Note on the Counterexamples for the Integral Tate Conjecture over Finite Fields

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Abstract. In this note we discuss some examples of non-torsion and non-algebraic cohomology classes for varieties over finite fields. The approach follows the construction of Atiyah-Hirzebruch and Totaro.

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1. Introduction

Let k be a finite field and let X be a smooth and projective variety over k. Let ℓ be a prime, $\ell \neq char(k)$. The Tate conjecture [20] predicts that the cycle class map

$$CH^i(X_{\bar{k}}) \otimes \mathbb{Q}_\ell \to \bigcup_H H^{2i}_{\acute{e}t}(X_{\bar{k}}, \mathbb{Q}_\ell(i))^H,$$

where the union is over all open subgroups H of $Gal(\bar{k}/k)$, is surjective.
In the integral version one is interested in the cokernel of the cycle class map

$$CH^i(X_{\bar{k}}) \otimes \mathbb{Z}_\ell \to \bigcup_H H^{2i}_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}_\ell(i))^H. \tag{1.1}$$

This map is not surjective in general: the counterexamples of Atiyah-Hirzebruch [1], revisited by Totaro [21], to the integral version of the Hodge conjecture, provide also counterexamples to the integral Tate conjecture [3]. More precisely, one constructs an ℓ-torsion class in $H^4_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}_\ell(2))$, which is not algebraic, for some smooth and projective variety X. However, one then wonders if there exists an example of a variety X over a finite field, such that the map

$$CH^i(X_{\bar{k}}) \otimes \mathbb{Z}_\ell \to \bigcup_H H^{2i}_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}_\ell(i))^H/torsion \tag{1.2}$$

is not surjective ([13, 3]). In the context of an integral version of the Hodge conjecture, Kollár [12] constructed such examples of curve classes. Over a finite field, Schoen [18] has proved that the map (1.2) is always surjective for curve classes, if the Tate conjecture holds for divisors on surfaces.
In this note we follow the approach of Atiyah-Hirzebruch and Totaro and we produce examples where the map (1.2) is not surjective for $\ell = 2, 3$ or 5.

Theorem 1.1. *Let ℓ be a prime from the following list: $\ell = 2, 3$ or 5. There exists a smooth and projective variety X over a finite field k, $char k \neq \ell$, such that the cycle class map*

$$CH^2(X_{\bar{k}}) \otimes \mathbb{Z}_\ell \to \bigcup_H H^4_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}_\ell(2))^H/torsion,$$

where the union is over all open subgroups H of $Gal(\bar{k}/k)$, is not surjective.

As in the examples of Atiyah-Hirzebruch and Totaro, our counterexamples are obtained as a projective approximation of the cohomology of classifying spaces of some simple simply connected groups, having ℓ-torsion in its cohomology. The non-algebraicity of a cohomology class is obtained by means of motivic cohomology operations: the operation Q_1 always vanishes on the algebraic classes and one establishes that it does not vanish on some class of degree 4. This is discussed in section 2. Next, in section 3 we investigate some properties of classifying spaces in our context and finally, following a suggestion of B. Totaro, we construct a projective variety approximating the cohomology of these spaces in small degrees in section 4.

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2. Motivic version of Atiyah-Hirzebruch arguments, revisited

2.1. Operations. Let k be a perfect field with $char(k) \neq \ell$ and let $\mathcal{H}.(k)$ be the motivic homotopy category of pointed k-spaces (see [15]). For $X \in \mathcal{H}.(k)$,

denote by $H^{*,*'}(X,\mathbb{Z}/\ell)$ the motivic cohomology groups with $\mathbb{Z}/\ell$-coefficients (*loc.cit.*). If X is a smooth variety over k (viewed as an object of $\mathcal{H}.(k)$), note that one has an isomorphism $CH^*(X)/\ell \xrightarrow{\sim} H^{2*,*}(X,\mathbb{Z}/\ell)$.
Voevodsky ([23], see also [17]) defined the reduced power operations P^i and the Milnor's operations Q_i on $H^{*,*'}(X,\mathbb{Z}/\ell)$:

$$P^i : H^{*,*'}(X,\mathbb{Z}/\ell) \to H^{*+2i(\ell-1),*'+i(\ell-1)}(X,\mathbb{Z}/\ell), i \geq 0$$

$$Q_i : H^{*,*'}(X,\mathbb{Z}/\ell) \to H^{*+2\ell^i-1,*'+(\ell^i-1)}(X,\mathbb{Z}/\ell), i \geq 0,$$

where $Q_0 = \beta$ is the Bockstein operation of degree $(1,0)$ induced from the short exact sequence $0 \to \mathbb{Z}/\ell \to \mathbb{Z}/\ell^2 \to \mathbb{Z}/\ell \to 0$.
One of the key ingredients for this construction is the following computation of the motivic cohomology of the classifying space $B_{\acute{e}t}\mu_\ell \in \mathcal{H}.(k)$:

LEMMA 2.1. *([23, §6]) For each object $X \in \mathcal{H}.(k)$, the graded algebra $H^{*,*'}(X \times B_{\acute{e}t}\mu_\ell,\mathbb{Z}/\ell)$ is generated over $H^{*,*'}(X,\mathbb{Z}/\ell)$ by elements x and y, $deg(x) = (1,1)$ and $deg(y) = (2,1)$, with $\beta(x) = y$ and*
$$x^2 = \begin{cases} 0 & \ell \text{ is odd} \\ \tau y + \rho x & \ell = 2 \end{cases}$$
where τ is a generator of $H^{0,1}(Spec(k),\mathbb{Z}/2) \cong \mu_2$ and ρ is the class of (-1) in $H^{1,1}(Spec(k),\mathbb{Z}/2) \simeq k^*/(k^*)^2$.

For what follows, we assume that k contains a primitive ℓ^2-th root of unity ξ, so that $B_{\acute{e}t}\mathbb{Z}/\ell \xrightarrow{\sim} B_{\acute{e}t}\mu_\ell$ and $\beta(\tau) = \xi^\ell$ $(= \rho$ for $p = 2)$ is zero in $k^*/(k^*)^\ell = H^{1,1}_{\acute{e}t}(Spec(k);\mathbb{Z}/\ell)$.

We will need the following properties:

PROPOSITION 2.2. *Let $X \in \mathcal{H}.(k)$.*

(i) $P^i(x) = 0$ *for $i > m-n$ and $i \geq n$ and $x \in H^{m,n}(X,\mathbb{Z}/\ell)$;*
(ii) $P^i(x) = x^\ell$ *for $x \in H^{2i,i}(X,\mathbb{Z}/\ell)$;*
(iii) *if X is a smooth variety over k, the operation*

$$Q_i : CH^m(X)/\ell = H^{2m,m}(X,\mathbb{Z}/\ell) \to H^{2m+2\ell^i-1,m+(\ell^i-1)}(X,\mathbb{Z}/\ell)$$

is zero ;
(iv) $Op.(\tau x) = \tau Op.(x)$ *for $Op. = \beta, Q_i$ or P^i;*
(v) $Q_i = [P^{\ell^{i-1}}, Q_{i-1}]$.

Proof. See [23, §9]. For (iii) one uses that $H^{m,n}(X,\mathbb{Z}/\ell) = 0$ if $m - 2n > 0$ and X is a smooth variety over k, (iv) follows from the Cartan formula for the motivic cohomology.

2.2. COMPUTATIONS FOR $B_{ét}\mathbb{Z}/\ell$. The computations in this section are similar to [1, 21, 22].

LEMMA 2.3. *In $H^{*,*'}(B_{ét}\mathbb{Z}/\ell,\mathbb{Z}/\ell)$, we have $Q_i(x) = y^{\ell^i}$ and $Q_i(y) = 0$.*

Proof. By definition $Q_0(x) = \beta(x) = y$. Using induction and Proposition 2.2, we compute

$$Q_i(x) = P^{\ell^{i-1}}Q_{i-1}(x) - Q_{i-1}P^{\ell^{i-1}}(x) = P^{\ell^{i-1}}Q_{i-1}(x)$$
$$= P^{\ell^{i-1}}(y^{\ell^{i-1}}) = y^{\ell^i}.$$

Then $Q_1(y) = -Q_0P^1(y) = -\beta(y^\ell) = 0$. For $i > 1$, using induction and Proposition 2.2 again, we conclude that $Q_i(y) = -Q_{i-1}P^{\ell^{i-1}}(y) = 0$. □

Let $G = (\mathbb{Z}/\ell)^3$. As above, we view $B_{ét}G$ as an object of the category $\mathcal{H}.(k)$ and we assume that k contains a primitive ℓ^2-th root of unity. From Lemma 2.1, we have an isomorphism of modules over $H^{*,*'}(Spec(k),\mathbb{Z}/\ell)$:

$$H^{*,*'}(B_{ét}G,\mathbb{Z}/\ell) \cong H^{*,*'}(Spec(k),\mathbb{Z}/\ell)[y_1,y_2,y_3] \otimes \Lambda(x_1,x_2,x_3)$$

where $\Lambda(x_1,x_2,x_3)$ is isomorphic to the $\mathbb{Z}/\ell$-module generated by 1 and $x_{i_1}...x_{i_s}$ for $i_1 < ... < i_s$, with relations $x_ix_j = -x_jx_i$ $(i \le j)$, $\beta(x_i) = y_i$ and $x_i^2 = \tau y_i$ for $\ell = 2$.

LEMMA 2.4. *Let $x = x_1x_2x_3$ in $H^{3,3}(B_{ét}G,\mathbb{Z}/\ell)$. Then*

$$Q_iQ_jQ_k(x) \neq 0 \in H^{2*,*}(B_{ét}G,\mathbb{Z}/\ell) \quad for\ i < j < k.$$

Proof. Using Proposition 2.2(v) and Cartan formula for the operations on cup-products ([23] Proposition 9.7 and Proposition 13.4), we first get $Q_k(x) = y_1^{\ell^k}x_2x_3 - y_2^{\ell^k}x_1x_3 + y_3^{\ell^k}x_1x_2$ and one then deduces

$$Q_iQ_jQ_k(x) = \sum_{\sigma\in S_3} \pm y_{\sigma(1)}^{\ell^k}y_{\sigma(2)}^{\ell^j}y_{\sigma(3)}^{\ell^i} \neq 0 \in \mathbb{Z}/\ell[y_1,y_2,y_3].$$

□

3. EXCEPTIONAL LIE GROUPS

Let (G,ℓ) be a simple simply connected Lie group and a prime number from the following list:

$$(3.1) \qquad (G,\ell) = \begin{cases} G_2, \ell = 2, \\ F_4, \ell = 3, \\ E_8, \ell = 5. \end{cases}$$

Then G is 2-connected and we have $H^3(G,\mathbb{Z}) \cong \mathbb{Z}$ for its (singular) cohomology group in degree 3. Hence BG, viewed as a topological space, is 3-connected and $H^4(BG,\mathbb{Z}) \cong \mathbb{Z}$ (see [14] for example). We write $x_4(G)$ for a generator of $H^4(BG,\mathbb{Z})$.

Given a field k with $char(k) \neq \ell$, let us denote by G_k the (split) reductive algebraic group over k corresponding to the Lie group G.
The Chow ring $CH^*(BG_k)$ has been defined by Totaro [22]. More precisely, one has

$$BG_k = \varinjlim(U/G_k), \tag{3.2}$$

where $U \subset W$ is an open set in a linear representation W of G_k, such that G_k acts freely on U. One can then identify $CH^i(BG_k)$ with the group $CH^i(U/G_k)$ if $\text{codim}_W(W \setminus U) > i$, the group $CH^i(BG_k)$ is then independent of a choice of such U and W. Similarly, one can define the étale cohomology groups $H^i_{ét}(BG_k, \mathbb{Z}_\ell(j))$ and the motivic cohomology groups $H^{*,*'}(BG_k, \mathbb{Z}/\ell)$ (see [8]), the latter coincide with the motivic cohomology groups of $B_{ét}G$ as in [15] (cf. [8, Proposition 2.29 and Proposition 3.10]). We also have the cycle class map

$$cl : CH^*(BG_{\bar{k}}) \otimes \mathbb{Z}_\ell \to \bigcup_H H^{2*}_{ét}(BG_{\bar{k}}, \mathbb{Z}_\ell(*))^H, \tag{3.3}$$

where the union is over all open subgroups H of $Gal(\bar{k}/k)$.
The following proposition is known.

PROPOSITION 3.1. *Let (G, ℓ) be a group and a prime number from the list (3.1). Then*

(i) *the group G has a maximal elementary non toral subgroup of rank 3:*
$$i : A \simeq (\mathbb{Z}/\ell)^3 \subset G;$$
(ii) *$H^4(BG, \mathbb{Z}/\ell) \simeq \mathbb{Z}/\ell$, generated by the image x_4 of the generator $x_4(G)$ of $H^4(BG, \mathbb{Z}) \simeq \mathbb{Z}$;*
(iii) *$Q_1(i^*x_4) = Q_1Q_0(x_1x_2x_3)$, in the notations of Lemma 2.4. In particular, $Q_1(i^*x_4)$ is nonzero.*

Proof. For (i) see [5], for the computation of the cohomology groups with $\mathbb{Z}/\ell$-coefficients in (ii) see [14] VII 5.12; (iii) follows from [11] for $\ell = 2$ and [9, Proposition 3.2] for $\ell = 3, 5$ (see [10] as well). The class $Q_1(i^*x_4)$ is nonzero by Lemma 2.4 (see also [8, Théorème 4.1]). □

4. ALGEBRAIC APPROXIMATION OF BG

Write

$$BG_k = \varinjlim(U/G_k) \tag{4.1}$$

as in the previous section. Using proposition 3.1 and a specialization argument, we will first construct a quasi-projective algebraic variety X over a finite field k as a quotient $X = U/G_k$ (where $codim_W(W \setminus U)$ is big enough), such that the cycle class map (1.2) is not surjective for such X. However, if one is interested only in quasi-projective counterexamples for the surjectivity of the map (1.2), one can produce more naive examples, for instance as a complement of some smooth hypersurfaces in a projective space. Hence we are interested to find an approximation of Chow groups and the étale cohomology of $BG_{\bar{k}}$ as a smooth and projective variety. In the case when the group G is finite, this is done in [3, Théorème 2.1]. In this section we give such an approximation for the groups we consider here, this construction is suggested by B. Totaro. We will proceed in three steps. First, we construct a quasi-projective approximation in a family parametrized by $Spec\,\mathbb{Z}$. Then, for the geometric generic fibre we produce a projective approximation, by a topological argument. We finish the construction by specialization.

Let G be a compact Lie group as in (3.1). Let $\mathcal{G}$ be a split reductive group over $Spec\,\mathbb{Z}$ corresponding to G, such a group exists by [SGA3] XXV 1.3.

Lemma 4.1. *For any fixed integer $s \geq 0$ there exists a projective scheme $\mathcal{Y}/\mathrm{Spec}\,\mathbb{Z}$ and an open subscheme $\mathcal{W} \subset \mathcal{Y}$ such that*

(i) *$W \to \mathrm{Spec}\,\mathbb{Z}$ is smooth and the complement of $\mathcal{W}$ is of codimension at least s in each fiber of $\mathcal{Y} \to \mathrm{Spec}\,\mathbb{Z}$;*

(ii) *for any point $t \in \mathrm{Spec}\,\mathbb{Z}$ with residue field $\kappa(t)$ there is a natural map $\mathcal{W}_t \to B(\mathbb{G}_m \times \mathcal{G})_t$ inducing an isomorphism*

$$H^i_{\acute{e}t}(\mathcal{W}_{\bar{t}}, \mathbb{Z}_\ell) \xrightarrow{\sim} H^i_{\acute{e}t}(B(\mathbb{G}_m \times \mathcal{G})_{\bar{t}}, \mathbb{Z}_\ell) \text{ for } i \leq s, \ell \neq char\,\kappa(t). \tag{4.2}$$

Proof. Write $T = Spec\,\mathbb{Z}$, as it is an affine scheme of dimension 1, we can embed $\mathcal{G}$ as a closed subgroup of $\mathcal{H} = GL_{d,T}$ for some d (see [SGA3] VI_B 13.2). Moreover, it induces an embedding $\mathcal{G} \hookrightarrow PGL_{d,T}$, as the center of $\mathcal{G}$ is trivial for groups we consider here.

By a construction of [22, Remark 1.4] and [2, Lemme 9.2], there exists $n > 0$, a linear $\mathcal{H}$-representation $\mathcal{O}_T^{\oplus n}$ and an $\mathcal{H}$-invariant open subset $\mathcal{U} \subset \mathcal{O}_T^{\oplus n}$, which one can assume flat over T, such that the action of $\mathcal{H}$ is free on $\mathcal{U}$. Let $\mathcal{V}_N = \mathcal{O}_T^{\oplus Nn}$. Then the group $PGL_{n,T}$ acts on $\mathbb{P}(\mathcal{V}_N)$ and, taking N sufficiently large, one can assume that the action is free outside a subset S of high codimension (with respect to s).

By restriction, the group $\mathcal{G}$ acts on $\mathbb{P}(\mathcal{V}_N)$ as well, let $\mathcal{Y} = \mathbb{P}(\mathcal{V}_N)//\mathcal{G}$ be the GIT quotient for this action [16, 19]. The scheme $\mathcal{Y}$ is projective over T and we fix an embedding $\mathcal{Y} \subset \mathbb{P}^M_T$. Let

$$f : \mathcal{W} \to T \tag{4.3}$$

be the open set of $\mathcal{Y}$ corresponding to the quotient of the open set $\mathcal{U}$ as above where $\mathcal{G}_T$ acts freely. From the construction, one can assume that $\mathcal{W}$ has codimension at least s in $\mathcal{Y}$ in each fibre over T.

For any point $t \in T$ the fibre $\mathcal{W}_t$ is a smooth quasi-projective variety and if N is big enough, we have isomorphisms (cf. p. 263 in [22])

$$\mathcal{W}_t \cong (\mathbb{P}(\mathcal{V}_N) - S)_t/\mathcal{G}_t \cong ((\mathcal{V}_N - \{0\})/\mathbb{G}_m - S)_t)/\mathcal{G}_t \cong (\mathcal{V}_N - S')_t/(\mathbb{G}_m \times \mathcal{G})_t$$

where $S' = pr^{-1}S \cup \{0\}$ for the projection $pr : (\mathcal{V}_N - \{0\}) \to \mathbb{P}(\mathcal{V}_N)$. Hence we have isomorphisms

$$H^i_{\acute{e}t}(\mathcal{W}_{\bar{t}}, \mathbb{Z}_\ell) \xrightarrow{\sim} H^i_{\acute{e}t}(B(\mathbb{G}_m \times \mathcal{G})_{\bar{t}}, \mathbb{Z}_\ell) \text{ for } i \leq s, \ell \neq char\,\kappa(b),$$

induced by a natural map $\mathcal{W}_t \to B(\mathbb{G}_m \times \mathcal{G})_t$ from the presentation (4.1). $\square$

Remark 4.2. More generally, in the statement above the map $\mathcal{W}_t \to B(\mathbb{G}_m \times \mathcal{G})_t$ induces an isomorphism $H^i_{\acute{e}t}(\mathcal{W}_F, \mathbb{Z}_\ell) \xrightarrow{\sim} H^i_{\acute{e}t}(B(\mathbb{G}_m \times \mathcal{G})_F, \mathbb{Z}_\ell)$, $i \le s, \ell \ne char\,\kappa(t)$ for any F-point of T over t.

Lemma 4.3. *Let $Y \subset \mathbb{P}^M_{\mathbb{C}}$ be a projective variety over $\mathbb{C}$ and let $W \subset Y$ be a dense open in Y. Assume that W is smooth. Then for a general linear subspace L in $\mathbb{P}^M$ of codimension equal to $1 + dim(Y - W)$, the scheme $X = L \cap W$ is smooth and projective and the natural maps $H^i(W, \mathbb{Z}) \to H^i(X, \mathbb{Z})$ are isomorphisms for $i < dim\, X$.*

Proof. We apply a version of the Lefschetz hyperplane theorem for quasi-projective varieties, established by Hamm (as a special case of Theorem II.1.2 in [4]): for $V \subset \mathbb{P}^M$ a closed complex subvariety of dimension d, not necessarily smooth, $Z \subset V$ a closed subset, and H a hyperplane in $\mathbb{P}^M$, if $V - (Z \cup H)$ is local complete intersection (e.g. $V - Z$ is smooth) then

$$\pi_i((V - Z) \cap H) \to \pi_i(V - Z)$$

is an isomorphism for $i < d - 1$ and surjective for $i = d - 1$. In particular, $H^i(V - Z, \mathbb{Z}) \to H^i((V - Z) \cap H, \mathbb{Z})$ is an isomorphism for $i < d - 1$ and surjective for $i = d - 1$ by the Whitehead theorem.
Applying this statement to W and to successive intersections of W with linear forms defining L, we then deduce that $H^i(W, \mathbb{Z}) \to H^i(X, \mathbb{Z})$ is an isomorphism for $i < dim\, X$.

$\square$

Proposition 4.4. *Let G be a compact Lie group as in (3.1).*
For all but finitely many primes p there exists a smooth and projective variety X_k over a finite field k with $char\, k = p$, an element $x_{4,\bar{k}} \in H^4_{\acute{e}t}(B(\mathbb{G}_m \times G_{\bar{k}}), \mathbb{Z}_\ell(2))$, invariant under the action of $Gal(\bar{k}/k)$ and a map $\iota : X_k \to B(\mathbb{G}_m \times G_k)$ in the category $\mathcal{H}.(k)$ such that

(i) *$\alpha_{\bar{k}} = \iota^* x_{4,\bar{k}}$ is a nonzero class in $H^4_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}_\ell(2))/torsion$;*
(ii) *the operation $Q_1(\bar{\alpha}_{\bar{k}})$ is nonzero, where we write $\bar{\alpha}_{\bar{k}}$ for the image of $\alpha_{\bar{k}}$ in $H^4_{\acute{e}t}(X_{\bar{k}}, \mu_\ell^{\otimes 2})$.*

Proof. Let $\mathcal{W} \subset \mathcal{Y} \subset \mathbb{P}^M_{\mathbb{Z}}$ be as in Lemma 4.1 for $s \ge 4$.
Let $Y = \mathcal{Y}_{\mathbb{C}}$ and $W = \mathcal{W}_{\mathbb{C}}$ be the geometric generic fibres of $\mathcal{Y}$ and $\mathcal{W}$. Consider a general linear space L in $\mathbb{P}^M$ of codimension equal to $1 + dim(Y - W)$. We deduce from Lemma 4.3 above, that the variety $X := L \cap W$ is smooth and projective, and

$$H^i(X, R) \simeq H^i(B(\mathbb{G}_m \times G), R) \text{ for } i \le s \text{ and } R = \mathbb{Z} \text{ or } \mathbb{Z}/n. \tag{4.4}$$

Hence $H^i_{\acute{e}t}(X, \mathbb{Z}/n) \simeq H^i_{\acute{e}t}(B(\mathbb{G}_m \times G), \mathbb{Z}/n), i \le s$. In particular, by functoriality of the isomorphisms $H^i_{\acute{e}t}(\cdot, \mathbb{Z}/n) \simeq H^i_{\acute{e}t}(\cdot, \mu_n^{\otimes j}), i \le s, j > 0$, for $\cdot = X$ and

$B(\mathbb{G}_m \times G)$, we get

$$H^i_{\acute{e}t}(X, \mu_n^{\otimes j}) \simeq H^i_{\acute{e}t}(B(\mathbb{G}_m \times G), \mu_n^{\otimes j}), i \leq s. \tag{4.5}$$

We can assume that we have an isomorphism as above for $i = 4$ and $i = 2\ell + 3$. Note that the cohomology of BG is a direct factor in the cohomology of $B(\mathbb{G}_m \times G)$ (cf. [8, Lemme 2.23]). Using Proposition 3.1, we then get an element $x_{4,\mathbb{C}}$ generating a direct factor isomorphic to $\mathbb{Z}_\ell$ in the cohomology group $H^4_{\acute{e}t}(B(\mathbb{G}_m \times G), \mathbb{Z}_\ell(2))$. Denote $\alpha_\mathbb{C}$ its image in $H^4_{\acute{e}t}(X, \mathbb{Z}_\ell(2))$.

We can now specialize the construction above to obtain the statement over a finite field. Note that one can assume that L is defined over $\mathbb{Q}$. One can then find an open $T' \subset \operatorname{Spec} \mathbb{Z}$ and a linear space $\mathcal{L} \subset \mathbb{P}^M_{T'}$ such that $\mathcal{L}_\mathbb{C} \simeq L$ and such that for any $t \in T'$ the fibre $\mathcal{X}_t$ of $\mathcal{X} = \mathcal{L} \cap \mathcal{T}$ is smooth. After passing to an étale cover T'' of T', one can assume that the inclusion $(\mathbb{Z}/\ell)^3 \subset G_\mathbb{C}$ from proposition 3.1 extends to an inclusion $i : \mathcal{A} = (\mathbb{Z}/\ell)^3_{T''} \hookrightarrow \mathcal{G}_{T''}$ (cf. [SGA3] XI.5.8).

Let $t \in T''$ and let $k = \kappa(t)$. As the schemes $\mathcal{X}_{T''}$, $\mathcal{W}_{T''}$ and $\mathcal{U}/\mathcal{A}$ are smooth over T'', we have the following commutative diagram, where the vertical maps are induced by the specialization maps (cf. [SGA4 1/2] Arcata V.3):

$$\begin{array}{ccccccc}
H^4_{\acute{e}t}(X, \mathbb{Z}_\ell(2)) & \longleftarrow & H^4_{\acute{e}t}(W, \mathbb{Z}_\ell(2)) & \longrightarrow & H^4_{\acute{e}t}(\mathcal{U}_\mathbb{C}/(\mathbb{Z}/\ell)^3, \mathbb{Z}/\ell) & \overset{\simeq}{\longleftarrow} & H^4_{\acute{e}t}(B(\mathbb{Z}/\ell)^3_\mathbb{C}, \mathbb{Z}/\ell) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \simeq \\
H^4_{\acute{e}t}(\mathcal{X}_{\bar{k}}, \mathbb{Z}_\ell(2)) & \longleftarrow & H^4_{\acute{e}t}(\mathcal{W}_{\bar{k}}, \mathbb{Z}_\ell(2)) & \longrightarrow & H^4_{\acute{e}t}(\mathcal{U}_{\bar{k}}/(\mathbb{Z}/\ell)^3, \mathbb{Z}/\ell) & \overset{\simeq}{\longleftarrow} & H^4_{\acute{e}t}(B(\mathbb{Z}/\ell)^3_{\bar{k}}, \mathbb{Z}/\ell)
\end{array}$$

The left vertical map is an isomorphism since $\mathcal{X}$ is proper, by a smooth-proper base change theorem. Hence we get a class $\alpha_{\bar{k}} \in H^4_{\acute{e}t}(\mathcal{X}_{\bar{k}}, \mathbb{Z}_\ell(2))$, corresponding to $\alpha_\mathbb{C} \in H^4_{\acute{e}t}(X, \mathbb{Z}_\ell(2))$. The map $H^4_{\acute{e}t}(W, \mathbb{Z}_\ell(2)) \to H^4_{\acute{e}t}(X, \mathbb{Z}_\ell(2))$ is an isomorphism by Lemma 4.3, so that $\alpha_{\bar{k}}$ comes from an element $x_{4,\bar{k}} \in H^4_{\acute{e}t}(\mathcal{W}_{\bar{k}}, \mathbb{Z}_\ell(2))$. Let $\bar{\alpha}_\mathbb{C} \in H^4_{\acute{e}t}(X, \mu_\ell^{\otimes 2})$ be the image of $\alpha_\mathbb{C}$ and let $\bar{\alpha}_{\bar{k}} \in H^4_{\acute{e}t}(\mathcal{X}_{\bar{k}}, \mu_\ell^{\otimes 2})$ be the image of $\alpha_{\bar{k}}$. As the operation Q_1 commutes with the isomorphisms $H^i_{\acute{e}t}(X, \mathbb{Z}/\ell) \to H^i_{\acute{e}t}(X, \mu_\ell^{\otimes j})$, we get $Q_1(\bar{\alpha}_\mathbb{C}) \neq 0$ by proposition 3.1. The étale cohomology operation Q_1 also commutes with the specialization maps (cf. [7]), since these maps are obtained as composite of the natural maps $\phi \circ \psi^{-1}$ on the étale cohomology groups with torsion coefficients $H^i_{\acute{e}t}(X_\mathbb{C}) \overset{\psi}{\leftarrow} H^i_{\acute{e}t}(\mathcal{X}_S) \overset{\phi}{\to} H^i_{\acute{e}t}(\mathcal{X}_{\bar{k}})$, where S is the strict henselization of T'' at t and ϕ is an isomorphism since $\mathcal{X}$ is smooth. Hence $Q_1(\bar{\alpha}_{\bar{k}})$ is nonzero as well. From the construction, the class $\alpha_{\bar{k}}$ generates a subgroup of $H^4_{\acute{e}t}(\mathcal{X}_{\bar{k}}, \mathbb{Z}_\ell(2))$, which is a direct factor isomorphic to $\mathbb{Z}_\ell$, and is Galois-invariant. Letting $X_k = \mathcal{X}_k$ this finishes the proof of the proposition.

□

Remark 4.5. For the purpose of this note, the proposition above is enough. See also [6] for a a general statement on a projective approximation of the cohomology of classifying spaces.

Theorem 1.1 now follows from the proposition above:

Proof of theorem 1.1.
For k a finite field and X_k as in the proposition above, we find a nontrivial class $\alpha_{\bar{k}}$ in its cohomology in degree 4 modulo torsion, which is not annihilated by the operation Q_1. This class cannot be algebraic by proposition 2.2(iii). □

Remark 4.6. We can also adapt the arguments of [3, Théorème 2.1] to produce projective examples with higher torsion non-algebraic classes, while in *loc.cit.* one constructs ℓ-torsion classes. Let $G(n)$ be the finite group $G(\mathbb{F}_{\ell^n})$, so that we have

$$\varprojlim H^*_{\acute{e}t}(BG(n), \mathbb{Z}_\ell) = H^*_{\acute{e}t}(BG_{\bar{k}}, \mathbb{Z}_\ell).$$

Then, following the construction in *loc.cit.* one gets

> *For any $n > 0$, there exists a positive integer i_n and a Godeaux-Serre variety $X_{n,\bar{k}}$ for the finite group $G(i_n)$ such that*
> (1) *there is an element $x \in H^4_{\acute{e}t}(X_{n,\bar{k}}; \mathbb{Z}_\ell(2))$ generating $\mathbb{Z}/\ell^{n'}$ for some $n' \geq n$;*
> (2) *x is not in the image of the cycle class map (1.1).*

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