L\xrightarrow{\sim}(V\times{}^\rho V\times{}^\theta V,\ L\times L\times L,\ Q\times{}^\rho Q\times{}^\theta Q,\ \widetilde{\rho},\ \diamond). $$

3. Trialitarian algebras

In this section, we assume that the characteristic of the base field F is different from 2. Trialitarian algebras are defined in [9, §43] as 4-tuples $T=(E,L,\sigma,\alpha)$ where L is a cubic étale F-algebra, (E,σ) is a central simple L-algebra of degree 8 with an orthogonal involution, and α is an isomorphism from the Clifford algebra $C(E,\sigma)$ to a certain twisted scalar extension of E. We just recall in detail the special case of trialitarian algebras of the form $\operatorname{End}\Gamma$ for Γ a cyclic composition, because this is the main case for the purposes of this paper.

Let $\Gamma = (V, L, Q, \rho, *)$ be a cyclic composition (of dimension 8) over F, with L a field, and let $\theta = \rho^2$. Let also σ_Q denote the orthogonal involution on $\operatorname{End}_L V$ adjoint to Q. We will use the product $*$ to see that the Clifford algebra $C(V, Q)$ is split and the even Clifford algebra $C_0(V, Q)$ decomposes into a direct product of two split central simple L-algebras of degree 8. Using the notation of Remark 2.3, to any $x \in V$ we associate L-linear maps

$$\ell_x \colon {}^{\rho}V \to {}^{\theta}V \quad \text{and} \quad r_x \colon {}^{\theta}V \to {}^{\rho}V$$

defined by

$$\ell_x({}^{\rho}y) = x *_{\theta} {}^{\rho}y = {}^{\theta}(x * y) \quad \text{and} \quad r_x({}^{\theta}z) = {}^{\theta}z *_{\rho} x = {}^{\rho}(z * x)$$

for y, $z \in V$. From (2) it follows that for $x \in V$ the L-linear map

$$\alpha_*(x) = \left(\begin{smallmatrix} 0 & r_x \\ \ell_x & 0 \end{smallmatrix}\right) \colon {}^{\rho}V \oplus {}^{\theta}V \to {}^{\rho}V \oplus {}^{\theta}V \quad \text{given by} \quad ({}^{\rho}y, {}^{\theta}z) \mapsto \big(r_x({}^{\theta}z), \ell_x({}^{\rho}y)\big)$$

satisfies $\alpha_*(x)^2 = Q(x)\operatorname{Id}$. Therefore, there is an induced L-algebra homomorphism

$$\alpha_* \colon C(V, Q) \to \operatorname{End}_L({}^{\rho}V \oplus {}^{\theta}V). \tag{4}$$

This homomorphism is injective because $C(V, Q)$ is a simple algebra, hence it is an isomorphism by dimension count. It restricts to an L-algebra isomorphism

$$\alpha_{*0} \colon C_0(V, Q) \xrightarrow{\sim} \operatorname{End}_L({}^{\rho}V) \times \operatorname{End}_L({}^{\theta}V),$$

see [9, (36.16)]. Note that we may identify $\operatorname{End}_L({}^{\rho}V)$ with the twisted algebra ${}^{\rho}(\operatorname{End}_L V)$ (where multiplication is defined by ${}^{\rho}f_1 \cdot {}^{\rho}f_2 = {}^{\rho}(f_1 \circ f_2)$) as follows: for $f \in \operatorname{End}_L V$, we identify ${}^{\rho}f$ with the map ${}^{\rho}V \to {}^{\rho}V$ such that ${}^{\rho}f({}^{\rho}x) = {}^{\rho}(f(x))$ for $x \in V$. On the other hand, let σ_Q be the orthogonal involution on $\operatorname{End}_L V$ adjoint to Q. The algebra $C_0(V, Q)$ is canonically isomorphic to the Clifford algebra $C(\operatorname{End}_L V, \sigma_Q)$ (see [9, (8.8)]), hence it depends only on $\operatorname{End}_L V$ and σ_Q. We may regard α_{*0} as an isomorphism of L-algebras

$$\alpha_{*0} \colon C(\operatorname{End}_L V, \sigma_Q) \xrightarrow{\sim} {}^{\rho}(\operatorname{End}_L V) \times {}^{\theta}(\operatorname{End}_L V).$$

Thus, α_{*0} depends only on $\operatorname{End}_L V$ and σ_Q. The trialitarian algebra $\operatorname{End}\Gamma$ is the 4-tuple

$$\operatorname{End}\Gamma = (\operatorname{End}_L V, L, \sigma_Q, \alpha_{*0}).$$

An *isomorphism* of trialitarian algebras $\operatorname{End}\Gamma \xrightarrow{\sim} \operatorname{End}\Gamma'$, for $\Gamma' = (V', L', Q', \rho', *')$ a cyclic composition, is defined to be an isomorphism of F-algebras with involution $\varphi \colon (\operatorname{End}_L V, \sigma_Q) \xrightarrow{\sim} (\operatorname{End}_{L'} V', \sigma_{Q'})$ subject to the following conditions:

(i) the restriction of φ to the center of $\operatorname{End}_L V$ is an isomorphism $\varphi|_L \colon (L, \rho) \xrightarrow{\sim} (L', \rho')$, and

(ii) the following diagram (where $\theta' = {\rho'}^2$) commutes:

$$\begin{array}{ccc} C(\operatorname{End}_L V, \sigma_Q) & \xrightarrow{\alpha_{*0}} & {}^{\rho}(\operatorname{End}_L V) \times {}^{\theta}(\operatorname{End}_L V) \\ \Big\downarrow{\scriptstyle C(\varphi)} & & \Big\downarrow{\scriptstyle {}^{\rho}\varphi \times {}^{\theta}\varphi} \\ C(\operatorname{End}_{L'} V', \sigma_{Q'}) & \xrightarrow{\alpha_{*'0}} & {}^{\rho'}(\operatorname{End}_{L'} V') \times {}^{\theta'}(\operatorname{End}_{L'} V') \end{array}$$

For example, it is straightforward to check that every isotopy $(\nu, f)\colon \Gamma \to \Gamma'$ induces an isomorphism $\operatorname{End}\Gamma \to \operatorname{End}\Gamma'$ mapping $g \in \operatorname{End}_L V$ to $f \circ g \circ f^{-1} \in \operatorname{End}_{L'} V'$. As part of the proof of the main theorem below, we show that every isomorphism $\operatorname{End}\Gamma \xrightarrow{\sim} \operatorname{End}\Gamma'$ is induced by an isotopy; see Lemma 3.4. (A cohomological proof that the trialitarian algebras $\operatorname{End}\Gamma$, $\operatorname{End}\Gamma'$ are isomorphic if and only if the cyclic compositions Γ, Γ' are isotopic is given in [9, (44.16)].)

We show that the trialitarian algebra $\operatorname{End}\Gamma$ admits a ρ-semilinear automorphism of order 3 if and only if Γ is induced. More precisely:

Theorem 3.1. *Let $\Gamma = (V, L, Q, \rho, *)$ be a cyclic composition over F, with L a field.*

(i) *If Σ is a symmetric composition over F and $f\colon \Sigma \otimes (L, \rho) \to \Gamma$ is an L-linear isotopy, then the automorphism $\tau_{(\Sigma,f)} = \operatorname{Int}(f \circ \widehat{\rho} \circ f^{-1})|_{\operatorname{End}_L V}$ of $\operatorname{End}\Gamma$, where $\widehat{\rho}$ is defined in (3), is such that $\tau^3_{(\Sigma,f)} = \operatorname{Id}$ and $\tau_{(\Sigma,f)}|_L = \rho$. The automorphism $\tau_{(\Sigma,f)}$ only depends, up to conjugation in $\operatorname{Aut}_F(\operatorname{End}\Gamma)$, on the isomorphism class of Σ.*

(ii) *If $\operatorname{End}\Gamma$ carries an F-automorphism τ such that $\tau|_L = \rho$ and $\tau^3 = \operatorname{Id}$, then Γ is induced. More precisely, there exists a symmetric composition Σ over F and an L-linear isotopy $f\colon \Sigma \otimes (L, \rho) \to \Gamma$ such that $\tau = \tau_{(\Sigma,f)}$.*

Proof. (i) It is clear that $\tau^3_{(\Sigma,f)} = \operatorname{Id}$ and $\tau_{(\Sigma,f)}|_L = \rho$. For the last claim, note that if $g\colon \Sigma \otimes (L, \rho) \to \Gamma$ is another L-linear isotopy, then $f \circ g^{-1}$ is an isotopy of Γ, hence $\operatorname{Int}(f \circ g^{-1})$ is an automorphism of $\operatorname{End}\Gamma$, and

$$\tau_{(\Sigma,f)} = \operatorname{Int}(f \circ g^{-1}) \circ \tau_{(\Sigma,g)} \circ \operatorname{Int}(f \circ g^{-1})^{-1}.$$

The proof of claim (ii) relies on three lemmas. Until the end of this section, we fix a cyclic composition $\Gamma = (V, L, Q, \rho, *)$, with L a field. We start with some general observations on ρ-semilinear automorphisms of $\operatorname{End}_L V$. For this, we consider the inclusions

$$L \hookrightarrow \operatorname{End}_L V \hookrightarrow \operatorname{End}_F V.$$

The field L is the center of $\operatorname{End}_L V$, hence every automorphism of $\operatorname{End}_L V$ restricts to an automorphism of L.

Lemma 3.2. *Let $\nu \in \{\operatorname{Id}_L, \rho, \theta\}$ be an arbitrary element in the Galois group $\operatorname{Gal}(L/F)$. For every F-linear automorphism φ of $\operatorname{End}_L V$ such that $\varphi|_L = \nu$, there exists an invertible transformation $u \in \operatorname{End}_F V$ such that $\varphi(f) = u \circ f \circ u^{-1}$ for all $f \in \operatorname{End}_L V$. The map u is uniquely determined up to a factor in $L^\times$;*

it is ν-semilinear, i.e., $u(x\lambda) = u(x)\nu(\lambda)$ for all $x \in V$ and $\lambda \in L$. Moreover, if $\varphi \circ \sigma_Q = \sigma_Q \circ \varphi$, then there exists $\mu \in L^\times$ such that

$$Q\big(u(x)\big) = \nu\big(\mu \cdot Q(x)\big) \qquad \textit{for all } x \in V.$$

Proof. The existence of u is a consequence of the Skolem–Noether theorem, since $\operatorname{End}_L V$ is a simple subalgebra of the simple algebra $\operatorname{End}_F V$: the automorphism φ extends to an inner automorphism $\operatorname{Int}(u)$ of $\operatorname{End}_F V$ for some invertible $u \in \operatorname{End}_F V$. Uniqueness of u up to a factor in $L^\times$ is clear because L is the centralizer of $\operatorname{End}_L V$ in $\operatorname{End}_F V$, and the ν-semilinearity of u follows from the equation $\varphi(f) = u \circ f \circ u^{-1}$ applied with f the scalar multiplication by an element in L.

Now, suppose φ commutes with σ_Q, hence for all $f \in \operatorname{End}_L V$

$$u \circ \sigma_Q(f) \circ u^{-1} = \sigma_Q(u \circ f \circ u^{-1}). \tag{5}$$

Let $\operatorname{Tr}_*(Q)$ denote the transfer of Q along the trace map $\operatorname{Tr}_{L/F}$, so $\operatorname{Tr}_*(Q)\colon V \to F$ is the quadratic form defined by $\operatorname{Tr}_*(Q)(x) = \operatorname{Tr}_{L/F}\big(Q(x)\big)$. The adjoint involution $\sigma_{\operatorname{Tr}_*(Q)}$ coincides on $\operatorname{End}_L V$ with σ_Q, hence from (5) it follows that $\sigma_{\operatorname{Tr}_*(Q)}(u)u$ centralizes $\operatorname{End}_L V$. Therefore, $\sigma_{\operatorname{Tr}_*(Q)}(u)u = \mu$ for some $\mu \in L^\times$. We then have $b_{\operatorname{Tr}_*(Q)}\big(u(x), u(y)\big) = b_{\operatorname{Tr}_*(Q)}(x, y\mu)$ for all x, $y \in V$, which means that

$$\operatorname{Tr}_{L/F}\big(b_Q(u(x), u(y))\big) = \operatorname{Tr}_{L/F}\big(\mu b_Q(x,y)\big). \tag{6}$$

Now, observe that since u is ν-semilinear, the map $c\colon V \times V \to L$ defined by $c(x,y) = \nu^{-1}\big(b_Q(u(x), u(y))\big)$ is L-bilinear. From (6), it follows that $c - \mu b_Q$ is a bilinear map on V that takes its values in the kernel of the trace map. But the value domain of an L-bilinear form is either L or $\{0\}$, and the trace map is not the zero map. Therefore, $c - \mu b_Q = 0$, which means that

$$\nu^{-1}\big(b_Q(u(x), u(y))\big) = \mu b_Q(x,y) \qquad \text{for all } x,\ y \in V,$$

hence $Q\big(u(x)\big) = \nu\big(\mu \cdot Q(x)\big)$ for all $x \in V$. □

Note that the arguments in the preceding proof apply to any quadratic space (V, Q) over L. By contrast, the next lemma uses the full cyclic composition structure: Let again $\nu \in \{\operatorname{Id}_L, \rho, \theta\}$. Given an invertible element $u \in \operatorname{End}_F V$ and $\mu \in L^\times$ such that for all $x \in V$ and $\lambda \in L$

$$u(x\lambda) = u(x)\nu(\lambda) \quad \text{and} \quad Q\big(u(x)\big) = \nu\big(\mu \cdot Q(x)\big),$$

we define an L-linear map $\beta_u\colon {}^\nu V \to \operatorname{End}_L({}^\rho V \oplus {}^\theta V)$ by

$$\beta_u({}^\nu x) = \begin{pmatrix} 0 & \nu(\mu)^{-1} r_{u(x)} \\ \ell_{u(x)} & 0 \end{pmatrix} \in \operatorname{End}_L({}^\rho V \oplus {}^\theta V) \quad \text{for } x \in V.$$

Then from (2) we get $\beta_u(x)^2 = \nu\big(Q(x)\big) = {}^\nu Q({}^\nu x)$. Therefore, the map β_u extends to an L-algebra homomorphism

$$\beta_u\colon C({}^\nu V, {}^\nu Q) \to \operatorname{End}_L({}^\rho V \oplus {}^\theta V).$$

Just like α_* in (4), the homomorphism β_u is an isomorphism. We also have an isomorphism of F-algebras $C({}^\nu\cdot)\colon C(V,Q)\to C({}^\nu V,{}^\nu Q)$ induced by the F-linear map $x\mapsto {}^\nu x$ for $x\in V$, so we may consider the F-automorphism ψ_u of $\operatorname{End}_L({}^\rho V\oplus{}^\theta V)$ that makes the following diagram commute:

$$\begin{array}{ccc} C(V,Q) & \xrightarrow{\alpha_*} & \operatorname{End}_L({}^\rho V\oplus{}^\theta V)\\ {\scriptstyle C({}^\nu\cdot)}\downarrow & & \downarrow{\scriptstyle\psi_u}\\ C({}^\nu V,{}^\nu Q) & \xrightarrow{\beta_u} & \operatorname{End}_L({}^\rho V\oplus{}^\theta V)\end{array}\tag{7}$$

Lemma 3.3. *The F-algebra automorphism ψ_u restricts to an F-algebra automorphism ψ_{u0} of $\operatorname{End}_L({}^\rho V)\times\operatorname{End}_L({}^\theta V)$. The restriction of ψ_{u0} to the center $L\times L$ is either $\nu\times\nu$ or $(\nu\times\nu)\circ\varepsilon$ where ε is the switch map $(\ell_1,\ell_2)\mapsto(\ell_2,\ell_1)$. Moreover, if $\psi_{u0}|_{L\times L}=\nu\times\nu$, then there exist invertible ν-semilinear transformations u_1, $u_2\in\operatorname{End}_F V$ such that*

$$\psi_u(f)=\begin{pmatrix}{}^\rho u_1 & 0\\ 0 & {}^\theta u_2\end{pmatrix}\circ f\circ\begin{pmatrix}{}^\rho u_1^{-1} & 0\\ 0 & {}^\theta u_2^{-1}\end{pmatrix}\quad\textit{for all } f\in\operatorname{End}_L({}^\rho V\oplus{}^\theta V).$$

For any pair (u_1,u_2) satisfying this condition, we have

$$u_2(x*y)=u(x)*u_1(y)\ \textit{and}\ u_1(x*y)=\big(u_2(x)*u(y)\big)\theta\nu(\mu)^{-1}\ \textit{for all } x,\,y\in V.$$

Proof. The maps α_* and β_u are isomorphisms of graded L-algebras for the usual $(\mathbb{Z}/2\mathbb{Z})$-gradings of $C(V,Q)$ and $C({}^\nu V,{}^\nu Q)$, and for the "checker-board" grading of $\operatorname{End}_L({}^\rho V\oplus{}^\theta V)$ defined by

$$\operatorname{End}_L({}^\rho V\oplus{}^\theta V)_0=\operatorname{End}_L({}^\rho V)\times\operatorname{End}_L({}^\theta V)$$

and

$$\operatorname{End}_L({}^\rho V\oplus{}^\theta V)_1=\begin{pmatrix}0 & \operatorname{Hom}_L({}^\theta V,{}^\rho V)\\ \operatorname{Hom}_L({}^\rho V,{}^\theta V) & 0\end{pmatrix}.$$

Therefore, ψ_u also preserves the grading, and it restricts to an automorphism ψ_{u0} of the degree 0 component. Because the map $C({}^\nu\cdot)$ is ν-semilinear, the map ψ_u also is ν-semilinear, hence its restriction to the center of the degree 0 component is either $\nu\times\nu$ or $(\nu\times\nu)\circ\varepsilon$.

Suppose $\psi_{u0}|_{L\times L}=\nu\times\nu$. By Lemma 3.2 (applied with ${}^\rho V\oplus{}^\theta V$ instead of V), there exists an invertible ν-semilinear transformation $v\in\operatorname{End}_F({}^\rho V\oplus{}^\theta V)$ such that $\psi_u(f)=v\circ f\circ v^{-1}$ for all $f\in\operatorname{End}_F({}^\rho V\oplus{}^\theta V)$. Since ψ_{u0} fixes $\left(\begin{smallmatrix}\mathrm{Id}_{{}^\rho V} & 0\\ 0 & 0\end{smallmatrix}\right)$, the element v centralizes $\left(\begin{smallmatrix}\mathrm{Id}_{{}^\rho V} & 0\\ 0 & 0\end{smallmatrix}\right)$, hence $v=\left(\begin{smallmatrix}{}^\rho u_1 & 0\\ 0 & {}^\theta u_2\end{smallmatrix}\right)$ for some invertible u_1, $u_2\in\operatorname{End}_F V$. The transformations u_1 and u_2 are ν-semilinear because v is ν-semilinear. From the commutativity of (7) we have $v\circ\alpha_*(x)=\beta_u({}^\nu x)\circ v=\alpha_*\big(u(x)\big)\circ v$ for all $x\in V$. By the definition of α_*, it follows that

$$u_1(z*x)=\theta\nu^{-1}(\mu)\big(u_2(z)*u(x)\big)\text{ and }u_2(x*y)=u(x)*u_1(y)\text{ for all }y,\,z\in V.$$

□

Lemma 3.4. *Let $\nu\in\{\mathrm{Id}_L,\rho,\theta\}$. For every F-linear automorphism φ of $\operatorname{End}\Gamma$ such that $\varphi|_L=\nu$, there exists an invertible transformation $u\in\operatorname{End}_F V$,* *uniquely determined up to a factor in $L^\times$, such that $\varphi(f) = u\circ f\circ u^{-1}$ for all $f\in \operatorname{End}_L V$. Every such u is a ν-semilinear isotopy $\Gamma\to\Gamma$.*

Proof. The existence of u, its uniqueness up to a factor in $L^\times$, and its ν-semilinearity, were established in Lemma 3.2. It only remains to show that u is an isotopy.

Since φ is an automorphism of $\operatorname{End}\Gamma$, it commutes with σ_Q, hence Lemma 3.2 yields $\mu\in L^\times$ such that $Q\big(u(x)\big) = \nu\big(\mu\cdot Q(x)\big)$ for all $x\in V$. We may therefore consider the maps β_u and ψ_u of Lemma 3.3. Now, recall from [9, (8.8)] that $C_0(V,Q) = C(\operatorname{End}_L V, \sigma_Q)$ by identifying $x\cdot y$ for x, $y\in V$ with the image in $C(\operatorname{End}_L V,\sigma_Q)$ of the linear transformation $x\otimes y$ defined by $z\mapsto x\cdot b_Q(y,z)$ for $z\in V$. We have

$$\varphi(x\otimes y) = u\circ(x\otimes y)\circ u^{-1}\colon z\mapsto u\big(x\cdot b_Q(y,u^{-1}(z))\big)\qquad \text{for } x,\ y,\ z\in V.$$

Since u is ν-semilinear and $Q\big(u(x)\big) = \nu\big(\mu\cdot Q(x)\big)$ for all $x\in V$, it follows that

$$u\big(x\cdot b_Q(y,u^{-1}(z))\big) = u(x)\cdot\nu\big(b_Q(y,u^{-1}(z))\big) = u(x)\cdot\nu(\mu)^{-1} b_Q(u(y),z).$$

Therefore, $\varphi(x\otimes y) = u(x)\otimes u(y)\nu(\mu)^{-1}$ for x, $y\in V$, hence the following diagram (where β_u and $C({}^\nu\cdot)$ are as in (7)) is commutative:

$$\begin{array}{ccc} C_0(V,Q) & \xrightarrow{C({}^\nu\cdot)|_{C_0(V,Q)}} & C_0({}^\nu V,{}^\nu Q) \\ {\scriptstyle C(\varphi)}\downarrow & & \downarrow{\scriptstyle \beta_u|_{C_0({}^\nu V,{}^\nu Q)}} \\ C_0(V,Q) & \xrightarrow{\alpha_{*0}} & \operatorname{End}_L({}^\rho V)\times\operatorname{End}_L({}^\theta V) \end{array}$$

On the other hand, the following diagram is commutative because φ is an automorphism of $\operatorname{End}\Gamma$:

$$\begin{array}{ccc} C_0(V,Q) & \xrightarrow{\alpha_{*0}} & \operatorname{End}_L({}^\rho V)\times\operatorname{End}_L({}^\theta V) \\ {\scriptstyle C(\varphi)}\downarrow & & \downarrow{\scriptstyle {}^\rho\varphi\times{}^\theta\varphi} \\ C_0(V,Q) & \xrightarrow{\alpha_{*0}} & \operatorname{End}_L({}^\rho V)\times\operatorname{End}_L({}^\theta V) \end{array}$$

Therefore, $\beta_u|_{C_0({}^\nu V,{}^\nu Q)}\circ C({}^\nu\cdot)|_{C_0(V,Q)} = ({}^\rho\varphi\times{}^\theta\varphi)\circ\alpha_{*0}$. By comparing with (7), we see that $\psi_{u0} = {}^\rho\varphi\times{}^\theta\varphi$, hence $\psi_{u0}|_{L\times L} = \nu\times\nu$. Lemma 3.3 then yields ν-semilinear transformations u_1, $u_2\in\operatorname{End}_F V$ such that

$$\psi_u(f) = \begin{pmatrix} {}^\rho u_1 & 0\\ 0 & {}^\theta u_2\end{pmatrix}\circ f\circ\begin{pmatrix} {}^\rho u_1^{-1} & 0\\ 0 & {}^\theta u_2^{-1}\end{pmatrix}\qquad \text{for all } f\in\operatorname{End}_L({}^\rho V\oplus{}^\theta V),$$

hence $\psi_{u0} = \operatorname{Int}({}^\rho u_1)\times\operatorname{Int}({}^\theta u_2)$. But we have $\psi_{u0} = {}^\rho\varphi\times{}^\theta\varphi = \operatorname{Int}({}^\rho u)\times\operatorname{Int}({}^\theta u)$. Therefore, multiplying (u_1,u_2) by a scalar in $L^\times$, we may assume $u=u_1$ and $u_2 = u\zeta$ for some $\zeta\in L^\times$. Lemma 3.3 then gives

$$u(x*y)\zeta = u(x)*u(y) \text{ and } u(x*y) = \big((\zeta u(x))*u(y)\big)\theta\nu(\mu)^{-1} \text{ for all } x,\ y\in V.$$

The second equation implies that $u(x*y) = \big(u(x)*u(y)\big)\rho(\zeta)\theta\nu(\mu)^{-1}$. By comparing with the first equation, we get $\rho(\zeta)\theta\nu(\mu)^{-1} = \zeta^{-1}$, hence $\nu(\mu) = \rho(\zeta)\theta(\zeta)$. Therefore, (ν,u) is an isotopy $\Gamma\to\Gamma$ with multiplier $\nu^{-1}(\zeta)$. □

We start with the proof of claim (ii) of Theorem 3.1. Suppose τ is an F-automorphism of $\operatorname{End}\Gamma$ such that $\tau|_L = \rho$ and $\tau^3 = \operatorname{Id}$. By Lemma 3.4, we may find an invertible ρ-semilinear transformation $t \in \operatorname{End}_F V$ such that $\tau(f) = t \circ f \circ t^{-1}$ for all $f \in \operatorname{End}_L V$, and every such t is an isotopy of Γ. Since $\tau^3 = \operatorname{Id}$, it follows that t^3 lies in the centralizer of $\operatorname{End}_L V$ in $\operatorname{End}_F V$, which is L. Let $t^3 = \xi \in L^\times$. We have $\rho(\xi) = t\xi t^{-1} = \xi$, hence $\xi \in F^\times$. The F-subalgebra of $\operatorname{End}_F V$ generated by L and t is a crossed product (L, ρ, ξ); its centralizer is the F-subalgebra $(\operatorname{End}_L V)^\tau$ fixed under τ, and we have

$$\operatorname{End}_F V \cong (L, \rho, \xi) \otimes_F (\operatorname{End}_L V)^\tau.$$

Now, $\deg(L, \rho, \xi) = 3$ and $\deg(\operatorname{End}_L V)^\tau = 8$, hence (L, ρ, ξ) is split. Therefore $\xi = N_{L/F}(\eta)$ for some $\eta \in L^\times$. Substituting $\eta^{-1}t$ for t, we get $t^3 = \operatorname{Id}_V$, and t is still a ρ-linear isotopy of Γ. Let $\mu \in L^\times$ be the corresponding multiplier, so that for all x, $y \in V$

$$Q\bigl(t(x)\bigr) = \rho\bigl(\rho(\mu)\theta(\mu)Q(x)\bigr) \quad \text{and} \quad t(x) * t(y) = t(x * y)\rho(\mu). \tag{8}$$

From the second equation we deduce that $t^3(x) * t^3(y) = t^3(x * y)N_{L/F}(\mu)$ for all x, $y \in V$, hence $N_{L/F}(\mu) = 1$ because $t^3 = \operatorname{Id}_V$. By Hilbert's Theorem 90, we may find $\zeta \in L^\times$ such that $\mu = \zeta\theta(\zeta)^{-1}$. Define $Q' = \rho(\zeta)\theta(\zeta)Q$ and let $x *' y = (x * y)\zeta$ for x, $y \in V$. Then Id_V is an isotopy $\Gamma \to \Gamma' = (V, L, Q', \rho, *')$ with multiplier ζ, and (8) implies that

$$Q'\bigl(t(x)\bigr) = \rho\bigl(Q'(x)\bigr) \quad \text{and} \quad t(x) *' t(y) = t(x *' y) \quad \text{for all } x,\, y \in V.$$

Now, observe that because t is ρ-semilinear and $t^3 = \operatorname{Id}_V$, the Galois group of L/F acts by semilinear automorphisms on V, hence we have a Galois descent (see [9, (18.1)]): the fixed point set $S = \{x \in V \mid t(x) = x\}$ is an F-vector space such that $V = S \otimes_F L$. Moreover, since $Q'\bigl(t(x)\bigr) = \rho\bigl(Q'(x)\bigr)$ for all $x \in V$, the restriction of Q' to S is a quadratic form $n\colon S \to F$, and we have $Q' = n_L$. Also, because $t(x *' y) = t(x) *' t(y)$ for all x, $y \in V$, the product $*'$ restricts to a product $\star$ on S, and $\Sigma = (S, n, \star)$ is a symmetric composition because Γ' is a cyclic composition. The canonical map $f\colon S \otimes_F L \to V$ yields an isomorphism of cyclic compositions $f\colon \Sigma \otimes (L, \rho) \xrightarrow{\sim} \Gamma'$, hence also an isotopy $f\colon \Sigma \otimes (L, \rho) \to \Gamma$. We have $t = f \circ \widehat{\rho} \circ f^{-1}$, hence τ is conjugation by $f \circ \widehat{\rho} \circ f^{-1}$. □

Theorem 3.5. *The assignment $\Sigma \mapsto \tau_{(\Sigma, f)}$ induces a bijection between the isomorphism classes of symmetric compositions Σ for which there exists an L-linear isotopy $f\colon \Sigma \otimes (L, \rho) \to \Gamma$ and conjugacy classes in $\operatorname{Aut}_F(\operatorname{End}\Gamma)$ of automorphisms τ of $\operatorname{End}\Gamma$ such that $\tau^3 = \operatorname{Id}$ and $\tau|_L = \rho$.*

Proof. We already know by Theorem 3.1 that the map induced by $\Sigma \mapsto \tau_{(\Sigma, f)}$ is onto. Therefore, it suffices to show that if the automorphisms $\tau_{(\Sigma, f)}$ and $\tau_{(\Sigma', f')}$ associated to symmetric compositions Σ and Σ' are conjugate, then Σ and Σ' are isomorphic. Assume $\tau_{(\Sigma', f')} = \varphi \circ \tau_{(\Sigma, f)} \circ \varphi^{-1}$ for some $\varphi \in \operatorname{Aut}_F(\operatorname{End}\Gamma)$, and let $t = f \circ \widehat{\rho} \circ f^{-1}$, $t' = f' \circ \widehat{\rho} \circ f'^{-1} \in \operatorname{End}\Gamma$ be the ρ-semilinear transformations such that $\tau_{(\Sigma, f)} = \operatorname{Int}(t)|_{\operatorname{End}_L V}$ and $\tau_{(\Sigma', f')} =$

$\mathrm{Int}(t')|_{\mathrm{End}_L V}$. By Lemma 3.4 we may find an isotopy $(\nu, u)\colon \Gamma \to \Gamma$ such that $\varphi = \mathrm{Int}(u)|_{\mathrm{End}_L V}$. The equation $\tau_{(\Sigma',f')} = \varphi \circ \tau_{(\Sigma,f)} \circ \varphi^{-1}$ then yields $\mathrm{Int}(t')|_{\mathrm{End}_L V} = \mathrm{Int}(u \circ t \circ u^{-1})|_{\mathrm{End}_L V}$, hence there exists $\xi \in L^\times$ such that $u \circ t \circ u^{-1} = \xi t'$. Because $t^3 = {t'}^3 = \mathrm{Id}_V$, we have $N_{L/F}(\xi) = 1$, hence Hilbert's Theorem 90 yields $\eta \in L^\times$ such that $\xi = \rho(\eta)\eta^{-1}$. Then $\eta^{-1}u\colon \Gamma \to \Gamma$ is a ν-semilinear isotopy such that $(\eta^{-1}u) \circ t \circ (\eta^{-1}u)^{-1} = t'$, and we have a commutative diagram

$$\begin{array}{ccc} \Sigma \otimes (L,\rho) & \xrightarrow{{f'}^{-1}\circ(\eta^{-1}u)\circ f} & \Sigma' \otimes (L,\rho) \\ \Big\downarrow \widehat{\rho} & & \Big\downarrow \widehat{\rho} \\ \Sigma \otimes (L,\rho) & \xrightarrow{{f'}^{-1}\circ(\eta^{-1}u)\circ f} & \Sigma' \otimes (L,\rho) \end{array}$$

The restriction of ${f'}^{-1} \circ (\eta^{-1}u) \circ f$ to Σ is an isotopy of symmetric compositions $\Sigma \to \Sigma'$; a scalar multiple of this map is an isomorphism $\Sigma \xrightarrow{\sim} \Sigma'$. □

4. Trialitarian automorphisms of groups of type D_4

Let F be a field of characteristic different from 2. By [9, (44.8)], for every adjoint simple group G of type D_4 over F there is a trialitarian algebra $T = (E, L, \sigma, \alpha)$ such that G is isomorphic to $\mathbf{Aut}_L(T)$.

Proposition 4.1. *The natural map $\Phi : \mathbf{Aut}_F(T) \to \mathbf{Aut}(G)$ induced by conjugation is an isomorphism of group schemes.*

Proof. The group G is the connected component of $\mathbf{Aut}_F(T)$ by construction. By [4, Exp. XXIV, Th. 1.3], the group $\mathbf{Aut}(G)$ is a smooth algebraic group scheme, and the conjugation homomorphism Φ is a homomorphism of algebraic groups. Since G is adjoint semisimple the restriction of Φ to the connected component is an injective homomorphism $G \to \mathbf{Aut}(G)$, hence by [9, (22.2)] the differential $d\Phi$ is injective. On the other hand, since the correspondence between trialitarian algebras and adjoint simple groups of type D_4 is actually shown in [9, (44.8)] to be an equivalence of groupoids, over an algebraic closure F_{alg} the map $\Phi_{\mathrm{alg}}\colon \mathbf{Aut}_F(T)(F_{\mathrm{alg}}) \to \mathbf{Aut}(G)(F_{\mathrm{alg}})$ is an isomorphism. By [9, (22.5)] it follows that Φ is an isomorphism of group schemes. □

We thus have a commutative diagram with exact rows:

$$(9) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{Aut}_L(T) & \longrightarrow & \mathbf{Aut}_F(T) & \longrightarrow & \mathbf{Aut}_F(L) & \longrightarrow & 1 \\ & & \Big\| & & \Big\downarrow \Phi & & \Big\| & & \\ 1 & \longrightarrow & G & \longrightarrow & \mathbf{Aut}(G) & \xrightarrow{\pi} & (\mathfrak{S}_3)_L & \longrightarrow & 1 \end{array}$$

where $(\mathfrak{S}_3)_L$ is a (non-constant) twisted form of the symmetric group $\mathfrak{S}_3$. Here $\mathbf{Aut}_F(L)$ is the group scheme given by $\mathbf{Aut}_F(L)(R) = \mathrm{Aut}_{R\text{-alg}}(L \otimes_F R)$ for any commutative F-algebra R. Thus, the type of the group G is related as follows to the type of L and to $\mathbf{Aut}_F(L)$:

(i) type $^1\mathsf{D}_4$: $L \cong F \times F \times F$ and $\mathbf{Aut}_F(L)(F) \cong \mathfrak{S}_3$;
(ii) type $^2\mathsf{D}_4$: $L \cong F \times \Delta$ (with Δ a quadratic field extension of F) and $\mathbf{Aut}_F(L)(F) \cong \mathfrak{S}_2$;
(iii) type $^3\mathsf{D}_4$: L a cyclic cubic field extension of F and $\mathbf{Aut}_F(L)(F) \cong \mathbb{Z}/3\mathbb{Z}$;
(iv) type $^6\mathsf{D}_4$: L a non-cyclic cubic field extension of F and $\mathbf{Aut}_F(L)(F) = 1$.

PROPOSITION 4.2. *Let G be an adjoint simple group of type D_4 over F. If $\mathbf{Aut}(G)(F)$ contains an outer automorphism φ such that φ^3 is inner, then G is of type $^1\mathsf{D}_4$ or $^3\mathsf{D}_4$, and in the trialitarian algebra $T = (E, L, \sigma, \alpha)$ such that $G \cong \mathbf{Aut}_L(T)$, the central simple L-algebra E is split.*

Proof. The exactness of the bottom row of (9) implies the exactness of

$$1 \longrightarrow G(F) \longrightarrow \mathbf{Aut}(G)(F) \xrightarrow{\pi} (\mathfrak{S}_3)_L(F) \tag{10}$$

Since the image $\pi(\varphi) \in (\mathfrak{S}_3)_L(F)$ has order 3, $\mathbf{Aut}_F(L)(F)$ must be isomorphic to $\mathfrak{S}_3$ or to $\mathbb{Z}/3\mathbb{Z}$ and hence the cases $^2\mathsf{D}_4$ and $^6\mathsf{D}_4$ can be ruled out from the characterization of the various types above. Therefore the type of G is $^1\mathsf{D}_4$ or $^3\mathsf{D}_4$. If G is of type $^1\mathsf{D}_4$, then the algebra E is split by [6, Example 17] or by [2, Theorem 13.1]. If G is of type $^3\mathsf{D}_4$, then after scalar extension to L the group G_L has type $^1\mathsf{D}_4$, so $E \otimes_F L$ is split. Therefore, the Brauer class of E has 3-torsion since it is split by a cubic extension. But it also has 2-torsion since E carries an orthogonal involution, hence E is split. □

For the rest of this section, we focus on trialitarian automorphisms (i.e., outer automorphisms of order 3) of groups of type $^3\mathsf{D}_4$. Let G be an adjoint simple group of type $^3\mathsf{D}_4$ over F, and let L be its associated cyclic cubic field extension of F. Thus,

$$(\mathfrak{S}_3)_L(F) = \mathrm{Gal}(L/F) \cong \mathbb{Z}/3\mathbb{Z}.$$

If G carries a trialitarian automorphism φ defined over F, then the map $\pi\colon \mathbf{Aut}(G)(F) \to \mathrm{Gal}(L/F)$ is a split surjection, hence $\mathbf{Aut}(G)(F) \cong G(F) \rtimes (\mathbb{Z}/3\mathbb{Z})$. Therefore, it is easy to see that for any other trialitarian automorphism φ' of G defined over F, the elements φ and φ' are conjugate in $\mathbf{Aut}(G)(F)$ if and only if there exists $g \in G(F)$ such that $\varphi' = \mathrm{Int}(g) \circ \varphi \circ \mathrm{Int}(g)^{-1}$. When this occurs, we have $\pi(\varphi) = \pi(\varphi')$.

THEOREM 4.3. (i) *Let G be an adjoint simple group of type $^3\mathsf{D}_4$ over F. The group G carries a trialitarian automorphism defined over F if and only if the trialitarian algebra $T = (E, L, \sigma, \alpha)$ (unique up to isomorphism) such that $G \cong \mathbf{Aut}_L(T)$ has the form $T \cong \mathrm{End}\,\Gamma$ for some induced cyclic composition Γ.*

(ii) *Let $G = \mathbf{Aut}_L(\mathrm{End}\,\Gamma)$ for some induced cyclic composition Γ. Every trialitarian automorphism φ of G has the form $\varphi = \mathrm{Int}(\tau)$ for some uniquely determined F-automorphism τ of $\mathrm{End}\,\Gamma$ such that $\tau^3 = \mathrm{Id}$ and $\tau|_L = \pi(\varphi)$. For a given nontrivial $\rho \in \mathrm{Gal}(L/F)$, the assignment $\Sigma \mapsto \mathrm{Int}(\tau_{(\Sigma,f)})$ defines a bijection between the isomorphism classes* *of symmetric compositions for which there exists an L-linear isotopy $f\colon \Sigma\otimes(L,\rho)\to\Gamma$ and conjugacy classes in $\mathbf{Aut}(G)(F)$ of trialitarian automorphisms φ of G such that $\pi(\varphi)=\rho$.*

Proof. Suppose first that φ is a trialitarian automorphism of G, and let $G=\mathbf{Aut}_L(T)$ for some trialitarian algebra $T=(E,L,\sigma,\alpha)$. Proposition 4.2 shows that the central simple L-algebra E is split, hence by [9, (44.16), (36.12)], we have $T=\operatorname{End}\Gamma$ for some cyclic composition $\Gamma=(V,L,Q,\rho,*)$ over F. Substituting φ^2 for φ if necessary, we may assume $\pi(\varphi)=\rho$. The preimage of φ under the isomorphism $\Phi_F\colon \mathbf{Aut}_F(T)(F)\xrightarrow{\sim}\mathbf{Aut}(G)(F)$ (from (9)) is an F-automorphism τ of T such that $\varphi=\operatorname{Int}(\tau)$, $\tau^3=\mathrm{Id}$, and $\tau|_L=\rho$. Since Φ_F is a bijection, τ is uniquely determined by φ. By Theorem 3.1(ii), the existence of τ implies that the cyclic composition Γ is induced.
Conversely, if Γ is induced, then by Theorem 3.1(i), the trialitarian algebra $\operatorname{End}\Gamma$ carries automorphisms τ such that $\tau^3=\mathrm{Id}$ and $\tau|_L\neq\mathrm{Id}_L$. For any such τ, conjugation by τ is a trialitarian automorphism of G.
The last statement in (ii) readily follows from Theorem 3.5 because trialitarian automorphisms $\operatorname{Int}(\tau)$, $\operatorname{Int}(\tau')$ are conjugate in $\mathbf{Aut}(G)(F)$ if and only if τ, τ' are conjugate in $\operatorname{Aut}_F(\operatorname{End}\Gamma)$. □

The following proposition shows that the algebraic subgroup of fixed points under a trialitarian automorphism of the form $\operatorname{Int}(\tau_{(\Sigma,f)})$ is isomorphic to $\mathbf{Aut}(\Sigma)$, hence in characteristic different from 2 and 3 it is a simple adjoint group of type G_2 or A_2, in view of the classification of symmetric compositions (see [3, §9]).

Proposition 4.4. *Let $G=\mathbf{Aut}_L\big(\operatorname{End}(\Sigma\otimes(L,\rho))\big)$ for some symmetric composition $\Sigma=(S,n,\star)$ over F and some cyclic cubic field extension L/F with nontrivial automorphism ρ. The subgroup of G fixed under the trialitarian automorphism $\operatorname{Int}(\widehat{\rho})$ is canonically isomorphic to $\mathbf{Aut}(\Sigma)$.*

Proof (Sketch). Mimicking the construction of the map α_* in (4), we may use the product $\star$ to construct an F-algebra isomorphism

$$\alpha_\star\colon C(S,n)\xrightarrow{\sim}\operatorname{End}_F(S\oplus S)$$

such that $\alpha_\star(x)(y,z)=(z\star x,x\star y)$ for x, y, $z\in S$. This isomorphism restricts to an isomorphism

$$\alpha_{\star 0}\colon C_0(S,n)\xrightarrow{\sim}(\operatorname{End}_F S)\times(\operatorname{End}_F S).$$

Let $\mathbf{Aut}(\operatorname{End}\Sigma)$ be the group scheme whose rational points are the F-algebra automorphisms φ of $(\operatorname{End}_F S,\sigma_n)$ that make the following diagram commute:

$$\begin{array}{ccc}
C(\operatorname{End}_F S,\sigma_n) & \xrightarrow{\alpha_{\star 0}} & (\operatorname{End}_F S)\times(\operatorname{End}_F S)\\
{\scriptstyle C(\varphi)}\downarrow & & \downarrow{\scriptstyle \varphi\times\varphi}\\
C(\operatorname{End}_F S,\sigma_n) & \xrightarrow{\alpha_{\star 0}} & (\operatorname{End}_F S)\times(\operatorname{End}_F S)
\end{array}$$

Arguing as in Lemma 3.4, one proves that every such automorphism has the form $\operatorname{Int}(u)$ for some isotopy u of Σ. But if u is an isotopy of Σ with multiplier μ, then $\mu^{-1}u$ is an automorphism of Σ. Therefore, mapping every automorphism u of Σ to $\operatorname{Int}(u)$ yields an isomorphism $\mathbf{Aut}(\Sigma) \xrightarrow{\sim} \mathbf{Aut}(\operatorname{End}\Sigma)$. The extension of scalars from F to L yields an isomorphism

$$\mathbf{PGL}(S) \xrightarrow{\sim} R_{L/F}\big(\mathbf{PGL}(S \otimes_F L)\big)^{\operatorname{Int}(\widehat{\rho})},$$

which carries the subgroup $\mathbf{Aut}(\operatorname{End}\Sigma)$ to $G^{\operatorname{Int}(\widehat{\rho})}$. □

To conclude, we briefly mention without proof the analogue of Theorem 4.3 for simply connected groups, which we could have considered instead of adjoint groups. (Among simple algebraic groups of type D_4, only adjoint and simply connected groups may admit trialitarian automorphisms.)

Theorem 4.5. (i) *For any cyclic composition $\Gamma = (V, L, Q, \rho, *)$ over F, with L a field, the group $\mathbf{Aut}_L(\Gamma)$ is simple simply connected of type $^3\mathsf{D}_4$, and there is an exact sequence of algebraic groups*

$$1 \longrightarrow \boldsymbol{\mu}_2^2 \longrightarrow \mathbf{Aut}_L(\Gamma) \xrightarrow{\operatorname{Int}} \mathbf{Aut}_L(\operatorname{End}\Gamma) \longrightarrow 1.$$

(ii) *A simple simply connected group of type $^3\mathsf{D}_4$ admits trialitarian automorphisms defined over F if and only if it is isomorphic to the automorphism group of an induced symmetric composition $\Gamma = (V, L, Q, \rho, *)$, with L a field. Conjugacy classes of trialitarian automorphisms of $\mathbf{Aut}_L(\Gamma)$ defined over F are in bijection with isomorphism classes of symmetric compositions Σ for which there is an isotopy $\Sigma \otimes (L, \rho) \to \Gamma$.*

Theorems 4.3 and 4.5 apply in particular to show that over a finite field of characteristic different from 2 and 3, every simple adjoint or simply connected group of type $^3\mathsf{D}_4$ admits trialitarian automorphisms. This follows because the Allen invariant is trivial and cyclic compositions are reduced, see [12, §4.8]. Note that the property holds without restriction on the characteristic (needed for the arguments in [12, §4.8]), and is a particular case of a more general result: every simple adjoint or simply connected linear algebraic group over a finite field is quasi-split by a theorem of Lang [10, Prop. 6.1], and therefore $\operatorname{Aut}(G)$ is a semidirect product, see [4, Exp. XXIV, 3.10] or [9, (31.4)].[4]

Examples 4.6. (i) Let $F = \mathbb{F}_q$ be the field with q elements, where q is odd and $q \equiv 1 \bmod 3$. As observed in Example 2.2(i), every symmetric composition over F is isomorphic either to the Okubo composition Σ or to the split para-Cayley composition $\widetilde{C}$, and (up to isomorphism) there is a unique cyclic composition $\Gamma \cong \widetilde{C} \otimes (L, \rho) \cong \Sigma \otimes (L, \rho)$ with cubic algebra (L, ρ). Therefore, the simply connected group $\mathbf{Aut}_L(\Gamma)$ and the adjoint group $\mathbf{Aut}_L(\operatorname{End}\Gamma)$ have exactly two conjugacy classes of trialitarian automorphisms defined over F. See also [8, (9.1)].

[4]We are indebted to Skip Garibaldi for this observation.

(ii) Example 2.2(ii) describes a cyclic composition induced by a unique (up to isomorphism) symmetric composition. Its automorphism group is a group of type ${}^3\mathsf{D}_4$ admitting a unique conjugacy class of trialitarian automorphisms.

(iii) In contrast to (i) and (ii) we get from Example 2.2(iii) examples of groups of type ${}^3\mathsf{D}_4$ with many conjugacy classes of trialitarian automorphisms.

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Max-Albert Knus
Department Mathematik
ETH Zentrum
CH-8092 Zürich
Switzerland
knus@math.ethz.ch

Jean-Pierre Tignol
ICTEAM Institute
Université catholique de Louvain
B-1348 Louvain-la-Neuve Belgium
jean-pierre.tignol@uclouvain.be