Rationally Isotropic Exceptional Projective Homogeneous Varieties Are Locally Isotropic

I. Panin, V. Petrov[1]

Received: September 20, 2014
Revised: May 31, 2015

Abstract. Assume that R is a regular local ring that contains an infinite field and whose field of fractions K has charactertistic $\neq 2$. Let X be an exceptional projective homogeneous scheme over R. We prove that in most cases the condition $X(K) \neq \emptyset$ implies $X(R) \neq \emptyset$.

2010 Mathematics Subject Classification: 14M17, 20G35
Keywords and Phrases: projective homogeneous varieties, rational points, exceptional groups

1. Introduction

The main result of the present article extends the main results of [Pa3] and [PP] to the case of exceptional groups. In the latter paper one can find historical remarks which might help the general reader. All the rings in the present paper are *commutative* and *Noetherian*. We prove the following theorem.

Theorem 1. *Let R be a regular local ring that contains an infinite field and whose field of fractions K has characteristic $\neq 2$. Let G be a split simple group of exceptional type (that is, E_6, E_7, E_8, F_4, or G_2), P be a parabolic subgroup of G, $[\xi]$ be a class from $\mathrm{H}^1(R,G)$, and $X = (G/P)_\xi$ be the corresponding homogeneous space over R. Assume that $P \neq P_7, P_8, P_{7,8}$ in case $G = E_8$, $P \neq P_7$ in case $G = E_7$, and $P \neq P_1$ in case $G = E_7^{ad}$. Then the condition $X(K) \neq \emptyset$ implies $X(R) \neq \emptyset$.*

The results of the present paper depend on the following yet unpublished results: [FP, Corollary of Theorem 1] and [Pa, Theorem 10.0.30].

[1]The research is supported by Russian Science Foundation grant 14-11-00456

2. Purity of some H^1 functors

Let R be a commutative noetherian domain of finite Krull dimension with a fraction field F. We say that a functor $\mathcal{F}$ from the category of commutative R-algebras to the category of sets *satisfies purity* for R if we have

$$\operatorname{Im}[\mathcal{F}(R) \to \mathcal{F}(F)] = \bigcap_{\operatorname{ht}\mathfrak{p}=1} \operatorname{Im}[\mathcal{F}(R_{\mathfrak{p}}) \to \mathcal{F}(F)].$$

An element $a \in \mathcal{F}(F)$ *is called R-unramified* if it belongs to $\bigcap_{\operatorname{ht}\mathfrak{p}=1} \operatorname{Im}[\mathcal{F}(R_{\mathfrak{p}}) \to \mathcal{F}(F)]$. If $\mathfrak{p}$ is a height one prime ideal in R, the element *a is called $\mathfrak{p}$-unramified*, if it belongs to $\operatorname{Im}[\mathcal{F}(R_{\mathfrak{p}}) \to \mathcal{F}(F)]$.
If $\mathcal{H}$ is an étale group sheaf we write $\mathrm{H}^i(-,\mathcal{H})$ for $\mathrm{H}^i_{\text{ét}}(-,\mathcal{H})$ below through the text.
The following theorem is proven in the characteristic zero case [Pa2, Theorem 4.0.3]. We extend it here to reductive group schemes. Let R be a commutative noetherian ring. Recall that an R-group scheme G is called *reductive*, if it is affine and smooth as an R-scheme and if, moreover, for each algebraically closed field Ω and for each ring homomorphism $R \to \Omega$ the scalar extension G_Ω is a connected reductive algebraic group over Ω. This definition of a reductive R-group scheme coincides with [SGA, Exp. XIX, Definition 2.7].

Theorem 2. *Let R be the local ring of a closed point on a smooth scheme over an infinite field. Let G be a reductive R-group scheme. Let $i\colon Z \hookrightarrow G$ be a closed subgroup scheme of the center* $\operatorname{Cent}(G)$. *It is known that Z is of multiplicative type. Let $G' = G/Z$ be the factor group, $\pi\colon G \to G'$ be the projection.*
If the functor $\mathrm{H}^1(-,G')$ *satisfies purity for R, then the functor* $\mathrm{H}^1(-,G)$ *satisfies purity for R as well.*

It is known that π is surjective and strictly flat. Thus the exact sequence of R-group schemes

$$(*) \qquad \{1\} \to Z \xrightarrow{i} G \xrightarrow{\pi} G' \to \{1\}$$

induces an exact sequence of group sheaves in the fppf-topology.

Lemma 1. *Consider the category of R-algebras. The functor*

$$R' \mapsto \mathcal{F}(R') = \mathrm{H}^1_{\mathrm{fppf}}(R', Z)/\operatorname{Im}(\delta_{R'}),$$

where δ is the connecting homomorphism associated to sequence (*), *satisfies purity for R.*

Proof. The lemma coincides with [Pa, Theorem 10.0.30]. □

Lemma 2. *The map*

$$\mathrm{H}^2_{\mathrm{fppf}}(R, Z) \to \mathrm{H}^2_{\mathrm{fppf}}(K, Z)$$

is injective.

Proof. See [C-TS, Theorem 4.3]. □

Proof of Theorem 2. Reproduce the diagram chase from the proof of [Pa2, Theorem 4.0.3]. For this purpose consider the commutative diagram

$$\begin{array}{ccccccccc}
\{1\} & \longrightarrow & \mathcal{F}(K) & \xrightarrow{\delta_K} & \mathrm{H}^1(K,G) & \xrightarrow{\pi_K} & \mathrm{H}^1(K,G') & \xrightarrow{\Delta_K} & \mathrm{H}^2_{\mathrm{fppf}}(K,Z) \\
 & & \uparrow & & \uparrow & & \uparrow & & \alpha\uparrow \\
\{1\} & \longrightarrow & \mathcal{F}(R) & \xrightarrow{\delta} & \mathrm{H}^1(R,G) & \xrightarrow{\pi} & \mathrm{H}^1(R,G') & \xrightarrow{\Delta} & \mathrm{H}^2_{\mathrm{fppf}}(R,Z)
\end{array}$$

Let $[\xi] \in \mathrm{H}^1(K,G)$ be an R-unramified class and let $[\bar{\xi}] = \pi_K([\xi])$. Clearly, $[\bar{\xi}] \in \mathrm{H}^1(K,G')$ is R-unramified. Thus there exists an element $[\bar{\xi}'] \in \mathrm{H}^1(R,G')$ such that $[\bar{\xi}']_K = [\bar{\xi}]$. The map α is injective by Lemma 2. One has $\Delta([\bar{\xi}']) = 0$, since $\Delta_K([\bar{\xi}]) = 0$. Thus there exists $[\xi'] \in \mathrm{H}^1(R,G)$ such that $\pi([\xi']) = [\bar{\xi}']$. Twisting G by ξ' we may assume that $[\bar{\xi}] = *$, so that $[\xi]$ comes from some $a \in \mathcal{F}(K)$.

LEMMA 3. *The above constructed element $a \in \mathcal{F}(K)$ is R-unramified.*

Assume Lemma 3; we use it to complete the proof of Theorem 2. By Lemma 1 the functor $\mathcal{F}$ satisfies the purity for regular local rings containing the field k. Thus there exists an element $a' \in \mathcal{F}(R)$ with $a'_K = a$. It is clear that $[\delta(a')]_K = [\xi]$. It remains to prove Lemma 3. First we need a small variation of Nisnevich's theorem.

LEMMA 4. *Let H be a reductive group scheme over a discrete valuation ring A. Let K be the fraction field of A. Then the map*

$$\mathrm{H}^1(A,H) \to \mathrm{H}^1(K,H)$$

is injective.

Proof. Let $[\xi_0], [\xi_1]$ be classes from $\mathrm{H}^1(A,H)$. Let $\mathcal{H}_0$ be a principal homogeneous H-bundle representing the class ξ_0. Let H_0 be the inner form of the group scheme H, corresponding to $\mathcal{H}_0$. Let $X = Spec(A)$. For each X-scheme S there is a well-known bijection $\phi_S\colon \mathrm{H}^1(S,H) \to \mathrm{H}^1(S,H_0)$ of non-pointed sets. That bijection takes the principal homogeneous H-bundle $\mathcal{H}_0 \times_X S$ to the trivial principal homogeneous H_0-bundle $H_0 \times_X S$. That bijection is functorial with respect to morphisms of X-schemes.
Assume that $[\xi_0]_K = [\xi_1]_K$. Then one has $* = \phi_K([\xi_0]_K) = \phi_K([\xi_1]_K) \in \mathrm{H}^1(K,H_0)$. The kernel of the map $\mathrm{H}^1(A,H_0) \to \mathrm{H}^1(K,H_0)$ is trivial by Nisnevich's theorem [Ni]. Thus $\phi_A([\xi]_1) = * = \phi_A([\xi]_0) \in \mathrm{H}^1(A,H_0)$. Whence $[\xi]_1 = [\xi]_0 \in \mathrm{H}^1(A,H)$. □

Now we go back to the proof of Lemma 3. Consider a height 1 prime ideal $\mathfrak{p}$ in R. Since $[\xi]$ is R-unramified there exists its lift up to an element $[\tilde{\xi}]$ in $\mathrm{H}^1(R_{\mathfrak{p}},G)$.
The map

$$\mathrm{H}^1(R_{\mathfrak{p}},G') \to \mathrm{H}^1(K,G')$$

is injective by Lemma 4. But

$$(\pi_{\mathfrak{p}}([\tilde{\xi}]))_K = \pi_K([\xi]) = *,$$

so $\pi_{\mathfrak{p}}[\tilde{\xi}] = *$. Therefore there exists a unique class $a_{\mathfrak{p}} \in \mathcal{F}(R_{\mathfrak{p}})$ such that $\delta(a_{\mathfrak{p}}) = [\tilde{\xi}] \in \mathrm{H}^1(R_{\mathfrak{p}}, G)$. So, $\delta_K(a_{\mathfrak{p},K}) = [\xi] \in \mathrm{H}^1(K, G)$ and finally $a = a_{\mathfrak{p},K}$. Lemma 3 is proven and Theorem 2 is proven as well. □

3. Purity of some H^1 functors, continued

Theorem 3. *Let R be such as in Theorem 1. The functor $\mathrm{H}^1(-, \mathrm{PGL}_n)$ satisfies purity for R.*

Proof. Let $[\xi] \in \mathrm{H}^1(K, \mathrm{PGL}_n)$ be an R-unramified element. Let $\delta\colon \mathrm{H}^1(-, \mathrm{PGL}_n) \to \mathrm{H}^2(-, \mathbb{G}_m)$ be the boundary map corresponding to the short exact sequence of étale group sheaves

$$1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1.$$

Let D_ξ be a central simple K-algebra of degree n corresponding ξ. If $D_\xi \cong M_l(D')$ for a skew-field D', then there exists $[\xi'] \in \mathrm{H}^1(K, \mathrm{PGL}_{n'})$ such that $D' = D_{\xi'}$. Then $\delta([\xi']) = [D'] = [D] = \delta(\xi)$. Replacing ξ by ξ', we may assume that $D := D_\xi$ is a central skew-field over K of degree n and the class $[D]$ is R-unramified. Since the functor $\mathrm{H}^2(-, \mathbb{G}_m)$ satisfies purity for R, there exists an Azumaya R-algebra A and an integer d such that $A_K = M_d(D)$.

There exists a projective left A-module P of finite rank such that each projective left A-module Q of finite rank is isomorphic to the left A-module P^m for an appropriative integer m (see [DeM, Cor.2]). In particular, two projective left A-modules of finite rank are isomorphic if they have the same rank as R-modules. One has an isomorphism $A \cong P^s$ of left A-modules for an integer s. Thus one has R-algebra isomorphisms $A \cong \mathrm{End}_A(P^s) \cong \mathrm{M}_s(\mathrm{End}_A(P))$. Set $B = \mathrm{End}_A(P)$. Observe, that $B_K = \mathrm{End}_{A_K}(P_K)$, since P is a finitely generated projective left A-module.

The class $[P_K]$ is a free generator of the group $K_0(A_K) = K_0(M_d(D)) \cong \mathbb{Z}$, since $[P]$ is a free generator of the group $K_0(A)$ and $K_0(A) = K_0(A_K)$. The P_K is a simple A_K-module, since $[P_K]$ is a free generator of $K_0(A_K)$. Thus $\mathrm{End}_{A_K}(P_K) = B_K$ is a skew-field.

We claim that the K-algebras B_K and D are isomorphic. In fact, $A_K = M_r(B_K)$ for an integer r, since P_K is a simple A_K-module. From the other side $A_K = M_d(D)$. As D, so B_K are skew-fields. Thus $r = d$ and D is isomorphic to B_K as K-algebras.

We claim further that B is an Azumaya R-algebra. That claim is local with respect to the étale topology on $\mathrm{Spec}(R)$. Thus it suffices to check the claim assuming that $\mathrm{Spec}(R)$ is strictly henselian local ring. In that case $A = M_l(R)$ and $P = (R^l)^m$ as an $M_l(R)$-module. Thus $B = \mathrm{End}_A(P) = M_m(R)$, which proves the claim.

Since B_K is isomorphic to D, one has $m = n$. So, B is an Azumaya R-algebra, and the K-algebra B_K is isomorphic to D. Let $[\zeta] \in \mathrm{H}^1(R, \mathrm{PGL}_n)$ be class corresponding to B. Then $[\zeta]_K = [\xi]$, since $\delta([\zeta])_K = [B_K] = [D] = \delta([\xi]) \in \mathrm{H}^2(K, \mathbb{G}_m)$. □

We denote by Sim_n the group of similitudes of a *split* quadratic form of rank n and by Sim_n^+ its connected component. Recall that $\mathrm{H}^1(-, \mathrm{Sim}_n)$ classifies similarity classes of nondegenerate quadratic forms of rank n (see [KMRT, (29.15)]).

Theorem 4. *Let R be such as in Theorem 1. The functor $\mathrm{H}^1(-, \mathrm{Sim}_n)$ satisfies purity for R.*

Proof. Let $[\xi] \in \mathrm{H}^1(K, \mathrm{Sim}_n)$ be an R-unramified element. Let φ be a quadratic form over K whose similarity class represents $[\xi]$. Diagonalizing φ we may assume that $\varphi = \sum_{i=1}^n f_i \cdot t_i^2$ for certain non-zero elements $f_1, f_2, \dots, f_n \in K$. For each i write f_i in the form $f_i = \frac{g_i}{h_i}$ with $g_i, h_i \in R$ and $h_i \neq 0$.
There are only finitely many height one prime ideals $\mathfrak{q}$ in R such that there exists $0 \leq i \leq n$ with f_i not in $R_\mathfrak{q}$. Let $\mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_s$ be all height one prime ideals in R with that property and let $\mathfrak{q}_i \neq \mathfrak{q}_j$ for $i \neq j$.
For all other height one prime ideals $\mathfrak{p}$ in R each f_i belongs to the group of units $R_\mathfrak{p}^\times$ of the ring $R_\mathfrak{p}$.
If $\mathfrak{p}$ is a height one prime ideal of R which is not from the list $\mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_s$, then $\varphi = \sum_{i=1}^n f_i \cdot t_i^2$ may be regarded as a quadratic space over $R_\mathfrak{p}$. We will write ${}_\mathfrak{p}\varphi$ for that quadratic space over $R_\mathfrak{p}$. Clearly, one has $({}_\mathfrak{p}\varphi) \otimes_{R_\mathfrak{p}} K = \varphi$ as quadratic spaces over K.
For each $j \in \{1, 2, \dots, s\}$ choose and fix a quadratic space ${}_j\varphi$ over $R_{\mathfrak{q}_j}$ and a non-zero element $\lambda_j \in K$ such that the quadratic spaces $({}_j\varphi) \otimes_{R_{\mathfrak{q}_j}} K$ and $\lambda_j \cdot \varphi$ are isomorphic over K. The ring R is factorial since it is regular and local. Thus for each $j \in \{1, 2, \dots, s\}$ we may choose an element $\pi_j \in R$ such that firstly π_j generates the only maximal ideal in $R_{\mathfrak{q}_j}$ and secondly π_j is an invertible element in $R_\mathfrak{n}$ for each height one prime ideal $\mathfrak{n}$ different from the ideal $\mathfrak{q}_j$.
Let $v_j \colon K^\times \to \mathbb{Z}$ be the discrete valuation of K corresponding to the prime ideal $\mathfrak{q}_j$. Set $\lambda = \prod_{i=1}^s \pi_j^{v_j(\lambda_j)}$ and

$$\varphi_{new} = \lambda \cdot \varphi.$$

Claim. The quadratic space φ_{new} is R-unramified. In fact, if a height one prime ideal $\mathfrak{p}$ is different from each of $\mathfrak{q}_j$'s, then $v_\mathfrak{p}(\lambda) = 0$. Thus, $\lambda \in R_\mathfrak{p}^\times$. In that case $\lambda \cdot ({}_\mathfrak{p}\varphi)$ is a quadratic space over $R_\mathfrak{p}$ and moreover one have isomorphisms of quadratic spaces $(\lambda \cdot ({}_\mathfrak{p}\varphi)) \otimes_{R_\mathfrak{p}} K = \lambda \cdot \varphi = \varphi_{new}$. If we take one of $\mathfrak{q}_j$'s, then $\frac{\lambda}{\lambda_j} \in R_{\mathfrak{q}_j}^\times$. Thus, $\frac{\lambda}{\lambda_j} \cdot ({}_j\varphi)$ is a quadratic space over $R_{\mathfrak{q}_j}$. Moreover, one has

$$\frac{\lambda}{\lambda_j} \cdot ({}_j\varphi) \otimes_{R_\mathfrak{q}} K = \frac{\lambda}{\lambda_j} \cdot \lambda_j \cdot \varphi = \varphi_{new}.$$

The Claim is proven.
By [PP, Corollary 3.1] there exists a quadratic space $\tilde{\varphi}$ over R such that the quadratic spaces $\tilde{\varphi} \otimes_R K$ and φ_{new} are isomorphic over K. This shows that the

similarity classes of the quadratic spaces $\tilde{\varphi} \otimes_R K$ and φ coincide. The theorem is proven. $\square$

Theorem 5. *Let R be such as in Theorem 1. The functor* $\mathrm{H}^1(-, \mathrm{Sim}_n^+)$ *satisfies purity for R.*

Proof. Consider an element $[\xi] \in \mathrm{H}^1(K, \mathrm{Sim}_n^+)$ such that for any $\mathfrak{p}$ of height 1 $[\xi]$ comes from $[\xi_{\mathfrak{p}}] \in \mathrm{H}^1(R_{\mathfrak{p}}, \mathrm{Sim}_n^+)$. Then the image of $[\xi]$ in $\mathrm{H}^1(K, \mathrm{Sim}_n)$ by Theorem 4 comes from some $[\zeta] \in \mathrm{H}^1(R, \mathrm{Sim}_n)$. We have a short exact sequence

$$1 \to \mathrm{Sim}_n^+ \to \mathrm{Sim}_n \to \mu_2 \to 1,$$

and $R^\times/(R^\times)^2$ injects into $K^\times/(K^\times)^2$. Thus the element $[\zeta]$ comes actually from some $[\zeta'] \in \mathrm{H}^1(R, \mathrm{Sim}_n^+)$. It remains to show that the map

$$\mathrm{H}^1(K, \mathrm{Sim}_n^+) \to \mathrm{H}^1(K, \mathrm{Sim}_n)$$

is injective, or, by twisting, that the map

$$\mathrm{H}^1(K, \mathrm{Sim}^+(q)) \to \mathrm{H}^1(K, \mathrm{Sim}(q))$$

has trivial kernel. The latter follows from the fact that the map

$$\mathrm{Sim}(q)(K) \to \mu_2(K)$$

is surjective (indeed, any reflection goes to $-1 \in \mu_2(K)$). $\square$

4. Proof Theorem 1

Till the end of the proof of Lemma 9 we suppose that R is the local ring of a closed point on a smooth scheme over an infinite field. Let $[\xi]$ be a class from $\mathrm{H}^1(R, G)$, and $X = (G/P)_\xi$ be the corresponding homogeneous space. Denote by L a Levi subgroup of P.

Lemma 5. *Consider a parabolic subgroup P_1 in PGO_n^+, which is the stabilizer of an isotropic line. A Levi subgroup of P_1 is isomorphic to* Sim_{n-2}^+.

Proof. Is is clear from the matrix representation that a Levi subgroup of a parabolic subgroup P_1 in O_n^+ is isomorphic to $\mathrm{O}_{n-2}^+ \times \mathbb{G}_m$. Now the homomorphism

$$\mathrm{O}_{n-2}^+ \times \mathbb{G}_m \to \mathrm{Sim}_{n-2}^+$$

induced by the natural inclusions is surjective in the sense of groups schemes, and its kernel is μ_2. The claim follows. $\square$

Recall that a subset Ψ of a root system Φ is called *closed* if for any $\alpha, \beta \in \Psi$ such that $\alpha + \beta \in \Phi$ we have $\alpha + \beta \in \Psi$.

Lemma 6. *Let L modulo its center be isomorphic to* PGO_{2m}^+ *(resp.,* PGO_{2m+1}^+ *or* $\mathrm{PGO}_{2m}^+ \times \mathrm{PGL}_2$*). Denote by Φ the root system of G with respect to T, and by Ψ the root system of L with respect to T, where T is a maximal split torus in L. Assume that there is a root $\lambda \in \Phi$ such that the smallest closed set of roots Ψ' containing Ψ and $\pm\lambda$ is a root subsystem of type D_{m+1} (resp. B_{m+1} or $D_{m+1} + A_1$), and Ψ is the standard subsystem of type D_m (resp.*

B_m or $D_m + A_1$) therein. Then there is a surjective map $L \to \mathrm{Sim}^+_{2m}$ (resp., $L \to \mathrm{Sim}^+_{2m+1}$ or $L \to \mathrm{Sim}^+_{2m} \times \mathrm{PGL}_2$) whose kernel is a central closed subgroup scheme in L. In particular, the functor $\mathrm{H}^1(-, L)$ satisfies purity for R.

Proof. Consider the subgroup $H_{\Psi'}$ of G corresponding to Ψ' in the sense of [SGA, Exp. XXII, Definition 5.4.2]. Then $H_{\Psi'}$ is split reductive of type D_{m+1} (resp. B_{m+1} or $D_{m+1} + A_1$) by [SGA, Exp. XXII, Proposition 5.10.1], so it maps onto the split adjoint group of the same type. Under this map L maps onto a Levi subgroup of a parabolic subgroup P_1, which is isomorphic to Sim^+_{2m} (resp. Sim^+_{2m+1} or $\mathrm{Sim}^+_{2m} \times \mathrm{PGL}_2$) by Lemma 5. The purity claim follows from Theorem 5, Theorem 3 and Theorem 2. □

Lemma 7. *For any semi-local R-algebra S the map*

$$\mathrm{H}^1(S, L) \to \mathrm{H}^1(S, G)$$

is injective. Moreover, $X(S) \neq \emptyset$ if and only if $[\xi]_S$ comes from $\mathrm{H}^1(S, L)$.

Proof. See [SGA, Exp. XXVI, Cor. 5.10]. □

Lemma 8. *Assume that the functor $\mathrm{H}^1(-, L)$ satisfies purity for R. Then $X(K) \neq \emptyset$ implies $X(R) \neq \emptyset$.*

Proof. By Lemma 7 $[\xi]_K$ comes from some $[\zeta] \in \mathrm{H}^1(K, L)$, which is uniquely determined. Since X is smooth projective, for any prime ideal $\mathfrak{p}$ of height 1 we have $X(R_\mathfrak{p}) \neq \emptyset$. By Lemma 7 $\xi_{R_\mathfrak{p}}$ comes from some $[\zeta_\mathfrak{p}] \in \mathrm{H}^1(R_\mathfrak{p}, L)$. Now $[\zeta_\mathfrak{p}]_K = [\zeta]$, and so by the purity assumption there is $[\zeta'] \in \mathrm{H}^1(R, L)$ such that $[\zeta']_K = [\zeta]$.
Set $[\xi']$ to be the image of ζ' in $\mathrm{H}^1(R, G)$. We claim that $[\xi'] = [\xi]$. Indeed, by the construction $[\xi']_K = [\xi]_K$. It remains to recall that the map $\mathrm{H}^1(R, G) \to \mathrm{H}^1(K, G_K)$ is injective by [FP, Corollary of Theorem 1]. □

Lemma 9. *Let $Q \leq P$ be another parabolic subgroup, $Y = (G/Q)_\xi$. Assume that $X(K) \neq \emptyset$ implies $Y(K) \neq \emptyset$, and $Y(K) \neq \emptyset$ implies $Y(R) \neq \emptyset$. Then $X(K) \neq \emptyset$ implies $X(R) \neq \emptyset$.*

Proof. Indeed, there is a map $Y \to X$, so $Y(R) \neq \emptyset$ implies $X(R) \neq \emptyset$. □

Proof of Theorem 1. We first suppose that R is the local ring of a closed point on a smooth scheme over an infinite field. By Lemma 9 we may assume that P_K is a minimal parabolic subgroup of $(G_\xi)_K$. All possible types of such P_K are listed in [T, Table II]: the Dynkin diagram with circled vertices erased corresponds to the type of L. We show case by case that $\mathrm{H}^1(-, L)$ satisfies purity for R, hence we are in the situation of Lemma 8.
If $P = B$ is the Borel subgroup, obviously $\mathrm{H}^1(S, L) = \{*\}$ for any semi-local R-algebra S. In the case of index $E^9_{7,4}$ (resp. ${}^1E^{16}_{6,2}$) L modulo its center is isomorphic to $\mathrm{PGL}_2 \times \mathrm{PGL}_2 \times \mathrm{PGL}_2$ (resp. $\mathrm{PGL}_3 \times \mathrm{PGL}_3$), and we may apply Theorem 2 and Theorem 3. In the all other cases we provide an element $\lambda \in \mathrm{X}^*(T)$ such that the assumption of Lemma 6 holds ($\tilde{\alpha}$ stands for the maximal root, enumeration follows [B]). The indices $E^{78}_{7,1}$, $E^{133}_{8,1}$ and $E^{78}_{8,2}$ are

not in the list below since in those cases the L does not belong to one of the type D_m, B_m, $D_m \times A_1$. The index $E^{66}_{7,1}$ is not in the list below since in that case we need a weight λ which is not in the root lattice. So, the indices $E^{78}_{7,1}$, $E^{133}_{8,1}$, $E^{78}_{8,2}$ and $E^{66}_{7,1}$ are the exceptions in the statement of the Theorem.

Index	${}^1E^{28}_{6,2}$	$E^{48}_{7,1}$	$E^{31}_{7,2}$	$E^{28}_{7,3}$	$E^{91}_{8,1}$	$E^{66}_{8,2}$	$E^{28}_{8,4}$	$F^{21}_{4,1}$
λ	α_1	$-\tilde{\alpha}$	α_1	α_1	$-\tilde{\alpha}$	α_8	α_1	$-\tilde{\alpha}$

It remains to settle the case $P = P_1$ for $G = E_7^{sc}$. Denote by $\tilde{E}_7$ a Levi subgroup of a parabolic subgroup P_8 in E_8. Comparing the exact sequences

$$\mathrm{H}^1(R, E_7^{sc}) \to \mathrm{H}^1(R, E_7^{ad}) \to \mathrm{H}^2(R, \mu_2)$$

and

$$\mathrm{H}^1(R, \tilde{E}_7^{sc}) \to \mathrm{H}^1(R, E_7^{ad}) \to \mathrm{H}^2(R, \mathbb{G}_m)$$

and one sees that the image of $[\xi]$ in $\mathrm{H}^1(R, E_7^{ad})$ comes from some $[\zeta] \in \mathrm{H}^1(R, \tilde{E}_7)$. Let $\tilde{P}_1$ denote the corresponding parabolic subgroup in $\tilde{E}_7$; then we have $(E_7^{sc}/P_1)_\xi \simeq (\tilde{E}_7/\tilde{P}_1)_\zeta$.
We claim that $\mathrm{H}^1(-, \tilde{L})$ satisfies purity for R, where $\tilde{L}$ is a Levi subgroup of $\tilde{P}_1$. Indeed, consider a Levi subgroup G' of a parabolic subgroup P_1 inside E_8; then G' has type D_7 and $\tilde{L}$ is a Levi subgroup of a parabolic subgroup P_1 in G'. The rest of the proof goes exactly the same way as in Lemma 6.
Now suppose that R is a regular local ring containing an infinite field k. We first prove a general lemma. Let k' be an infinite field, X be a k'-smooth irreducible affine variety, Denote by $k'[X]$ the ring of regular functions on X and by $k'(X)$ the field of rational functions on X. Let $\mathfrak{p}$ be prime ideal in $k'[X]$, and let $\mathcal{O}_\mathfrak{p}$ be the corresponding local ring.

Lemma 10. *Theorem 1 holds for the local ring $\mathcal{O}_\mathfrak{p}$.*

Proof. Choose a maximal ideal $\mathfrak{m} \subset k'[X]$ containing $\mathfrak{p}$. One has inclusions of k'-algebras $\mathcal{O}_\mathfrak{m} \subset \mathcal{O}_\mathfrak{p} \subset k'(X)$. We already proved Theorem 1 for the ring $\mathcal{O}_\mathfrak{m}$. Thus Theorem 1 holds for the ring $\mathcal{O}_\mathfrak{p}$. □

The rest of the proof of Theorem 1 follows the arguments from [FP, page 5], which we reproduce here. Namely, let $\mathfrak{m}$ be the maximal ideal of R. Let k' be the algebraic closure of the prime field of R in k. Note that k' is perfect. It follows from Popescu's theorem ([P, Sw]) that R is a filtered inductive limit of smooth k'-algebras R_α. Modifying the inductive system R_α if necessary, we can assume that each R_α is integral. There are an index α, a 1-cocycle $\xi_\alpha \in Z^1(R_\alpha, G)$, and an element $f_\alpha \in R_\alpha$ such that $\xi = \varphi_\alpha(\xi_\alpha)$, f is the image of f_α under the homomorphism $\phi_\alpha : R_\alpha \to R$, the homogeneous space $X_\alpha := (G/H)_{\xi_\alpha}$ over R_α has a section over $(R_\alpha)_{f_\alpha}$.
If the field k' is infinite, then set $\mathfrak{p} = \phi_\alpha^{-1}(\mathfrak{m})$. The homomorphism ϕ_α induces a homomorphism of local rings $(R_\alpha)_\mathfrak{p} \to R$. By Lemma 10 one has $X_\alpha(R_\alpha) \neq \emptyset$, whence $X(R) \neq \emptyset$.

If the field k' is finite, then k contains an element t transcendental over k'. Thus R contains the subfield $k'(t)$ of rational functions in the variable t. So, if $R'_\alpha := R_\alpha \otimes_{k'} k'(t)$, then ϕ_α can be decomposed as follows $R_\alpha \xrightarrow{i_\alpha} R_\alpha \otimes_{k'} k'(t) = R'_\alpha \xrightarrow{\psi_\alpha} R$. Let $\xi' = i_\alpha(\xi_\alpha)$, $f'_\alpha = f_\alpha \otimes 1 \in R'_\alpha$, then the homogeneous space $X'_\alpha := (G/H)_{\xi'_\alpha}$ over R'_α has a section over $(R'_\alpha)_{f'_\alpha}$.
Let $\mathfrak{q} = \psi_\alpha^{-1}(\mathfrak{m})$. The ring R'_α is a $k'(t)$-smooth algebra over the infinite field $k'(t)$, and the homogeneous space $X'_\alpha := (G/H)_{\xi'_\alpha}$ over R'_α has a section over $(R'_\alpha)_{f'_\alpha}$. By Lemma 10 one has $X'_\alpha((R'_\alpha)_\mathfrak{q}) \neq \emptyset$. The homomorphism ψ_α can be factored as $R'_\alpha \to (R'_\alpha)_\mathfrak{q} \to R$. Thus $X(R) \neq \emptyset$. □

The authors heartily thank Anastasia Stavrova for discussion on the earlier version of this work.

References

[B] N. Bourbaki, *Groupes et algèbres de Lie. Chapitres 4, 5 et 6*, Masson, Paris, 1981.

[C-TS] J.-L. Colliot-Thélѐne, J.-J. Sansuc, *Principal homogeneous spaces under flasque tori: applications*, J. Algebra 106 (1987), 148–205.

[SGA] M. Demazure, A. Grothendieck, *Schémas en groupes*, Lecture Notes in Mathematics, Vol. 151–153, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[DeM] F.R. DeMeyer, *Projective modules over central separable algebras*, Canad. J. Math. 21 (1969), 39–43.

[Gr2] A. Grothendieck. Le group de Brauer II, in *Dix exposés sur la cohomologique de schémes*, Amsterdam,North-Holland, 1968.

[Gr3] A. Grothendieck. Le group de Brauer III: Exemples et compl'ements, in *Dix exposés sur la cohomologique de schémes*, Amsterdam,North-Holland, 1968.

[FP] R. Fedorov, I. Panin, *A proof of Grothendieck–Serre conjecture on principal bundles over a semilocal regular ring containing an infinite field*, Preprint, April 2013, http://www.arxiv.org/abs/1211.2678v2.

[KMRT] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The Book of Involutions*, AMS Colloquium Pub. 44, Providence, RI, 1998.

[Ni] Y. Nisnevich, *Rationally Trivial Principal Homogeneous Spaces and Arithmetic of Reductive Group Schemes Over Dedekind Rings*, C. R. Acad. Sci. Paris, Série I, 299, no. 1, 5–8 (1984).

[Pa] I. Panin, *On Grothendieck—Serre's conjecture concerning principal G-bundles over reductive group schemes:II*, Preprint (2013), http://www.math.org/0905.1423v3.

[Pa2] I. Panin, *Purity conjecture for reductive groups*, in Russian, Vestnik SPbGU ser. I, no. 1 (2010), 51–56.

[Pa3] I. Panin, *Rationally isotropic quadratic spaces are locally isotropic*, Invent. math. 176 (2009), 397-403.

[PP] I. Panin, K. Pimenov, *Rationally Isotropic Quadratic Spaces Are Locally Isotropic: II*, Documenta Mathematica, Vol. Extra Volume: 5. Andrei A. Suslin's Sixtieth Birthday, P. 515-523, 2010.

[P] D. Popescu. *General Néron desingularization and approximation*, Nagoya Math. Journal, 104 (1986), 85–115.

[Sw] R.G. Swan. *Néron-Popescu desingularization*, Algebra and Geometry (Taipei, 1995), Lect. Algebra Geom. Vol. 2, Internat. Press, Cambridge, MA, 1998, pp. 135–192.

[T] J. Tits, *Classification of algebraic semisimple groups*, Algebraic groups and discontinuous subgroups, Proc. Sympos. Pure Math., 9, Amer. Math. Soc., Providence RI, 1966, 33–62.

I. Panin, V. Petrov
St. Petersburg Department
of the V. A. Steklov Institute of Mathematics
of the Russian Academy of Sciences
27 Fontanka
St. Petersburg
Russia
paniniv@gmail.com
victorapetrov@gmail.com