Homology Stability for the Special Linear Group of a Field and Milnor-Witt K-theory

Dedicated to Andrei Suslin

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Abstract. Let F be a field of characteristic zero and let $f_{t,n}$ be the stabilization homomorphism from the nth integral homology of $\mathrm{SL}_t(F)$ to the nth integral homology of $\mathrm{SL}_{t+1}(F)$. We prove the following results: For all n, $f_{t,n}$ is an isomorphism if $t \geq n+1$ and is surjective for $t = n$, confirming a conjecture of C-H. Sah. $f_{n,n}$ is an isomorphism when n is odd and when n is even the kernel is isomorphic to the $(n+1)$st power of the fundamental ideal of the Witt Ring of F. When n is even the cokernel of $f_{n-1,n}$ is isomorphic to the nth Milnor-Witt K-theory group of F. When n is odd, the cokernel of $f_{n-1,n}$ is isomorphic to the square of the nth Milnor K-group of F.

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1. Introduction

Given a family of groups $\{G_t\}_{t\in\mathbb{N}}$ with canonical homomorphisms $G_t \to G_{t+1}$, we say that the family has homology stability if there exist constants $K(n)$ such that the natural maps $\mathrm{H}_n(G_t,\mathbb{Z}) \to \mathrm{H}_n(G_{t+1},\mathbb{Z})$ are isomorphisms for $t \geq K(n)$. The question of homology stability for families of linear groups over a ring R - general linear groups, special linear groups, symplectic, orthogonal and unitary groups - has been studied since the 1970s in connection with applications to algebraic K-theory, algebraic topology, the scissors congruence problem, and the homology of Lie groups. These families of linear groups are known to have homology stability at least when the rings satisfy some appropriate finiteness condition, and in particular in the case of fields and local rings

([4],[26],[27],[25], [5],[2], [21],[15],[14]). It seems to be a delicate - but interesting and apparently important - question, however, to decide the minimal possible value of $K(n)$ for a particular class of linear groups (with coefficients in a given class of rings) and the nature of the obstruction to extending the stability range further.

The best illustration of this last remark are the results of Suslin on the integral homology of the general linear group of a field in the paper [23]. He proved that, for an infinite field F, the maps $\mathrm{H}_n(\mathrm{GL}_t(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{GL}_{t+1}(F),\mathbb{Z})$ are isomorphisms for $t \geq n$ (so that $K(n) = n$ in this case), while the cokernel of the map $\mathrm{H}_n(\mathrm{GL}_{n-1}(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{GL}_n(F),\mathbb{Z})$ is naturally isomorphic to the nth Milnor K-group, $K_n^{\mathrm{M}}(F)$. In fact, if we let

$$H_n(F) := \mathrm{Coker}(\mathrm{H}_n(\mathrm{GL}_{n-1}(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{GL}_n(F),\mathbb{Z})),$$

his arguments show that there is an isomorphism of graded rings $H_\bullet(F) \cong K_\bullet^{\mathrm{M}}(F)$ (where the multiplication on the first term comes from direct sum of matrices and cross product on homology). In particular, the non-negatively graded ring $H_\bullet(F)$ is generated in dimension 1.

Recent work of Barge and Morel ([1]) suggested that Milnor-Witt K-theory may play a somewhat analogous role for the homology of the special linear group. The Milnor-Witt K-theory of F is a $\mathbb{Z}$-graded ring $K_\bullet^{\mathrm{MW}}(F)$ surjecting naturally onto Milnor K-theory. It arises as a ring of operations in stable motivic homotopy theory. (For a definition see section 2 below, and for more details see [17, 18, 19].) Let $SH_n(F) := \mathrm{Coker}(\mathrm{H}_n(\mathrm{SL}_{n-1}(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z}))$ for $n \geq 1$, and let $SH_0(F) = \mathbb{Z}[F^\times]$ for convenience. Barge and Morel construct a map of graded algebras $SH_\bullet(F) \to K_\bullet^{\mathrm{MW}}(F)$ for which the square

$$\begin{tikzcd} SH_\bullet(F) \arrow[r] \arrow[d] & K_\bullet^{\mathrm{MW}}(F) \arrow[d] \\ H_\bullet(F) \arrow[r] & K_\bullet^{\mathrm{M}}(F) \end{tikzcd}$$

commutes.

A result of Suslin ([24]) implies that the map $\mathrm{H}_2(\mathrm{SL}_2(F),\mathbb{Z}) = SH_2(F) \to K_2^{\mathrm{MW}}(F)$ is an isomorphism. Since positive-dimensional Milnor-Witt K-theory is generated by elements of degree 1, it follows that the map of Barge and Morel is surjective in even dimensions greater than or equal to 2. They ask the question whether it is in fact an isomorphism in even dimensions.

As to the question of the range of homology stability for the special linear groups of an infinite field, as far as the authors are aware the most general result to date is still that of van der Kallen [25], whose results apply to much more general classes of rings. In the case of a field, he proves homology stability for $\mathrm{H}_n(\mathrm{SL}_t(F),\mathbb{Z})$ in the range $t \geq 2n+1$. On the other hand, known results when n is small suggest a much larger range. For example, the theorems of Matsumoto and Moore imply that the maps $\mathrm{H}_2(\mathrm{SL}_t(F),\mathbb{Z}) \to \mathrm{H}_2(\mathrm{SL}_{t+1}(F),\mathbb{Z})$ are isomorphisms for $t \geq 3$ and are surjective for $t = 2$. In the paper [22] (Conjecture 2.6), C-H. Sah conjectured that for an infinite field F (and more generally for a division algebra with infinite centre), the homomorphism $\mathrm{H}_n(\mathrm{SL}_t(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_{t+1}(F),\ )$ is an isomorphism if $t \geq n+1$ and is surjective for $t = n$.

The present paper addresses the above questions of Barge/Morel and Sah in the case of a field of characteristic zero. We prove the following results about the homology stability for special linear groups:

THEOREM 1.1. *Let F be a field of characteristic 0. For $n, t \geq 1$, let $f_{t,n}$ be the stabilization homomorphism $\mathrm{H}_n(\mathrm{SL}_t(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_{t+1}(F),\mathbb{Z})$*

(1) *$f_{t,n}$ is an isomorphism for $t \geq n+1$ and is surjective for $t = n$.*
(2) *If n is odd $f_{n,n}$ is an isomorphism*
(3) *If n is even the kernel of $f_{n,n}$ is isomorphic to $I^{n+1}(F)$.*
(4) *For even n the cokernel of $f_{n-1,n}$ is naturally isomorphic to $K_n^{\mathrm{MW}}(F)$.*
(5) *For odd $n \geq 3$ the cokernel of $f_{n-1,n}$ is naturally isomorphic to $2K_n^{\mathrm{M}}(F)$.*

Proof. The proofs of these statements can be found below as follows:

(1) Corollary 5.11.
(2) Corollary 6.12.
(3) Corollary 6.13.
(4) Corollary 6.11.
(5) Corollary 6.13

□

Our strategy is to adapt Suslin's argument for the general linear group in [23] to the case of the special linear group. Suslin's argument is an ingenious variation on the method of van der Kallen in [25], in turn based on ideas of Quillen. The broad idea is to find a highly connected simplicial complex on which the group G_t acts and for which the stabilizers of simplices are (approximately) the groups G_r, with $r \leq t$, and then to use this to construct a spectral sequence calculating the homology of the G_n in terms of the homology of the G_r. Suslin constructs a family $\mathcal{E}(n)$ of such spectral sequences, calculating the homology of $\mathrm{GL}_n(F)$. He constructs partially-defined products $\mathcal{E}(n) \times \mathcal{E}(m) \to \mathcal{E}(n+m)$ and then proves some periodicity and decomposabilty properties which allow him to conclude by an easy induction.

Initially, the attempt to extend these arguments to the case of $\mathrm{SL}_n(F)$ does not appear very promising. Two obstacles to extending Suslin's arguments become quickly apparent.

The main obstacle is Suslin's Theorem 1.8 which says that a certain inclusion of a block diagonal linear group in a block triangular group is a homology isomorphism. The corresponding statement for subgroups of the special linear group is emphatically false, as elementary calculations easily show. Much of Suslin's subsequent results - in particular, the periodicity and decomposability properties of the spectral sequences $\mathcal{E}(n)$ and of the graded algebra $S_\bullet(F)$ which plays a central role - depend on this theorem. And, indeed, the analogous spectral sequences and graded algebra which arise when we replace the general linear with the special linear group do not have these periodicity and decomposability properties.

However, it turns out - at least when the characteristic is zero - that the failure of Suslin's Theorem 1.8 is not fatal. A crucial additional structure is available to us in the case of the special linear group; almost everything in sight in a $\mathbb{Z}[F^\times]$-module. In the analogue of Theorem 1.8, the map of homology groups is a split inclusion whose cokernel has a completely different character as a $\mathbb{Z}[F^\times]$-module than the homology of the block diagonal group. The former is 'additive ', while the latter is 'multiplicative ', notions which we define and explore in section 4 below. This leads us to introduce the concept of '$\mathcal{AM}$ modules', which decompose in a canonical way into a direct sum of an additive factor and a multiplicative factor. This decomposition is sufficiently canonical that in our graded ring structures the additive and multiplicative parts are each ideals. By working modulo the messy additive factors and projecting onto multiplicative parts, we recover an analogue of Suslin's Theorem 1.8 (Theorem 4.23 below), which we then use to prove the necessary periodicity (Theorem 5.10) and decomposability (Theorem 6.8) results.

A second obstacle to emulating the case of the general linear group is the vanishing of the groups $\mathrm{H}_1(\mathrm{SL}_n(F),\mathbb{Z})$. The algebra $H_\bullet(F)$, according to Suslin's arguments, is generated by degree 1. On the other hand, $SH_1(F) = 0 = \mathrm{H}_1(\mathrm{SL}_1(F),\mathbb{Z}) = 0$. This means that the best we can hope for in the case of the special linear group is that the algebra $SH_\bullet(F)$ is generated by degrees 2 and 3. This indeed turns out to be essentially the case, but it means we have to work harder to get our induction off the ground. The necessary arguments in degree $n = 2$ amount to the Theorem of Matsumoto and Moore, as well as variations due to Suslin ([24]) and Mazzoleni ([11]). The argument in degree $n = 3$ was supplied recently in a paper by the present authors ([8]).

We make some remarks on the hypothesis of characteristic zero in this paper: This assumption is used in our definition of $\mathcal{AM}$-modules and the derivation of their properties in section 4 below. In fact, a careful reading of the proofs in that section will show that at any given point all that is required is that the prime subfield be sufficiently large; it must contain an element of order not dividing m for some appropriate m. Thus in fact our arguments can easily be adapted to show that our main results on homology stability for the nth homology group of the special linear groups are true provided the prime field is sufficiently large (in a way that depends on n). However, we have not attempted here to make this more explicit. To do so would make the statements of the results unappealingly complicated, and we will leave it instead to a later paper to deal with the case of positive characteristic. We believe that an appropriate extension of the notion of $\mathcal{AM}$-module will unlock the characteristic $p > 0$ case.

As to our restriction to fields rather than more general rings, we note that Daniel Guin [5] has extended Suslin's results to a larger class of rings with many units. We have not yet investigated a similar extension of the results below to this larger class of rings.

2. Notation and Background Results

2.1. Group Rings and Grothendieck-Witt Rings. For a group G, we let $\mathbb{Z}[G]$ denote the corresponding integral group ring. It has an additive $\mathbb{Z}$-basis consisting of the elements $g \in G$, and is made into a ring by linearly extending the multiplication of group elements. In the case that the group G is the multiplicative group, $F^\times$, of a field F, we will denote the basis elements by $\langle a\rangle$, for $a \in F^\times$. We use this notation in order, for example, to distinguish the elements $\langle 1-a\rangle$ from $1-\langle a\rangle$, or $\langle -a\rangle$ from $-\langle a\rangle$, and also because it coincides, conveniently for our purposes, with the notation for generators of the Grothendieck-Witt ring (see below). There is an augmentation homomorphism $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$, $\langle g\rangle \mapsto 1$, whose kernel is the augmentation ideal $\mathcal{I}_G$, generated by the elements $g-1$. Again, if $G = F^\times$, we denote these generators by $\langle\langle a\rangle\rangle := \langle a\rangle - 1$.

The Grothendieck-Witt ring of a field F is the Grothendieck group, $\mathrm{GW}(F)$, of the set of isometry classes of nondgenerate symmetric bilinear forms under orthogonal sum. Tensor product of forms induces a natural multiplication on the group. As an abstract ring, this can be described as the quotient of the ring $\mathbb{Z}\left[F^\times/(F^\times)^2\right]$ by the ideal generated by the elements $\langle\langle a\rangle\rangle \cdot \langle\langle 1-a\rangle\rangle$, $a \neq 0, 1$. (This is just a mild reformulation of the presentation given in Lam, [9], Chapter II, Theorem 4.1.) Here, the induced ring homomorphism $\mathbb{Z}[F^\times] \to \mathbb{Z}\left[F^\times/(F^\times)^2\right] \to \mathrm{GW}(F)$, sends $\langle a\rangle$ to the class of the 1-dimensional form with matrix $[a]$. This class is (also) denoted $\langle a\rangle$. $\mathrm{GW}(F)$ is again an augmented ring and the augmentation ideal, $I(F)$, - also called the *fundamental ideal* - is generated by *Pfister* 1*-forms*, $\langle\langle a\rangle\rangle$. It follows that the n-th power, $I^n(F)$, of this ideal is generated by *Pfister* n-forms $\langle\langle a_1, \ldots, a_n\rangle\rangle := \langle\langle a_1\rangle\rangle \cdots \langle\langle a_n\rangle\rangle$.

Now let $\mathrm{H} := \langle 1\rangle + \langle -1\rangle = \langle\langle -1\rangle\rangle + 2 \in \mathrm{GW}(F)$. Then $\mathrm{H} \cdot I(F) = 0$, and the *Witt ring* of F is the ring

$$W(F) := \frac{\mathrm{GW}(F)}{\langle \mathrm{H}\rangle} = \frac{\mathrm{GW}(F)}{\mathrm{H}\cdot\mathbb{Z}}.$$

Since $\mathrm{H} \mapsto 2$ under the augmentation, there is a natural ring homomorphism $W(F) \to \mathbb{Z}/2$. The fundamental ideal $I(F)$ of $\mathrm{GW}(F)$ maps isomorphically to the kernel of this ring homomorphism under the map $\mathrm{GW}(F) \to W(F)$, and we also let $I(F)$ denote this ideal.

For $n \leq 0$, we define $I^n(F) := W(F)$. The graded additive group $I^\bullet(F) = \{I^n(F)\}_{n\in\mathbb{Z}}$ is given the structure of a commutative graded ring using the natural graded multiplication induced from the multiplication on $W(F)$. In particular, if we let $\eta \in I^{-1}(F)$ be the element corresponding to $1 \in W(F)$, then multiplication by $\eta : I^{n+1}(F) \to I^n(F)$ is just the natural inclusion.

2.2. Milnor K-theory and Milnor-Witt K-theory. The Milnor ring of a field F (see [12]) is the graded ring $K^{\mathrm{M}}_\bullet(F)$ with the following presentation:

Generators: $\{a\}$, $a \in F^\times$, in dimension 1.

Relations:

(a) $\{ab\} = \{a\} + \{b\}$ for all $a, b \in F^\times$.

(b) $\{a\}\cdot\{1-a\}=0$ for all $a\in F^\times\setminus\{1\}$.

The product $\{a_1\}\cdots\{a_n\}$ in $K_n^{\mathrm{M}}(F)$ is also written $\{a_1,\ldots,a_n\}$. So $K_0^{\mathrm{M}}(F)=\mathbb{Z}$ and $K_1^{\mathrm{M}}(F)$ is an additive group isomorphic to $F^\times$.

We let $k_\bullet^{\mathrm{M}}(F)$ denote the graded ring $K_\bullet^{\mathrm{M}}(F)/2$ and let $i^n(F):=I^n(F)/I^{n+1}(F)$, so that $i^\bullet(F)$ is a non-negatively graded ring.

In the 1990s, Voevodsky and his collaborators proved a fundamental and deep theorem - originally conjectured by Milnor ([13]) - relating Milnor K-theory to quadratic form theory:

Theorem 2.1 ([20]). *There is a natural isomorphism of graded rings $k_\bullet^{\mathrm{M}}(F)\cong i^\bullet(F)$ sending $\{a\}$ to $\langle\langle a\rangle\rangle$.*

In particular for all $n\ge 1$ we have a natural identification of $k_n^{\mathrm{M}}(F)$ and $i^n(F)$ under which the symbol $\{a_1,\ldots,a_n\}$ corresponds to the class of the form $\langle\langle a_1,\ldots,a_n\rangle\rangle$.

The Milnor-Witt K-theory of a field is the graded ring $K_\bullet^{\mathrm{MW}}(F)$ with the following presentation (due to F. Morel and M. Hopkins, see [17]):

Generators: $[a]$, $a\in F^\times$, in dimension 1 and a further generator η in dimension -1.

Relations:

(a) $[ab]=[a]+[b]+\eta\cdot[a]\cdot[b]$ for all $a,b\in F^\times$

(b) $[a]\cdot[1-a]=0$ for all $a\in F^\times\setminus\{1\}$

(c) $\eta\cdot[a]=[a]\cdot\eta$ for all $a\in F^\times$

(d) $\eta\cdot h=0$, where $h=\eta\cdot[-1]+2\in K_0^{\mathrm{MW}}(F)$.

Clearly there is a unique surjective homomorphism of graded rings $K_\bullet^{\mathrm{MW}}(F)\to K_\bullet^{\mathrm{M}}(F)$ sending $[a]$ to $\{a\}$ and inducing an isomorphism

$$\frac{K_\bullet^{\mathrm{MW}}(F)}{\langle\eta\rangle}\cong K_\bullet^{\mathrm{M}}(F).$$

Furthermore, there is a natural surjective homomorphism of graded rings $K_\bullet^{\mathrm{MW}}(F)\to I^\bullet(F)$ sending $[a]$ to $\langle\langle a\rangle\rangle$ and η to η. Morel shows that there is an induced isomorphism of graded rings

$$\frac{K_\bullet^{\mathrm{MW}}(F)}{\langle h\rangle}\cong I^\bullet(F).$$

The main structure theorem on Milnor-Witt K-theory is the following theorem of Morel:

Theorem 2.2 (Morel, [18]). *The commutative square of graded rings*

$$\begin{tikzcd} K_\bullet^{\mathrm{MW}}(F) \arrow[r] \arrow[d] & K_\bullet^{\mathrm{M}}(F) \arrow[d] \\ I^\bullet(F) \arrow[r] & i^\bullet(F) \end{tikzcd}$$

is cartesian.

Thus for each $n \in \mathbb{Z}$ we have an isomorphism

$$K_n^{\mathrm{MW}}(F) \cong K_n^{\mathrm{M}}(F) \times_{i^n(F)} I^n(F).$$

It follows that for all n there is a natural short exact sequence

$$0 \to I^{n+1}(F) \to K_n^{\mathrm{MW}}(F) \to K_n^{\mathrm{M}}(F) \to 0$$

where the inclusion $I^{n+1}(F) \to K_n^{\mathrm{MW}}(F)$ is given by

$$\langle\langle a_1, \ldots, a_{n+1} \rangle\rangle \mapsto \eta[a_1]\cdots[a_n].$$

Similarly, for $n \geq 0$, there is a short exact sequence

$$0 \to 2K_n^{\mathrm{M}}(F) \to K_n^{\mathrm{MW}}(F) \to I^n(F) \to 0$$

where the inclusion $2K_n^{\mathrm{M}}(F) \to K_n^{\mathrm{MW}}(F)$ is given (for $n \geq 1$) by

$$2\{a_1, \ldots, a_n\} \mapsto h[a_1]\cdots[a_n].$$

Observe that, when $n \geq 2$,

$$h[a_1][a_2]\cdots[a_n] = ([a_1][a_2] - [a_2][a_1])[a_3]\cdots[a_n] = [a_1^2][a_2]\cdots[a_n].$$

(The first equality follows from Lemma 2.3 (3) below, the second from the observation that $[a_1^2]\cdots[a_n] \in \mathrm{Ker}(K_n^{\mathrm{MW}}(F) \to I^n(F)) = 2K_n^{\mathrm{M}}(F)$ and the fact, which follows from Morel's theorem, that the composite $2K_n^{\mathrm{M}}(F) \to K_n^{\mathrm{MW}}(F) \to K_n^{\mathrm{M}}(F)$ is the natural inclusion map.)
When $n = 0$ we have an isomorphism of rings

$$\mathrm{GW}(F) \cong W(F) \times_{\mathbb{Z}/2} \mathbb{Z} \cong K_0^{\mathrm{MW}}(F).$$

Under this isomorphism $\langle\langle a \rangle\rangle$ corresponds to $\eta[a]$ and $\langle a \rangle$ corresponds to $\eta[a] + 1$. (Observe that with this identification, $h = \eta[-1] + 2 = \langle 1 \rangle + \langle -1 \rangle \in K_0^{\mathrm{MW}}(F) = \mathrm{GW}(F)$, as expected.)
Thus each $K_n^{\mathrm{MW}}(F)$ has the structure of a $\mathrm{GW}(F)$-module (and hence also of a $\mathbb{Z}[F^\times]$-module), with the action given by $\langle\langle a \rangle\rangle \cdot ([a_1]\cdots[a_n]) = \eta[a][a_1]\cdots[a_n]$.
We record here some elementary identities in Milnor-Witt K-theory which we will need below.

Lemma 2.3. *Let $a, b \in F^\times$. The following identities hold in the Milnor-Witt K-theory of F:*

(1) $[a][-1] = [a][a]$.
(2) $[ab] = [a] + \langle a \rangle [b]$.
(3) $[a][b] = -\langle -1 \rangle [b][a]$.

Proof.

(1) See, for example, the proof of Lemma 2.7 in [7].
(2) $\langle a \rangle b = (\eta[a] + 1)[b] = \eta[a][b] + [b] = [ab] - [a]$.
(3) See [7], Lemma 2.7.

□

2.3. Homology of Groups. Given a group G and a $\mathbb{Z}[G]$-module M, $\mathrm{H}_n(G,M)$ will denote the nth homology group of G with coefficients in the module M. $B_\bullet(G)$ will denote the *right bar resolution of* G: $B_n(G)$ is the free right $\mathbb{Z}[G]$-module with basis the elements $[g_1|\cdots|g_n]$, $g_i \in G$. ($B_0(G)$ is isomorphic to $\mathbb{Z}[G]$ with generator the symbol $[\ ]$.) The boundary $d = d_n : B_n(G) \to B_{n-1}(G)$, $n \geq 1$, is given by

$$d([g_1|\cdots|g_n]) = \sum_{i=o}^{n-1}(-1)^i[g_1|\cdots|\hat{g}_i|\cdots|g_n] + (-1)^n[g_1|\cdots|g_{n-1}]\langle g_n\rangle.$$

The augmentation $B_0(G) \to \mathbb{Z}$ makes $B_\bullet(G)$ into a free resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$, and thus $\mathrm{H}_n(G,M) = H_n(B_\bullet(G)\otimes_{\mathbb{Z}[G]} M)$.
If $C_\bullet = (C_q, d)$ is a non-negative complex of $\mathbb{Z}[G]$-modules, then $E_{\bullet,\bullet} := B_\bullet(G)\otimes_{\mathbb{Z}[G]} C_\bullet$ is a double complex of abelian groups. Each of the two filtrations on $E_{\bullet,\bullet}$ gives a spectral sequence converging to the homology of the total complex of $E_{\bullet,\bullet}$, which is by definition, $\mathrm{H}_\bullet(G,C)$. (see, for example, Brown, [3], Chapter VII).
The first spectral sequence has the form

$$E^2_{p,q} = \mathrm{H}_p(G, H_q(C)) \Longrightarrow \mathrm{H}_{p+q}(G,C).$$

In the special case that there is a weak equivalence $C_\bullet \to \mathbb{Z}$ (the complex consisting of the trivial module $\mathbb{Z}$ concentrated in dimension 0), it follows that $\mathrm{H}_\bullet(G,C) = \mathrm{H}_\bullet(G,\mathbb{Z})$.
The second spectral sequence has the form

$$E^1_{p,q} = \mathrm{H}_p(G, C_q) \Longrightarrow \mathrm{H}_{p+q}(G,C).$$

Thus, if $C_\bullet$ is weakly equivalent to $\mathbb{Z}$, this gives a spectral sequence converging to $\mathrm{H}_\bullet(G,\mathbb{Z})$.
Our analysis of the homology of special linear groups will exploit the action of these groups on certain permutation modules. It is straightforward to compute the map induced on homology groups by a map of permutation modules. We recall the following basic principles (see, for example, [6]): If G is a group and if X is a G-set, then Shapiro's Lemma says that

$$\mathrm{H}_p(G,\mathbb{Z}[X]) \cong \bigoplus_{y\in X/G} \mathrm{H}_p(G_y,\mathbb{Z}),$$

the isomorphism being induced by the maps

$$\mathrm{H}_p(G_y,\mathbb{Z}) \to \mathrm{H}_p(G,\mathbb{Z}[X])$$

described at the level of chains by

$$B_p\otimes_{\mathbb{Z}[G_y]}\mathbb{Z} \to B_p\otimes_{\mathbb{Z}[G]}\mathbb{Z}[X], \quad z\otimes 1 \mapsto z\otimes y.$$

Let X_i, $i=1,2$ be transitive G-sets. Let $x_i \in X_i$ and let H_i be the stabiliser of x_i, $i=1,2$. Let $\phi : \mathbb{Z}[X_1] \to \mathbb{Z}[X_2]$ be a map of $\mathbb{Z}[G]$-modules with

$$\phi(x_1) = \sum_{g\in G/H_2} n_g g x_2, \qquad \text{with } n_g \in \mathbb{Z}.$$

Then the induced map $\phi_\bullet : \mathrm{H}_\bullet(H_1, \mathbb{Z}) \to \mathrm{H}_\bullet(H_2, \mathbb{Z})$ is given by the formula

$$\phi_\bullet(z) = \sum_{g \in H_1 \backslash G / H_2} n_g \mathrm{cor}^{H_2}_{g^{-1}H_1 g \cap H_2} \mathrm{res}^{g^{-1}H_1 g}_{g^{-1}H_1 g \cap H_2} \left(g^{-1} \cdot z\right) \tag{1}$$

There is an obvious extension of this formula to non-transitive G-sets.

2.4. HOMOLOGY OF $\mathrm{SL}_n(F)$ AND MILNOR-WITT K-THEORY. Let F be an infinite field.

The theorem of Matsumoto and Moore ([10], [16]) gives a presentation of the group $\mathrm{H}_2(\mathrm{SL}_2(F), \mathbb{Z})$. It has the following form: The generators are symbols $\langle a_1, a_1 \rangle$, $a_i \in F^\times$, subject to the relations:

(i) $\langle a_1, a_2 \rangle = 0$ if $a_i = 1$ for some i
(ii) $\langle a_1, a_2 \rangle = \langle a_2^{-1}, a_1 \rangle$
(iii) $\langle a_1, a_2 b_2 \rangle + \langle a_2, b_2 \rangle = \langle a_1 a_2, b_2 \rangle + \langle a_1, a_2 \rangle$
(iv) $\langle a_1, a_2 \rangle = \langle a_1, -a_1 a_2 \rangle$
(v) $\langle a_1, a_2 \rangle = \langle a_1, (1 - a_1) a_2 \rangle$

It can be shown that for all $n \geq 2$, $K_n^{\mathrm{MW}}(F)$ admits a (generalised) Matsumoto-Moore presentation:

THEOREM 2.4 ([7], Theorem 2.5). *For $n \geq 2$, $K_n^{\mathrm{MW}}(F)$ admits the following presentation as an additive group:*
Generators: The elements $[a_1][a_2] \cdots [a_n]$, $a_i \in F^\times$.
Relations:

(i) $[a_1][a_2] \cdots [a_n] = 0$ *if $a_i = 1$ for some i.*
(ii) $[a_1] \cdots [a_{i-1}][a_i] \cdots [a_n] = [a_1] \cdots [a_i^{-1}][a_{i-1}] \cdots [a_n]$
(iii) $[a_1] \cdots [a_{n-1}][a_n b_n] + [a_1] \cdots \widehat{[a_{n-1}]}[a_n][b_n] = [a_1] \cdots [a_{n-1} a_n][b_n] + [a_1] \cdots [a_{n-1}][a_n]$
(iv) $[a_1] \cdots [a_{n-1}][a_n] = [a_1] \cdots [a_{n-1}][-a_{n-1} a_n]$
(v) $[a_1] \cdots [a_{n-1}][a_n] = [a_1] \cdots [a_{n-1}][(1 - a_{n-1}) a_n]$

In particular, it follows when $n = 2$ that there is a natural isomorphism $K_2^{\mathrm{MW}}(F) \cong \mathrm{H}_2(\mathrm{SL}_2(F), \mathbb{Z})$. This last fact is essentially due to Suslin ([24]). A more recent proof, which we will need to invoke below, has been given by Mazzoleni ([11]).

Recall that Suslin ([23]) has constructed a natural surjective homomorphism $\mathrm{H}_n(\mathrm{GL}_n(F), \mathbb{Z}) \to K_n^{\mathrm{M}}(F)$ whose kernel is the image of $\mathrm{H}_n(\mathrm{GL}_{n-1}(F), \mathbb{Z})$.

In [8], the authors proved that the map $\mathrm{H}_3(\mathrm{SL}_3(F), \mathbb{Z}) \to \mathrm{H}_3(\mathrm{GL}_3(F), \mathbb{Z})$ is injective, that the image of the composite $\mathrm{H}_3(\mathrm{SL}_3(F), \mathbb{Z}) \to \mathrm{H}_3(\mathrm{GL}_3(F), \mathbb{Z}) \to K_3^{\mathrm{M}}(F)$ is $2K_3^{\mathrm{M}}(F)$ and that the kernel of this composite is precisely the image of $\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z})$.

In the next section we will construct natural homomorphisms $T_n \circ \epsilon_n : \mathrm{H}_n(\mathrm{SL}_n(F), \mathbb{Z}) \to K_n^{\mathrm{MW}}(F)$, in a manner entirely analogous to Suslin's construction. In particular, the image of $\mathrm{H}_n(\mathrm{SL}_{n-1}(F), \mathbb{Z})$ is contained in the kernel of $T_n \circ \epsilon_n$ and the diagrams

$$\begin{tikzcd} \mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z}) \arrow[r] \arrow[d] & K_n^{\mathrm{MW}}(F) \arrow[d] \\ \mathrm{H}_n(\mathrm{GL}_n(F),\mathbb{Z}) \arrow[r] & K_n^{\mathrm{M}}(F) \end{tikzcd}$$

commute. It follows that the image of $T_3 \circ \epsilon_3$ is $2K_3^{\mathrm{M}}(F) \subset K_3^{\mathrm{MW}}(F)$, and its kernel is the image of $\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z})$.

3. The algebra $\tilde{S}(F^\bullet)$

In this section we introduce a graded algebra functorially associated to F which admits a natural homomorphism to Milnor-Witt K-theory and from the homology of $\mathrm{SL}_n(F)$. It is the analogue of Suslin's algebra $S_\bullet(F)$ in [24], which admits homomorphisms to Milnor K-theory and from the homology of $\mathrm{GL}_n(F)$. However, we will need to modify this algebra in the later sections below, by projecting onto the 'multiplicative ' part, in order to derive our results about the homology of $\mathrm{SL}_n(F)$.

We say that a finite set of vectors $v_1, \ldots, v_q$ in an n-dimensional vector space V *are in general position* if every subset of size $\min(q, n)$ is linearly independent. If $v_1, \ldots, v_q$ are elements of the n-dimensional vector space V and if $\mathcal{E}$ is an ordered basis of V, we let $[v_1|\cdots|v_q]_{\mathcal{E}}$ denote the $n \times q$ matrix whose i-th column is the components of v_i with respect to the basis $\mathcal{E}$.

3.1. Definitions. For a field F and finite-dimensional vector spaces V and W, we let $X_p(W,V)$ denote the set of all ordered p-tuples of the form

$$((w_1,v_1),\ldots,(w_p,v_p))$$

where $(w_i,v_i) \in W \oplus V$ and the v_i are in general position. We also define $X_0(W,V) := \emptyset$. $X_p(W,V)$ is naturally an $\mathrm{A}(W,V)$-module, where

$$\mathrm{A}(W,V) := \begin{pmatrix} \mathrm{Id}_W & \mathrm{Hom}(V,W) \\ 0 & \mathrm{GL}(V) \end{pmatrix} \subset \mathrm{GL}(W \oplus V)$$

Let $C_p(W,V) = \mathbb{Z}[X_p(W,V)]$, the free abelian group with basis the elements of $X_p(W,V)$. We obtain a complex, $C_\bullet(W,V)$, of $\mathrm{A}(W,V)$-modules by introducing the natural simplicial boundary map

$$\begin{array}{rcl} d_{p+1} : C_{p+1}(W,V) & \to & C_p(W,V) \\ ((w_1,v_1),\ldots,(w_{p+1},v_{p+1})) & \mapsto & \\ & & \displaystyle\sum_{i=1}^{p+1} (-1)^{i+1}((w_1,v_1),\ldots,\widehat{(w_i,v_i)},\ldots,(w_{p+1},v_{p+1})) \end{array}$$

Lemma 3.1. *If F is infinite, then $\mathrm{H}_p(C_\bullet(W,V)) = 0$ for all p.*

Proof. If

$$z = \sum_i n_i((w_1^i, v_1^i), \ldots, (w_p^i, v_p^i)) \in C_p(W, V)$$

is a cycle, then since F is infinite, it is possible to choose $v \in V$ such that $v, v_1^i, \ldots, v_p^i$ are in general position for all i. Then $z = d_{p+1}((-1)^p s_v(z))$ where s_v is the 'partial homotopy operator' defined by $s_v((w_1, v_1), \ldots, (w_p, v_p)) =$

$$\begin{cases} ((w_1, v_1), \ldots, (w_p, v_p), (0, v)), & \text{if } v, v_1, \ldots v_p \text{ are in general position,} \\ 0, & \text{otherwise} \end{cases}$$

□

We will assume our field F is infinite for the remainder of this section. (In later sections, it will even be assumed to be of characteristic zero.)

If $n = \dim_F(V)$, we let $H(W, V) := \mathrm{Ker}(d_n) = \mathrm{Im}(d_{n+1})$. This is an $\mathrm{A}(W, V)$-submodule of $C_n(W, V)$. Let $\tilde{\mathrm{S}}(W, V) := \mathrm{H}_0(\mathrm{SA}(W, V), H(W, V)) = H(W, V)_{\mathrm{SA}(W,V)}$ where $\mathrm{SA}(W, V) := \mathrm{A}(W, V) \cap \mathrm{SL}(W \oplus V)$.

If $W' \subset W$, there are natural inclusions $X_p(W', V) \to X_p(W, V)$ inducing a map of complexes of $\mathrm{A}(W', V)$-modules $C_\bullet(W', V) \to C_\bullet(W, V)$.

When $W = 0$, we will use the notation, $X_p(V)$, $C_p(V)$, $H(V)$ and $\tilde{\mathrm{S}}(V)$ instead of $X_p(0, V)$, $C_p(0, V)$, $H(0, V)$ and $\tilde{\mathrm{S}}(0, V)$

Since, $\mathrm{A}(W, V)/\mathrm{SA}(W, V) \cong F^\times$, any homology group of the form

$$\mathrm{H}_i(\mathrm{SA}(W, V), M), \text{ where } M \text{ is a } \mathrm{A}(W, V)\text{-module,}$$

is naturally a $\mathbb{Z}[F^\times]$-module: If $a \in F^\times$ and if $g \in \mathrm{A}(W, V)$ is any element of determinant a, then the action of a is the map on homology induced by conjugation by g on $\mathrm{A}(W, V)$ and multiplication by g on M. In particular, the groups $\tilde{\mathrm{S}}(W, V)$ are $\mathbb{Z}[F^\times]$-modules.

Let $e_1, \ldots, e_n$ denote the standard basis of F^n. Given $a_1, \ldots, a_n \in F^\times$, we let $\lfloor a_1, \ldots, a_n \rceil$ denote the class of $d_{n+1}(e_1, \ldots, e_n, a_1e_1 + \cdots + a_ne_n)$ in $\tilde{\mathrm{S}}(F^n)$. If $b \in F^\times$, then $\langle b \rangle \cdot \lfloor a_1, \ldots, a_n \rceil$ is represented by

$$d_{n+1}(e_1, \ldots, be_i, \ldots, e_n, a_1e_1 + \cdots a_ibe_i \cdots + a_ne_n)$$

for any i. (As a lifting of $b \in F^\times$, choose the diagonal matrix with b in the (i, i)-position and 1 in all other diagonal positions.)

Remark 3.2. Given $x = (v_1, \ldots, v_v, v) \in X_{n+1}(F^n)$, let $A = [v_1 | \cdots | v_n] \in \mathrm{GL}_n(F)$ of determinant $\det A$ and let $A' = \mathrm{diag}(1, \ldots, 1, \det A)$. Then $B = A'A^{-1} \in \mathrm{SL}_n(F)$ and thus x is in the $\mathrm{SL}_n(F)$-orbit of

$$(e_1, \ldots, e_{n-1}, \det Ae_n, A'w) \text{ with } w = A^{-1}v,$$

and hence $d_{n+1}(x)$ represents the element $\langle \det A \rangle \lfloor w \rceil$ in $\tilde{S}(F^n)$.

Theorem 3.3. $\tilde{\mathrm{S}}(F^n)$ *has the following presentation as a* $\mathbb{Z}[F^\times]$*-module:*
Generators: The elements $\lfloor a_1, \ldots, a_n \rceil$, $a_i \in F^\times$

Relations: For all $a_1,\ldots,a_n \in F^\times$ and for all $b_1,\ldots,b_n \in F^\times$ with $b_i \neq b_j$ for $i \neq j$

$$\lfloor b_1a_1,\ldots,b_na_n\rceil - \lfloor a_1,\ldots,a_n\rceil =$$
$$\sum_{i=1}^{n}(-1)^{n+i}\left\langle(-1)^{n+i}a_i\right\rangle \lfloor a_1(b_1-b_i),\ldots,a_i\widehat{(b_i-b_i)},\ldots,a_n(b_n-b_i),b_i\rceil.$$

Proof. Taking $\mathrm{SL}_n(F)$-coinvariants of the exact sequence of $\mathbb{Z}[\mathrm{GL}_n(F)]$-modules

$$C_{n+2}(F^n) \xrightarrow{d_{n+2}} C_{n+1}(F^n) \xrightarrow{d_{n+1}} H(F^n) \longrightarrow 0$$

gives the exact sequence of $\mathbb{Z}[F^\times]$-modules

$$C_{n+2}(F^n)_{\mathrm{SL}_n(F)} \xrightarrow{d_{n+2}} C_{n+1}(F^n)_{\mathrm{SL}_n(F)} \xrightarrow{d_{n+1}} \tilde{S}(F^n) \longrightarrow 0.$$

It is straightforward to verify that

$$X_{n+1}(F^n) \cong \coprod_{a=(a_1,\ldots,a_n),a_i\neq 0} \mathrm{GL}_n(F)\cdot(e_1,\ldots,e_n,a)$$

as a $\mathrm{GL}_n(F)$-set. It follows that

$$C_{n+1}(F^n) \cong \bigoplus_a \mathbb{Z}[\mathrm{GL}_n(F)]\cdot(e_1,\ldots,e_n,a)$$

as a $\mathbb{Z}[\mathrm{GL}_n(F)]$-module, and thus that

$$C_{n+1}(F^n)_{\mathrm{SL}_n(F)} \cong \bigoplus_a \mathbb{Z}[F^\times]\cdot(e_1,\ldots,e_n,a)$$

as a $\mathbb{Z}[F^\times]$-module.

Similarly, every element of $X_{n+2}(F^n)$ is in the $\mathrm{GL}_n(F)$-orbit of a unique element of the form $(e_1,\ldots,e_n,a,b\cdot a)$ where $a=(a_1,\ldots,a_n)$ with $a_i \neq 0$ for all i and $b=(b_1,\ldots,b_n)$ with $b_i \neq 0$ for all i and $b_i \neq b_j$ for all $i \neq j$, and $b\cdot a := (b_1a_1,\ldots,b_na_n)$. Thus

$$X_{n+2}(F^n) \cong \coprod_{(a,b)} \mathrm{GL}_n(F)\cdot(e_1,\ldots,e_n,a,b\cdot a)$$

as a $\mathrm{GL}_n(F)$-set and

$$C_{n+2}(F^n)_{\mathrm{SL}_n(F)} \cong \bigoplus_{(a,b)} \mathbb{Z}[F^\times]\cdot(e_1,\ldots,e_n,a,b\cdot a)$$

as a $\mathbb{Z}[F^\times]$-module.

So d_{n+1} induces an isomorphism

$$\frac{\oplus\mathbb{Z}[F^\times]\cdot(e_1,\ldots,e_n,a)}{\langle d_{n+2}(e_1,\ldots,e_n,a,b\cdot a)|(a,b)\rangle} \cong \tilde{S}(F^n).$$

Now $d_{n+2}(e_1,\ldots,e_n,a,b\cdot a) =$

$$\sum_{i=1}^{n}(-1)^{i+1}(e_1,\ldots,\hat{e_i},\ldots,e_n,a,b\cdot a) + (-1)^i\big((e_1,\ldots,e_n,b\cdot a)-(e_1,\ldots,e_n,a)\big).$$

Applying the idea of Remark 3.2 to the terms $(e_1,\ldots,\hat{e_i},\ldots,e_n,a,b\cdot a)$ in the sum above, we let $M_i(a) := [e_1|\cdots|\hat{e_i}|\cdots|e_n|a]$ and $\delta_i = \det M_i(a) = (-1)^{n-i}a_i$. Since

$$M_i(a)^{-1} = \begin{pmatrix} 1 & \ldots & 0 & -a_1/a_i & 0 & \ldots & 0 \\ 0 & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 1 & -a_{i-1}/a_i & 0 & \ldots & 0 \\ 0 & \ldots & 0 & -a_{i+1}a_i & 1 & \ldots & 0 \\ 0 & \ldots & 0 & \vdots & 0 & \ddots & 0 \\ 0 & \ldots & 0 & -a_n/a_i & 0 & \ldots & 1 \\ 0 & \ldots & 0 & 1/a_i & 0 & \ldots & 0 \end{pmatrix}$$

it follows that $d_{n+1}(e_1,\ldots,\hat{e_i},\ldots,e_n,a,b{\cdot}a)$ represents $\langle\delta_i\rangle\lfloor w_i\rceil \in \tilde{S}(F^n)$ where $w_i = M_i(a)^{-1}(b\cdot a) = (a_1(b_1-b_i),\ldots,a_i\widehat{(b_i-b_i)},\ldots,a_n(b_n-b_i),b_i)$. This proves the theorem. □

3.2. Products. If $W' \subset W$, there is a natural bilinear pairing

$$C_p(W',V)\times C_q(W)\to C_{p+q}(W\oplus V), \qquad (x,y)\mapsto x*y$$

defined on the basis elements by

$$((w_1',v_1),\ldots,(w_p',v_p))*(w_1,\ldots,w_q) := \Big((w_1',v_1),\ldots,(w_p',v_p),(w_1,0),\ldots,(w_q,0)\Big).$$

This pairing satisfies $d_{p+q}(x*y) = d_p(x)*y+(-1)^p x*d_q(y)$.
Furthermore, if $\alpha\in \mathrm{A}(W',V)\subset \mathrm{GL}(W\oplus V)$ then $(\alpha x)*y = \alpha(x*y)$, and if $\alpha\in\mathrm{GL}(V)\subset \mathrm{A}(W',V)\subset\mathrm{GL}(W\oplus V)$ and $\beta\in\mathrm{GL}(W)\subset\mathrm{GL}(W\oplus V)$, then $(\alpha x)*(\beta y) = (\alpha\cdot\beta)(x*y)$. (However, if $W'\neq 0$ then the images of $\mathrm{A}(W',V)$ and $\mathrm{GL}(W)$ in $\mathrm{GL}(W\oplus V)$ don't commute.)
In particular, there are induced pairings on homology groups

$$H(W',V)\otimes H(W)\to H(W\oplus V),$$

which in turn induce well-defined pairings

$$\tilde{\mathrm{S}}(W',V)\otimes H(W)\to\tilde{\mathrm{S}}(W,V) \text{ and } \tilde{\mathrm{S}}(V)\otimes\tilde{\mathrm{S}}(W)\to\tilde{\mathrm{S}}(W\oplus V).$$

Observe further that this latter pairing is $\mathbb{Z}[F^\times]$-balanced: If $a\in F^\times$, $x\in\tilde{\mathrm{S}}(W)$ and $y\in\tilde{\mathrm{S}}(V)$, then $(\langle a\rangle x)*y = x*(\langle a\rangle y) = \langle a\rangle(x*y)$. Thus there is a well-defined map

$$\tilde{\mathrm{S}}(V)\otimes_{\mathbb{Z}[F^\times]}\tilde{\mathrm{S}}(W)\to\tilde{\mathrm{S}}(W\oplus V).$$

In particular, the groups $\{H(F^n)\}_{n\geq 0}$ form a natural graded (associative) algebra, and the groups $\{\tilde{\mathrm{S}}(F^n)\}_{n\geq 0} = \tilde{S}(F^\bullet)$ form a graded associative $\mathbb{Z}[F^\times]$-algebra.
The following explicit formula for the product in $\tilde{S}(F^\bullet)$ will be needed below:

Lemma 3.4. *Let $a_1,\ldots,a_n$ and $a_1',\ldots,a_m'$ be elements of $F^\times$. Let $b_1,\ldots,b_n$, $b_1',\ldots,b_m'$ be any elements of $F^\times$ satisfying $b_i\neq b_j$ for $i\neq j$ and $b_s'\neq b_t'$ for $s\neq t$.*

Then

$$\lfloor a_1,\dots,a_n\rceil * \lfloor a'_1,\dots,a'_m\rceil =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m}(-1)^{m+n+i+j}\left\langle(-1)^{i+j}a_ia'_j\right\rangle\times$$

$$\times\lfloor a_1(b_1-b_i),\dots,a_i\widehat{(b_i-b_i)},\dots,b_i,a'_1(b'_1-b'_j),\dots,a'_j\widehat{(b'_j-b'_j)},\dots,b'_j\rceil$$

$$+(-1)^n\sum_{i=1}^{n}(-1)^{i+1}\left\langle(-1)^{i+1}a_i\right\rangle\lfloor a_1(b_1-b_i),\dots,a_i\widehat{(b_i-b_i)},\dots,b_i,b'_1a'_1,\dots,b'_ma'_m\rceil$$

$$+(-1)^m\sum_{j=1}^{m}(-1)^{j+1}\left\langle(-1)^{j+1}a'_j\right\rangle\lfloor b_1a_1,\dots,b_na_n,a'_1(b'_1-b'_j),\dots,a'_j\widehat{(b'_j-b'_j)},\dots,b'_j\rceil$$

$$+\lfloor b_1a_1,\dots,b_na_n,b'_1a'_1,\dots,b'_ma'_m\rceil$$

Proof. This is an entirely straightforward calculation using the definition of the product, Remark 3.2, the matrices $M_i(a)$, $M_j(a')$ as in the proof of Theorem 3.3, and the partial homotopy operators s_v with $v=(a_1b_1,\dots,a_nb_n,a'_1b'_1,\dots,a'_mb'_m)$. □

3.3. THE MAPS ϵ_V. If $\dim_F(V)=n$, then the exact sequence of $\mathrm{GL}(V)$-modules

$$0\longrightarrow H(V)\longrightarrow C_n(V)\xrightarrow{d_n}C_{n-1}(V)\xrightarrow{d_{n-1}}\cdots\xrightarrow{d_1}C_0(V)=\mathbb{Z}\longrightarrow 0$$

gives rise to an iterated connecting homomorphism

$$\epsilon_V:\mathrm{H}_n(\mathrm{SL}(V),\mathbb{Z})\to\mathrm{H}_0(\mathrm{SL}(V),H(V))=\tilde{\mathrm{S}}(V).$$

Note that the collection of groups $\{\mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z})\}$ form a graded $\mathbb{Z}[F^\times]$-algebra under the graded product induced by exterior product on homology, together with the obvious direct sum homomorphism $\mathrm{SL}_n(F)\times\mathrm{SL}_m(F)\to\mathrm{SL}_{n+m}(F)$.

LEMMA 3.5. *The maps $\epsilon_n:\mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z})\to\tilde{\mathrm{S}}(F^n)$, $n\ge 0$, give a well-defined homomorphism of graded $\mathbb{Z}[F^\times]$-algebras; i.e.*

(1) *If $a\in F^\times$ and $z\in\mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z})$, then $\epsilon_n(\langle a\rangle z)=\langle a\rangle\,\epsilon_n(z)$ in $\tilde{\mathrm{S}}(F^n)$, and*
(2) *If $z\in\mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z})$ and $w\in\mathrm{H}_m(\mathrm{SL}_m(F),\mathbb{Z})$ then*

$$\epsilon_{n+m}(z\times w)=\epsilon_n(z)*\epsilon_m(w)\ \text{ in }\tilde{\mathrm{S}}(F^{n+m}).$$

Proof.

(1) The exact sequence above is a sequence of $\mathrm{GL}(V)$-modules and hence all of the connecting homomorphisms $\delta_i:\mathrm{H}_{n-i+1}(\mathrm{SL}(V),\mathrm{Im}(d_i))\to\mathrm{H}_{n-i}(\mathrm{SL}(V),\mathrm{Ker}(d_i))$ are $F^\times$-equivariant.
(2) Let $\mathcal{C}^\tau_\bullet(V)$ denote the truncated complex.

$$\mathcal{C}^\tau_p(V)=\begin{cases}C_p(V), & p\le\dim_F(V)\\ 0, & p>\dim_F(V)\end{cases}$$

Thus $H(V) \to \mathcal{C}^{\tau}_{\bullet}(V)$ is a weak equivalence of complexes (where we regard $H(V)$ as a complex concentrated in dimension $\dim(V)$). Since the complexes $\mathcal{C}^{\tau}_{\bullet}(V)$ are complexes of free abelian groups, it follows that for two vector spaces V and W, the map $H(V)\otimes_{\mathbb{Z}} H(W) \to T_{\bullet}(V,W)$ is an equivalence of complexes, where $T_{\bullet}(V,W)$ is the total complex of the double complex $\mathcal{C}^{\tau}_{\bullet}(V) \otimes_{\mathbb{Z}} \mathcal{C}^{\tau}_{\bullet}(W)$. Now $T_{\bullet}(V,W)$ is a complex of $\mathrm{SL}(V) \times \mathrm{SL}(W)$-modules, and the product $*$ induces a commutative diagram of complexes of $\mathrm{SL}(V) \times \mathrm{SL}(W)$-complexes:

$$\begin{tikzcd}
H(V) \otimes_{\mathbb{Z}} H(W) \arrow[r] \arrow[d, "*"] & \mathcal{C}^{\tau}_{\bullet}(V) \otimes \mathcal{C}^{\tau}_{\bullet}(W) \arrow[d, "*"] \\
H(V \oplus W) \arrow[r] & \mathcal{C}^{\tau}_{\bullet}(V \oplus W)
\end{tikzcd}$$

which, in turn, induces a commutative diagram

$$\begin{tikzcd}
\mathrm{H}_n(\mathrm{SL}(V),\mathbb{Z}) \otimes \mathrm{H}_m(\mathrm{SL}(W),\mathbb{Z}) \arrow[r, "\epsilon_V \otimes \epsilon_W"] \arrow[d, "\times"] & \mathrm{H}_0(\mathrm{SL}(V),H(V)) \otimes \mathrm{H}_0(\mathrm{SL}(W),H(W)) \arrow[d, "\times"] \\
\mathrm{H}_{n+m}(\mathrm{SL}(V) \times \mathrm{SL}(W),\mathbb{Z}\otimes\mathbb{Z}) \arrow[r, "\epsilon_{T_\bullet}"] \arrow[d] & \mathrm{H}_0(\mathrm{SL}(V) \times \mathrm{SL}(W), H(V)\otimes H(W)) \arrow[d] \\
\mathrm{H}_{n+m}(\mathrm{SL}(V \oplus W),\mathbb{Z}) \arrow[r, "\epsilon_{V\oplus W}"] & \mathrm{H}_0(\mathrm{SL}(V\oplus W), H(V \oplus W))
\end{tikzcd}$$

(where $n = \dim(V)$ and $m = \dim(W)$).

□

Lemma 3.6. *If $V = W \oplus W'$ with $W' \neq 0$, then the composite*

$$\mathrm{H}_n(\mathrm{SL}(W),\mathbb{Z}) \longrightarrow \mathrm{H}_n(\mathrm{SL}(V),\mathbb{Z}) \xrightarrow{\epsilon_V} \tilde{\mathrm{S}}(V)$$

is zero.

Proof. The exact sequence of $\mathrm{SL}(V)$-modules

$$0 \to \mathrm{Ker}(d_1) \to C_1(V) \to \mathbb{Z} \to 0$$

is split as a sequence of $\mathrm{SL}(W)$-modules via the map $\mathbb{Z} \to C_1(V), m \mapsto m \cdot e$ where e is any nonzero element of W'. It follows that the connecting homomorphism $\delta_1 : \mathrm{H}_n(\mathrm{SL}(W),\mathbb{Z}) \to \mathrm{H}_{n-1}(\mathrm{SL}(W),\mathrm{Ker}(d_1))$ is zero. □

Let $\mathrm{SH}_n(F)$ denote the cokernel of the map $\mathrm{H}_n(\mathrm{SL}_{n-1}(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z})$. It follows that the maps ϵ_n give well-defined homomorphisms $\mathrm{SH}_n(F) \to \tilde{\mathrm{S}}(F^n)$, which yield a homomorphism of graded $\mathbb{Z}[F^{\times}]$-algebras $\epsilon_{\bullet} : \mathrm{SH}_{\bullet}(F) \to \tilde{\mathrm{S}}(F^{\bullet})$.

3.4. The maps D_V. Suppose now that W and V are vector spaces and that $\dim(V) = n$. Fix a basis $\mathcal{E}$ of V. The group $\mathrm{A}(W,V)$ acts transitively on $X_n(W,V)$ (with trivial stabilizers), while the orbits of $\mathrm{SA}(W,V)$ are in one-to-one correspondence with the points of $F^{\times}$ via the correspondence

$$X_n(W,V) \to F^{\times}, \quad ((w_1,v_1),\ldots,(w_n,v_n)) \mapsto \det\left([v_1|\cdots|v_n]_{\mathcal{E}}\right).$$

Thus we have an induced isomorphism

$$\mathrm{H}_0(\mathrm{SA}(W,V), C_n(W,V)) \xrightarrow[\cong]{\det} \mathbb{Z}[F^\times].$$

Taking $\mathrm{SA}(W,V)$-coinvariants of the inclusion $H(W,V) \to C_n(W,V)$ then yields a homomorphism of $\mathbb{Z}[F^\times]$-modules

$$D_{W,V} : \tilde{\mathrm{S}}(W,V) \to \mathbb{Z}[F^\times].$$

In particular, for each $n \geq 1$ we have a homomorphism of $\mathbb{Z}[F^\times]$-modules $D_n : \tilde{\mathrm{S}}(F^n) \to \mathbb{Z}[F^\times]$.
We will also set $D_0 : \tilde{\mathrm{S}}(F^0) = \mathbb{Z} \to \mathbb{Z}$ equal to the identity map. Here $\mathbb{Z}$ is a trivial $F^\times$-module.
We set

$$\mathcal{A}_n = \begin{cases} \mathbb{Z}, & n = 0 \\ \mathcal{I}_{F^\times}, & n \text{ odd} \\ \mathbb{Z}[F^\times], & n > 0 \text{ even} \end{cases}$$

We have $\mathcal{A}_n \subset \mathbb{Z}[F^\times]$ for all n and we make $\mathcal{A}_\bullet$ into a graded algebra by using the multiplication on $\mathbb{Z}[F^\times]$.

Lemma 3.7.

(1) *The image of D_n is $\mathcal{A}_n$.*
(2) *The maps $D_\bullet : \tilde{\mathrm{S}}(F^\bullet) \to \mathcal{A}_\bullet$ define a homomorphism of graded $\mathbb{Z}[F^\times]$-algebras.*
(3) *For each $n \geq 0$, the surjective map $D_n : \tilde{\mathrm{S}}(F^n) \to \mathcal{A}_n$ has a $\mathbb{Z}[F^\times]$-splitting.*

Proof.

(1) Consider a generator $\lfloor a_1, \ldots, a_n \rceil$ of $\tilde{\mathrm{S}}(F^n)$.
Let $e_1, \ldots, e_n$ be the standard basis of F^n. Let $a := a_1e_1 + \cdots + a_ne_n$. Then

$$\begin{aligned} \lfloor a_1, \ldots, a_n \rceil &= d_{n+1}(e_1, \ldots, e_n, a) \\ &= \sum_{i=1}^{n} (-1)^{i+1}(e_1, \ldots, \widehat{e_i}, \ldots, e_n, a) + (-1)^n(e_1, \ldots, e_n). \end{aligned}$$

Thus

$$\begin{aligned} D_n(\lfloor a_1, \ldots, a_n \rceil) &= \sum_{i=1}^{n} (-1)^{i+1} \langle \det([e_1|\cdots|\widehat{e_i}|\cdots|e_n|a]) \rangle + (-1)^n \langle 1 \rangle \\ &= \begin{cases} \langle a_1 \rangle - \langle -a_2 \rangle + \cdots + \langle a_n \rangle - \langle 1 \rangle, & n \text{ odd} \\ \langle -a_1 \rangle - \langle a_2 \rangle + \cdots - \langle a_n \rangle + \langle 1 \rangle, & n > 0 \text{ even} \end{cases} \end{aligned}$$

Thus, when n is even, $D_n(\lfloor -1, 1, -1, \ldots, -1, 1 \rceil) = \langle 1 \rangle$ and D_n maps onto $\mathbb{Z}[F^\times]$.
When n is odd, clearly, $D_n(\lfloor a_1, \ldots, a_n \rceil) \in \mathcal{I}_{F^\times}$. However, for any $a \in F^\times$, $D_n(\lfloor a, -1, 1, \ldots, -1, 1 \rceil) = \langle\langle a \rangle\rangle \in \mathcal{A}_n = \mathcal{I}_{F^\times}$.

(2) Note that $C_n(F^n) \cong \mathbb{Z}[\mathrm{GL}_n(F)]$ naturally. Let μ be the homomorphism of additive groups

$$\begin{aligned} \mu : \mathbb{Z}[\mathrm{GL}_n(F)] \otimes \mathbb{Z}[\mathrm{GL}_m(F)] &\to \mathbb{Z}[\mathrm{GL}_{n+m}(F)], \\ A \otimes B &\mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \end{aligned}$$

The formula $D_{m+n}(x * y) = D_n(x) \cdot D_m(y)$ now follows from the commutative diagram

$$\begin{tikzcd} H(F^n) \otimes H(F^m) \arrow[r, "*"] \arrow[d] & H(F^{n+m}) \arrow[d] \\ C_n(F^n) \otimes C_m(F^m) \arrow[r, "*"] \arrow[d, "\cong"] & C_{n+m}(F^{n+m}) \arrow[d, "\cong"] \\ \mathbb{Z}[\mathrm{GL}_n(F)] \otimes \mathbb{Z}[\mathrm{GL}_m(F)] \arrow[r, "\mu"] \arrow[d, "\det \otimes \det"] & \mathbb{Z}[\mathrm{GL}_{n+m}(F)] \arrow[d, "\det"] \\ \mathbb{Z}[F^\times] \otimes \mathbb{Z}[F^\times] \arrow[r, "\cdot"] & \mathbb{Z}[F^\times] \end{tikzcd}$$

(3) When n is even the maps D_n are split surjections, since the image is a free module of rank 1.

It is easy to verify that the map $D_1 : \tilde{\mathrm{S}}(F) \to \mathcal{A}_1 = \mathcal{I}_{F^\times}$ is an isomorphism. Now let $E \in \tilde{\mathrm{S}}(F^2)$ be any element satisfying $D_2(E) = \langle 1 \rangle$ (eg. we can take $E = \lfloor -1, 1 \rceil$). Then for $n = 2m+1$ odd, the composite $\tilde{\mathrm{S}}(F) * E^{*m} \to \tilde{\mathrm{S}}(F^n) \to \mathcal{I}_{F^\times} = \mathcal{A}_n$ is an isomorphism.

□

We will let $\tilde{S}(W,V)^+ = \mathrm{Ker}(D_{W,V})$. Thus $\tilde{S}(F^n) \cong \tilde{S}(F^n)^+ \oplus \mathcal{A}_n$ as a $\mathbb{Z}[F^\times]$-module by the results above.
Observe that it follows directly from the definitions that the image of ϵ_V is contained in $\tilde{S}(V)^+$ for any vector space V.

3.5. THE MAPS T_n.

LEMMA 3.8. *If $n \geq 2$ and $b_1, \ldots, b_n$ are distinct elements of $F^\times$ then*

$$[b_1][b_2]\cdots[b_n] = \sum_{i=1}^{n} [b_1 - b_i]\cdots[b_{i-1} - b_i][b_i][b_{i+1} - b_i]\cdots[b_n - b_i] \text{ in } K_n^{\mathrm{MW}}(F).$$

Proof. We will use induction on n starting with $n=2$: Suppose that $b_1 \neq b_2 \in F^\times$. Then

$$\begin{aligned}
&[b_1-b_2]([b_1]-[b_2])\\
&\quad = \left([b_1]+\langle b_1\rangle\left[1-\frac{b_2}{b_1}\right]\right)\left(-\langle b_1\rangle\left[\frac{b_2}{b_1}\right]\right) \text{ by Lemma 2.3 (2)}\\
&\quad = -\langle b_1\rangle[b_1]\left[\frac{b_2}{b_1}\right] \text{ since } [x][1-x]=0\\
&\quad = [b_1]([b_1]-[b_2]) \text{ by Lemma 2.3(2) again}\\
&\quad = [b_1]([-1]-[b_2]) \text{ by Lemma 2.3 (1)}\\
&\quad = [b_1](-\langle -1\rangle[-b_2])\\
&\quad = [-b_2][b_1] \text{ by Lemma 2.3 (3).}
\end{aligned}$$

Thus

$$\begin{aligned}
[b_1][b_2-b_1]+[b_1-b_2][b_2] &= -\langle -1\rangle[b_2-b_1][b_1]+[b_1-b_2][b_2]\\
&= -([b_1-b_2]-[-1])[b_1]+[b_1-b_2][b_2]\\
&= -[b_1-b_2]([b_1]-[b_2])+[-1][b_1]\\
&= -[-b_2][b_1]+[-1][b_1]=([-1]-[-b_2])[b_1]\\
&= -\langle -1\rangle[b_2][b_1]=[b_1][b_2]
\end{aligned}$$

proving the case $n=2$.
Now suppose that $n>2$ and that the result holds for $n-1$. Let $b_1,\ldots,b_n$ be distinct elements of $F^\times$.We wish to prove that

$$\left(\sum_{i=1}^{n-1}[b_1-b_i]\cdots[b_i]\cdots[b_{n-1}-b_i]\right)[b_n]=\sum_{i=1}^{n}[b_1-b_i]\cdots[b_i]\cdots[b_n-b_i].$$

We re-write this as:

$$\sum_{i=1}^{n-1}[b_1-b_i]\cdots[b_i]\cdots[b_{n-1}-b_i]([b_n]-[b_n-b_i])=[b_1-b_n]\cdots[b_{n-1}-b_n][b_n].$$

Now

$$\begin{aligned}
&[b_1-b_i]\cdots[b_i]\cdots[b_{n-1}-b_i]([b_n]-[b_n-b_i])\\
&\quad = (-\langle -1\rangle)^{n-i}[b_1-b_i]\cdots[b_{n-1}-b_i]\Big([b_i]([b_n]-[b_n-b_i])\Big)\\
&\quad = (-\langle -1\rangle)^{n-i}[b_1-b_i]\cdots[b_{n-1}-b_i]\Big([b_i-b_n][b_n]\Big)\\
&\quad = [b_1-b_i]\cdots[b_i-b_n]\cdots[b_{n-1}-b_i][b_n].
\end{aligned}$$

So the identity to be proved reduces to

$$\left(\sum_{i=1}^{n-1}[b_1-b_i]\cdots[b_i-b_n]\cdots[b_{n-1}-b_i]\right)[b_n]=[b_1-b_n]\cdots[b_{n-1}-b_n][b_n].$$

Letting $b'_i = b_i - b_n$ for $1 \le i \le n-1$, then $b_j - b_i = b'_j - b'_i$ for $i, j \le n-1$ and this reduces to the case $n-1$. $\square$

Theorem 3.9.

(1) *For all $n \ge 1$, there is a well-defined homomorphism of $\mathbb{Z}[F^\times]$-modules*

$$T_n : \tilde{\mathrm{S}}(F^n) \to K_n^{\mathrm{MW}}(F)$$

sending $\lfloor a_1, \ldots, a_n \rceil$ to $[a_1]\cdots[a_n]$.

(2) *The maps $\{T_n\}$ define a homomorphism of graded $\mathbb{Z}[F^\times]$-algebras $\tilde{\mathrm{S}}(F^\bullet) \to K_\bullet^{\mathrm{MW}}(F)$: We have*

$$T_{n+m}(x * y) = T_n(x) \cdot T_m(y), \qquad \text{for all } x \in \tilde{\mathrm{S}}(F^n), y \in \tilde{\mathrm{S}}(F^m).$$

Proof.

(1) By Theorem 3.3, in order to show that T_n is well-defined we must prove the identity

$$[b_1 a_1]\cdots[b_n a_n] - [a_1]\cdots[a_n] =$$
$$\sum_{i=1}^{n} (-\langle -1\rangle)^{n+i} \langle a_i\rangle [a_1(b_1 - b_i)]\cdots[\widehat{a_i(b_i - b_i)}]\cdots[a_n(b_n - b_i][b_i]$$

in $K_n^{\mathrm{MW}}(F)$.

Writing $[b_i a_i] = [a_i] + \langle a_i\rangle[b_i]$ and $[a_j(b_j - b_i)] = [a_j] + \langle a_j\rangle[b_j - b_i]$ and expanding the products on both sides and using (3) of Lemma 2.3 to permute terms, this identity can be rewritten as

$$\sum_{\emptyset \ne I \subset \{1,\ldots,n\}} (-\langle -1\rangle)^{\mathrm{sgn}(\sigma_I)} \langle a_{i_1}\cdots a_{i_k}\rangle [a_{j_1}]\cdots[a_{j_s}][b_{i_1}]\cdots[b_{i_k}] =$$
$$\sum_{\emptyset \ne I \subset \{1,\ldots,n\}} (-\langle -1\rangle)^{\mathrm{sgn}(\sigma_I)} \langle a_{i_1}\cdots a_{i_k}\rangle [a_{j_1}]\cdots[a_{j_s}] \times$$
$$\times \left(\sum_{t=1}^{k} [b_{i_1} - b_{i_t}]\cdots[b_{i_t}]\cdots[b_{i_k} - b_{i_t}] \right)$$

where $I = \{i_1 < \cdots < i_k\}$ and the complement of I is $\{j_1 < \cdots < j_s\}$ (so that $k + s = n$) and σ_I is the permutation

$$\begin{pmatrix} 1 & \ldots & s & s+1 & \ldots & n \\ j_1 & \ldots & j_s & i_1 & \ldots & i_k \end{pmatrix}.$$

The result now follows from the identity of Lemma 3.8.

(2) We can assume that $x = \lfloor a_1, \ldots, a_n \rceil$ and $y = \lfloor a'_1, \ldots, a'_m \rceil$ with $a_i, a'_j \in F^\times$. From the definition of T_{n+m} and the formula of Lemma 3.4,

$$
\begin{aligned}
&T_{n+m}(x*y) = \\
&\sum_{i=1}^{n}\sum_{j=1}^{m}(-1)^{n+m+i+j}\left\langle(-1)^{i+j}a_i a'_j\right\rangle \times \\
&\qquad \times [a_1(b_1-b_i)]\cdots[a_i\widehat{(b_i-b_i)}]\cdots[b_i][a'_1(b'_1-b'_j)]\cdots[a'_j\widehat{(b'_j-b'_j)}]\cdots[b'_j] \\
&\quad +(-1)^n\sum_{i=1}^{n}(-1)^{i+1}\left\langle(-1)^{i+1}a_i\right\rangle[a_1(b_1-b_i)]\cdots[a_i\widehat{(b_i-b_i)}]\cdots[b_i][b'_1a'_1]\cdots[b'_m a'_m] \\
&\quad +(-1)^m\sum_{j=1}^{m}(-1)^{j+1}\left\langle(-1)^{j+1}a'_j\right\rangle[b_1a_1]\cdots[b_na_n][a'_1(b'_1-b'_j)]\cdots[a'_j\widehat{(b'_j-b'_j)}]\cdots[b'_j] \\
&\qquad +[b_1a_1]\cdots[b_na_n][b_i][b'_1a'_1]\cdots[b'_m a'_m]
\end{aligned}
$$

which factors as $X\cdot Y$ with $X =$

$$
\sum_{i=1}^{n}(-1)^{n+i+1}\left\langle(-1)^{i+1}a_i\right\rangle[a_1(b_1-b_i)]\cdots[a_i\widehat{(b_i-b_i)}]\cdots[b_i]+[b_1a_1]\cdots[b_na_n]
$$

$$
=[a_1]\cdots[a_n]=T_n(x) \text{ by part (1)}
$$

and $Y =$

$$
\sum_{j=1}^{m}(-1)^{m+j+1}\left\langle(-1)^{j+1}a'_j\right\rangle[a'_1(b'_1-b'_j)]\cdots[a'_j\widehat{(b'_j-b'_j)}]\cdots[b'_j]+[b'_1a'_1]\cdots[b'_m a'_m]
$$

$$
=[a'_1]\cdots[a'_m]=T_m(y) \text{ by (1) again.}
$$

□

Note that T_1 is the natural surjective map $\tilde{S}(F)\cong\mathcal{I}_{F^\times}\to K_1^{\mathrm{MW}}(F)$, $\lfloor a\rceil\leftrightarrow\langle\langle a\rangle\rangle\mapsto[a]$. It has a nontrivial kernel in general.
Note furthermore that $\mathrm{SH}_2(F)=\mathrm{H}_2(\mathrm{SL}_2(F),\mathbb{Z})$. It is well-known ([24],[11], and [7]) that $\mathrm{H}_2(\mathrm{SL}_2(F),\mathbb{Z})\cong K_2^{\mathrm{M}}(F)\times_{k_2^{\mathrm{M}}(F)} I^2(F)\cong K_2^{\mathrm{MW}}(F)$.
In fact we have:

Theorem 3.10. *The composite $T_2\circ\epsilon_2 : \mathrm{H}_2(\mathrm{SL}_2(F),\mathbb{Z})\to K_2^{\mathrm{MW}}(F)$ is an isomorphism.*

Proof. For $p\geq 1$, let $\bar{X}_p(F)$ denote the set of all p-tuples $(x_1,\ldots,x_p)$ of points of $\mathbb{P}^1(F)$ and let $\bar{X}_0(F)=\emptyset$. We let $\bar{C}_p(F)$ denote the $\mathrm{GL}_2(F)$ permutation module $\mathbb{Z}[\bar{X}_p(F)]$ and form a complex $\bar{C}_\bullet(F)$ using the natural simplicial boundary maps, $\bar{d}_p$. This complex is acyclic and the map $F^2\setminus\{0\}\to\mathbb{P}^1(F)$, $v\mapsto\overline{v}$ induces a map of complexes $C_\bullet(F^2)\to\bar{C}_\bullet(F)$.
Let $\bar{H}_2(F):=\mathrm{Ker}(\bar{d}_2:\bar{C}_2(F)\to\bar{C}_1(F))$ and let $\bar{S}_2(F)=\mathrm{H}_0(\mathrm{SL}_2(F),\bar{H}_2(F))$.
We obtain a commutative diagram of $\mathrm{SL}_2(F)$-modules with exact rows:

$$
\begin{tikzcd}
C_4(F^2) \arrow[r, "d_4"] \arrow[d] & C_3(F^2) \arrow[r, "d_3"] \arrow[d] & H(F^2) \arrow[r] \arrow[d] & 0 \\
\bar{C}_4(F) \arrow[r, "\bar{d}_4"] & \bar{C}_3(F) \arrow[r, "\bar{d}_3"] & \bar{H}_2(F) \arrow[r] & 0
\end{tikzcd}
$$

Taking $\mathrm{SL}_2(F)$-coinvariants gives the diagram

$$\begin{tikzcd}
\mathrm{H}_0(\mathrm{SL}_2(F),C_4(F^2)) \arrow[r, "d_4"] \arrow[d] & \mathrm{H}_0(\mathrm{SL}_2(F),C_3(F^2)) \arrow[r, "d_3"] \arrow[d] & \tilde{S}(F^2) \arrow[r] \arrow[d, "\phi"] & 0 \\
\mathrm{H}_0(\mathrm{SL}_2(F),\bar{C}_4(F)) \arrow[r, "\bar{d}_4"] & \mathrm{H}_0(\mathrm{SL}_2(F),\bar{C}_3(F)) \arrow[r, "\bar{d}_3"] & \bar{S}_2(F) \arrow[r] & 0
\end{tikzcd}$$

Now the calculations of Mazzoleni, [11], show that $\mathrm{H}_0(\mathrm{SL}_2(F),\bar{C}_3(F)) \cong \mathbb{Z}[F^\times/(F^\times)^2]$ via

$$\text{class of } (\infty,0,a) \mapsto \langle a\rangle \in \mathbb{Z}[F^\times/(F^\times)^2],$$

where $a \in \mathbb{P}^1(F) = \overline{e_1+ae_2}$ and $\infty := \overline{e_1}$. Furthermore $\bar{S}_2(F) \cong \mathrm{GW}(F)$ in such a way that the induced map $\mathbb{Z}[F^\times/(F^\times)^2] \to \mathrm{GW}(F)$ is the natural one. Since $\lfloor a,b\rceil = d_3(e_1,e_2,ae_1+be_2)$, it follows that $\phi(\lfloor a,b\rceil) = \langle a/b\rangle = \langle ab\rangle$ in $\mathrm{GW}(F)$.

Associated to the complex $\bar{C}_\bullet(F)$ we have an iterated connecting homomorphism $\omega : \mathrm{H}_2(\mathrm{SL}_2(F),\mathbb{Z}) \to \bar{S}_2(F) = \mathrm{GW}(F)$. Observe that $\omega = \phi\circ\epsilon_2$. In fact, (Mazzoleni, [11], Lemma 5) the image of ω is $I^2(F) \subset \mathrm{GW}(F)$.

On the other hand, the module $\tilde{S}(F^2)^+$ is generated by the elements $[[a,b]] := \lfloor a,b\rceil - D_2(\lfloor a,b\rceil)\cdot E$ (where E, as above, denotes the element $\lfloor -1,1\rceil$). Note that $T_2([[a,b]]) = T_2(\lfloor a,b\rceil) = [a][b]$ since $T_2(E) = [-1][1] = 0$ in $K_2^{\mathrm{MW}}(F)$.

Furthermore,

$$\begin{aligned}
\phi([[a,b]]) &= \phi(\lfloor a,b\rceil) - D_2(\lfloor a,b\rceil)\phi(E)\\
&= \langle ab\rangle - (\langle -a\rangle - \langle b\rangle + \langle 1\rangle)\langle -1\rangle\\
&= \langle ab\rangle - \langle a\rangle + \langle -b\rangle - \langle -1\rangle\\
&= \langle ab\rangle - \langle a\rangle - \langle b\rangle + \langle 1\rangle\\
&= \langle\langle a,b\rangle\rangle
\end{aligned}$$

(using the identity $\langle b\rangle + \langle -b\rangle = \langle 1\rangle + \langle -1\rangle$ in $\mathrm{GW}(F)$).

Using these calculations we thus obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{H}_2(\mathrm{SL}_2(F),\mathbb{Z}) \arrow[r, "\epsilon_2"] \arrow[dr, "\omega"'] & \tilde{S}(F^2)^+ \arrow[r, "T_2"] \arrow[d, "\phi"] & K_2^{\mathrm{MW}}(F) \arrow[dl] \\
& I^2(F) &
\end{tikzcd}$$

Now, the natural embedding $F^\times \to \mathrm{SL}_2(F)$, $a \mapsto \mathrm{diag}(a,a^{-1}) := \tilde{a}$ induces a homomorphism, μ:

$$\begin{aligned}
\bigwedge^2 (F^\times) \cong \mathrm{H}_2(F^\times,\mathbb{Z}) &\to \mathrm{H}_2(\mathrm{SL}_2(F),\mathbb{Z}),\\
a\wedge b &\mapsto \left([\tilde{a}|\tilde{b}] - [\tilde{b}|\tilde{a}]\right)\otimes 1 \in B_2(\mathrm{SL}_2(F))\otimes_{\mathbb{Z}[\mathrm{SL}_2(F)]}\mathbb{Z}.
\end{aligned}$$

Mazzoleni's calculations (see [11], Lemma 6) show that $\mu(\bigwedge^2(F^\times)) = \mathrm{Ker}(\omega)$ and that there is an isomorphism $\mu(\bigwedge^2(F^\times)) \cong 2\cdot K_2^{\mathrm{M}}(F)$ given by $\mu(a\wedge b) \mapsto 2\{a,b\}$.

On the other hand, a straightforward calculation shows that $\epsilon_2\left(\mu(a\wedge b)\right) =$

$$\langle a\rangle \left\lfloor b, \frac{1}{ab}\right\rceil - \left\lfloor b, \frac{1}{b}\right\rceil - \langle a\rangle \left\lfloor 1, \frac{1}{a}\right\rceil + \langle b\rangle \left\lfloor 1, \frac{1}{b}\right\rceil + \left\lfloor a, \frac{1}{a}\right\rceil - \langle b\rangle \left\lfloor a, \frac{1}{ab}\right\rceil := C_{a,b}$$

Now by the diagram above,

$$T_2(C_{a,b}) = T_2(\epsilon_2\left(\mu(a\wedge b)\right)) \in \mathrm{Ker}(K_2^{\mathrm{MW}}(F) \to I^2(F)) \cong 2K_2^{\mathrm{M}}(F).$$

Recall that the natural embedding $2K_2^{\mathrm{M}}(F) \to K_2^{\mathrm{MW}}(F)$ is given by $2\{a,b\} \mapsto [a^2][b] = [a][b] - [b][a]$ and the composite

$$2K_2^{\mathrm{M}}(F) \longrightarrow K_2^{\mathrm{MW}}(F) \xrightarrow{\kappa_2} K_2^{\mathrm{M}}(F)$$

is the natural inclusion map. Since

$$\begin{aligned} \kappa_2\left(T_2(C_{a,b})\right) &= \left\{b, \frac{1}{ab}\right\} - \left\{b, \frac{1}{b}\right\} - \left\{1, \frac{1}{a}\right\} + \left\{1, \frac{1}{b}\right\} + \left\{a, \frac{1}{a}\right\} - \left\{a, \frac{1}{ab}\right\} \\ &= \{a,b\} - \{b,a\} = 2\{a,b\}, \end{aligned}$$

it follows that we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mu(\bigwedge^2(F^\times)) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}_2(\mathrm{SL}_2(F),\mathbb{Z}) \arrow[r, "\omega"] \arrow[d, "T_2\circ\epsilon_2"] & I^2(F) \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & 2K_2^{\mathrm{M}}(F) \arrow[r] & K_2^{\mathrm{MW}}(F) \arrow[r] & I^2(F) \arrow[r] & 0 \end{tikzcd}$$

proving the theorem. □

4. $\mathcal{AM}$-modules

From the results of the last section, it follows that there is a $\mathbb{Z}[F^\times]$-decomposition

$$\tilde{S}(F^2) \cong K_2^{\mathrm{MW}}(F) \oplus \mathbb{Z}[F^\times] \oplus ?$$

It is not difficult to determine that the missing factor is isomorphic to the 1-dimensional vector space F (with the tautological $F^\times$-action). However, as we will see, this extra term will not play any role in the calculations of $\mathrm{H}_n(\mathrm{SL}_k(F),\mathbb{Z})$.

As $\mathbb{Z}[F^\times]$-modules, our main objects of interest (Milnor-Witt K-theory, the homology of the special linear group, the powers of the fundamental ideal in the Grothendieck-Witt ring) are what we call below 'multiplicative '; there exists $m \geq 1$ such that, for all $a \in F^\times$, $\langle a^m\rangle$ acts trivially. This is certainly not true of the vector space F above. In this section we formalise this difference, and use this formalism to prove an analogue of Suslin's Theorem 1.8 ([23]) (see Theorem 4.23 below).

Throughout the remainder of this article, F will denote a field of characteristic 0.

Let $\mathcal{S}_F \subset \mathbb{Z}[F^\times]$ denote the multiplicative set generated by the elements $\{\langle\langle a\rangle\rangle = \langle a\rangle - 1 \mid a \in F^\times \setminus \{1\}\}$. Note that $0 \notin \mathcal{S}_F$, since the elements of $\mathcal{S}_F$ map to units under the natural ring homomorphism $\mathbb{Z}[F^\times] \to F$. We will also let $\mathcal{S}_{\mathbb{Q}}^+ \subset \mathbb{Z}[\mathbb{Q}^\times]$ denote the multiplicative set generated by $\{\langle\langle a\rangle\rangle = \langle a\rangle - 1 \mid a \in \mathbb{Q}^\times \setminus \{\pm 1\}\}$.

DEFINITION 4.1. A $\mathbb{Z}[F^\times]$-module M is said to be *multiplicative* if there exists $s \in \mathcal{S}_{\mathbb{Q}}^+$ with $sM = 0$.

DEFINITION 4.2. We will say that a $\mathbb{Z}[F^\times]$-module is *additive* if every $s \in \mathcal{S}_{\mathbb{Q}}^+$ acts as an automorphism on M.

EXAMPLE 4.3. Any trivial $\mathbb{Z}[F^\times]$-module M is multiplicative , since $\langle\langle a\rangle\rangle$ annihilates M for all $a \neq 1$.

EXAMPLE 4.4. $\mathrm{GW}(F)$, and more generally $I^n(F)$, is multiplicative since $\langle\langle a^2\rangle\rangle$ annihilates these modules for all $a \in F^\times$.

EXAMPLE 4.5. Similarly, the groups $\mathrm{H}_n(\mathrm{SL}_n(F), \mathbb{Z})$ are multiplicative since they are annihilated by the elements $\langle\langle a^m\rangle\rangle$.

EXAMPLE 4.6. Any vector space over F, with the induced action of $\mathbb{Z}[F^\times]$, is additive since all elements of $\mathcal{S}_F$ act as automorphisms.

EXAMPLE 4.7. More generally, if V is a vector space over F, then for all $r \geq 1$, the rth tensor power $\mathrm{T}^r_{\mathbb{Z}}(V) = \mathrm{T}^r_{\mathbb{Q}}(V)$ is an additive module since, if $a \in \mathbb{Q} \setminus \{\pm 1\}$, $\langle a\rangle$ acts as multiplication by a^r and hence $\langle\langle a\rangle\rangle$ acts as multiplication by $a^r - 1$. For the same reasons, the rth exterior power, $\bigwedge^r_{\mathbb{Z}}(V)$, is an additive module.

REMARK 4.8. Observe that if $\langle\langle a^m\rangle\rangle$ acts as an automorphism of the $\mathbb{Z}[F^\times]$-module M for some $a \in F^\times$, $m > 1$, then so does $\langle\langle a\rangle\rangle$, since $\langle\langle a^m\rangle\rangle = \langle\langle a\rangle\rangle(\langle a^{m-1}\rangle + \cdots + \langle a\rangle + 1) = (\langle a^{m-1}\rangle + \cdots + \langle a\rangle + 1)\langle\langle a\rangle\rangle$ in $\mathbb{Z}[F^\times]$.

LEMMA 4.9. *Let*

$$0 \to M_1 \to M \to M_2 \to 0$$

be a short exact sequence of $\mathbb{Z}[F^\times]$*-modules.*
Then M is multiplicative if and only if M_1 and M_2 are.

Proof. Suppose M is multiplicative . If $s \in \mathcal{S}_{\mathbb{Q}}^+$ satisfies $sM = 0$, it follows that $sM_1 = sM_2 = 0$.
Conversely, if M_1 and M_2 are multiplicative then there exist $s_1, s_2 \in \mathcal{S}_{\mathbb{Q}}^+$ with $s_iM_i = 0$ for $i = 1, 2$. It follows that $sM = 0$ for $s = s_1s_2 \in \mathcal{S}_{\mathbb{Q}}^+$. □

LEMMA 4.10. *Let*

$$0 \to A_1 \to A \to A_2 \to 0$$

be a short exact sequence of $\mathbb{Z}[F^\times]$*-modules. If A_1 and A_2 are additive modules, then so is A.*

Proof. This is immediate from the definition. □

Lemma 4.11. *Let* $\phi : M \to N$ *be a homomorphism of* $\mathbb{Z}[F^\times]$*-modules.*

(1) *If* M *and* N *are multiplicative , then so are* $\mathrm{Ker}(\phi)$ *and* $\mathrm{Coker}(\phi)$*.*
(2) *If* M *and* N *are additive , then so are* $\mathrm{Ker}(\phi)$ *and* $\mathrm{Coker}(\phi)$*.*

Proof. (1) This follows from Lemma 4.9 above.
(2) If $s \in \mathcal{S}_{\mathbb{Q}}^+$, then s acts as an automorphism of M and N, and hence of $\mathrm{Coker}(\phi)$ and $\mathrm{Ker}(\phi)$.

□

Corollary 4.12. *Let* $C = (C_\bullet, d)$ *be a complex of* $\mathbb{Z}[F^\times]$*-modules. If* $C_\bullet$ *is additive (i.e. if each* C_n *is an additive module), then each* $H_n(C)$ *is an additive module. If each* C_n *is multiplicative then each* $H_n(C)$ *is a multiplicative module.*

Lemma 4.13. *Let* M *be a multiplicative* $\mathbb{Z}[F^\times]$*-module and* A *an additive* $\mathbb{Z}[F^\times]$*-module. Then* $\mathrm{Hom}_{\mathbb{Z}[F^\times]}(M, A) = 0$ *and* $\mathrm{Hom}_{\mathbb{Z}[F^\times]}(A, M) = 0$*.*

Proof. Let $f : M \to A$ be a $\mathbb{Z}[F^\times]$-homomorphism. Every $s \in \mathcal{S}_{\mathbb{Q}}^+$ acts as an automorphism of A. However, there exists $s \in \mathcal{S}_{\mathbb{Q}}^+$ with $sM = 0$. Thus, for $m \in M$, $0 = f(sm) = sf(m) \Longrightarrow f(m) = 0$.
Let $g : A \to M$ be a $\mathbb{Z}[F^\times]$-homomorphism. Again, choose $s \in \mathcal{S}_{\mathbb{Q}}^+$ acting as an automorphism of A and annihilating M. If $a \in A$, then there exists $b \in a$ with $a = sb$. Hence $g(a) = sg(b) = 0$ in M. □

Lemma 4.14. *If* P *is a* $\mathbb{Z}[F^\times]$*-module and if* A *is an additive submodule and* M *a multiplicative submodule, then* $A \cap M = 0$*.*

Proof. There exists $s \in \mathbb{Z}[\mathbb{Q}^\times]$ which annihilates any submodule of M but is injective on any submodule of A. □

Lemma 4.15.

(1) *If*
$$0 \longrightarrow M \longrightarrow H \xrightarrow{\pi} A \longrightarrow 0$$
is an exact sequence of $\mathbb{Z}[F^\times]$*-modules with* M *multiplicative and* A *additive then the sequence splits (over* $\mathbb{Z}[F^\times]$*).*
(2) *Similarly, if*
$$0 \longrightarrow A \longrightarrow H \longrightarrow M \longrightarrow 0$$
is an exact sequence of $\mathbb{Z}[F^\times]$*-modules with* M *multiplicative and* A *additive then the sequence splits.*

Proof. As above we can find $s \in \mathbb{Z}[\mathbb{Q}^\times]$ such that $s \cdot M = 0$ and s acts as an automorphism of A.

(1) Then sH is a $\mathbb{Z}[F^\times]$-submodule of H and π induces an isomorphism $sH \cong A$, since $\pi(sH) = s\pi(H) = sA = A$ and if $\pi(sh) = 0$ then $s\pi(h) = 0$ in A, so that $\pi(h) = 0$ and $h \in M$.
(2) We have $sH = A$ and multiplication by s gives an automorphism, α, of A. Thus the $\mathbb{Z}[F^\times]$-homomorphism $H \to A, h \mapsto \alpha^{-1}(s \cdot h)$ splits the sequence.

$\square$

Definition 4.16. We will say that a $\mathbb{Z}[F^\times]$-module H is *an $\mathcal{AM}$ module* if there exists a multiplicative $\mathbb{Z}[F^\times]$-module M and an additive $\mathbb{Z}[F^\times]$ module A and an isomorphism of $\mathbb{Z}[F^\times]$-modules $H \cong A \oplus M$.

Lemma 4.17. *Let H be an $\mathcal{AM}$ module and let $\phi : H \to A \oplus M$ be an isomorphism of $\mathbb{Z}[F^\times]$-modules, with M multiplicative and A additive .*
Then

$$\phi^{-1}(A) = \bigcup_{A' \subset H, A' \text{additive}} A' \qquad \text{and} \qquad \phi^{-1}(M) = \bigcup_{M' \subset H, M' \text{multiplicative}} M'$$

Proof. Let $M' \subset H$ be multiplicative. Then the composite

$$M' \longrightarrow H \xrightarrow{\phi} A \oplus M \longrightarrow A$$

is zero by Lemma 4.13, and thus $M' \subset \phi^{-1}(M)$.
An analogous argument can be applied to $\phi^{-1}(A)$. $\square$

It follows that the submodules $\phi^{-1}(A)$ and $\phi^{-1}(M)$ are independent of the choice of ϕ, A and M. We will denote the first as $H_{\mathcal{A}}$ and the second as $H_{\mathcal{M}}$. Thus if H is an $\mathcal{AM}$ module then there is a canonical decomposition $H = H_{\mathcal{A}} \oplus H_{\mathcal{M}}$, where $H_{\mathcal{A}}$ (resp. $H_{\mathcal{M}}$) is the maximal additive (resp. multiplicative) submodule of H. We have canonical projections

$$\pi_{\mathcal{A}} : H \to H_{\mathcal{A}}, \qquad \pi_{\mathcal{M}} : H \to H_{\mathcal{M}}.$$

Lemma 4.18. *Let H be a $\mathcal{AM}$ module. Suppose that H is also a module over a ring R and that the action of R commutes with that of $\mathbb{Z}[F^\times]$. Then $H_{\mathcal{A}}$ and $H_{\mathcal{M}}$ are R-submodules of H.*

Proof. Let $r \in R$. Then the composite

$$H_{\mathcal{A}} \xrightarrow{r\cdot} H \xrightarrow{\pi_{\mathcal{M}}} H_{\mathcal{M}}$$

is a $\mathbb{Z}[F^\times]$-homomorphism and thus is 0 by Lemma 4.13. It follows that $r \cdot H_{\mathcal{A}} \subset \mathrm{Ker}(\pi_{\mathcal{M}}) = H_{\mathcal{A}}$. $\square$

Lemma 4.19. *Let $f : H \to H'$ be a $\mathbb{Z}[F^\times]$-homomorphism of $\mathcal{AM}$ modules.*
Then there exist $\mathbb{Z}[F^\times]$-homomorphisms $f_{\mathcal{A}} : H_{\mathcal{A}} \to H'_{\mathcal{A}}$ and $f_{\mathcal{M}} : H_{\mathcal{M}} \to H'_{\mathcal{M}}$ such that $f = f_{\mathcal{A}} \oplus f_{\mathcal{M}}$.
Suppose that H and H' are modules over a ring R and that the R-action commutes with the $\mathbb{Z}[F^\times]$-action in each case. If f is an R-homomorphism, then so are $f_{\mathcal{A}}$ and $f_{\mathcal{M}}$.

Proof. This is immediate from Lemmas 4.13 and 4.18. $\square$

Lemma 4.20. *If*

$$0 \longrightarrow L \xrightarrow{j} H \xrightarrow{\pi} K \longrightarrow 0$$

is a short exact sequence of $\mathbb{Z}[F^\times]$-modules and if L and K are $\mathcal{AM}$ modules, then so is H.

Proof. Let $\tilde{H} = \pi^{-1}(K_{\mathcal{M}})$. Then the exact sequence

$$0 \to L \to \tilde{H} \to K_{\mathcal{M}} \to 0$$

gives the exact sequence

$$0 \to \frac{L}{L_{\mathcal{M}}} \to \frac{\tilde{H}}{j(L_{\mathcal{M}})} \to K_{\mathcal{M}} \to 0.$$

Since $L/L_{\mathcal{M}} \cong L_{\mathcal{A}}$ is additive , this latter sequence is split, by Lemma 4.15 (2).
So $\tilde{H}/j(L_{\mathcal{M}})$ is a $\mathcal{AM}$ module, and there is a $\mathbb{Z}[k^{\times}]$-isomorphism

$$\tilde{H}/j(L_{\mathcal{M}}) \xrightarrow[\cong]{\phi} L_{\mathcal{A}} \oplus K_{\mathcal{M}}.$$

Let $\bar{\phi}$ be the composite

$$\tilde{H} \longrightarrow \tilde{H}/j(L_{\mathcal{M}}) \xrightarrow{\phi} L_{\mathcal{A}} \oplus K_{\mathcal{M}}.$$

Let $H_m = \bar{\phi}^{-1}(K_{\mathcal{M}}) \subset \tilde{H} \subset H$. Then, we have an exact sequence

$$0 \to L_{\mathcal{M}} \to H_m \to K_{\mathcal{M}} \to 0$$

so that H_m is multiplicative .
On the other hand, since $\tilde{H}/H_m \cong L_{\mathcal{A}}$ and $H/\tilde{H} \cong K_{\mathcal{A}}$, we have a short exact sequence

$$0 \to L_{\mathcal{A}} \to \frac{H}{H_m} \to K_{\mathcal{A}} \to 0.$$

This implies that H/H_m is additive , and thus H is $\mathcal{AM}$ by Lemma 4.15 (1). □

Lemma 4.21. *Let $(C_\bullet, d)$ be a complex of $\mathbb{Z}[k^{\times}]$-modules. If each C_n is $\mathcal{AM}$, then $\mathrm{H}_\bullet(C)$ is $\mathcal{AM}$, and furthermore*

$$\mathrm{H}_\bullet(C_{\mathcal{A}}) = \mathrm{H}_\bullet(C)_{\mathcal{A}}$$
$$\mathrm{H}_\bullet(C_{\mathcal{M}}) = \mathrm{H}_\bullet(C)_{\mathcal{M}}$$

Proof. The differentials d decompose as $d = d_{\mathcal{A}} \oplus d_{\mathcal{M}}$ by Lemma 4.19. □

Theorem 4.22. *Let (E^r, d^r) be a first quadrant spectral sequence of $\mathbb{Z}[k^{\times}]$-modules converging to the $\mathbb{Z}[k^{\times}]$-module $H_\bullet = \{H_n\}_{n\geq 0}$.*
If for some $r_0 \geq 1$ all of the modules $E^{r_0}_{p,q}$ are $\mathcal{AM}$, then the same holds for all the modules $E^r_{p,q}$ for all $r \geq r_0$ and hence for the modules $E^{\infty}_{p,q}$.
Furthermore, $H_\bullet$ is $\mathcal{AM}$ and the spectral sequence decomposes as a direct sum $E^r = {E^r}_{\mathcal{A}} \oplus {E^r}_{\mathcal{M}}$ $(r \geq r_0)$ with ${E^r}_{\mathcal{A}}$ converging to $H_{\bullet\mathcal{A}}$ and ${E^r}_{\mathcal{M}}$ converging to $H_{\bullet\mathcal{M}}$.

Proof. Since $E^{r+1} = \mathrm{H}(E^r, d^r)$ for all r, the first statement follows from Lemma 4.21.
Since E^r is a first quadrant spectral sequence (and, in particular, is bounded), it follows that for any fixed (p,q), $E^\infty_{p,q} = E^r_{p,q}$ for all sufficiently large r. Thus E^∞ is also $\mathcal{AM}$.
Now H_n admits a filtration $0 = F_0H_n \subset \cdots \subset F_nH_n = H_n$ with corresponding quotients $\mathrm{gr}_pH_n \cong E^\infty_{p,n-p}$.
Since all the quotients are $\mathcal{AM}$, it follows by Lemma 4.20, together with an induction on the filtration length, that H_n is $\mathcal{AM}$.
The final two statements follow again from Lemma 4.21. □

If G is a subgroup of $\mathrm{GL}(V)$, we let SG denote $G \cap \mathrm{SL}(V)$.

THEOREM 4.23. *Let V, W be finite-dimensional vector spaces over F and let $G_1 \subset \mathrm{GL}(W)$, $G_2 \subset \mathrm{GL}(V)$ be subgroups and suppose that G_2 contains the group $F^\times$ of scalar matrices.*
Let M be a subspace of $\mathrm{Hom}_F(V,W)$ for which $G_1M = M = MG_2$.
Let
$$G = \begin{pmatrix} G_1 & M \\ 0 & G_2 \end{pmatrix} \subset \mathrm{GL}(W \oplus V).$$
Then, for $i \geq 1$, the groups $\mathrm{H}_i(SG,\mathbb{Z})$ are $\mathcal{AM}$ and the natural embedding $j : S(G_1 \times G_2) \to SG$ induces an isomorphism
$$\mathrm{H}_i(S(G_1 \times G_2),\mathbb{Z}) \cong \mathrm{H}_i(SG,\mathbb{Z})_{\mathcal{M}}.$$

Proof. We begin by noting that the groups $\mathrm{H}_i(SG,\mathbb{Z})$ are $\mathbb{Z}[F^\times]$-modules: The action of $F^\times$ is derived from the short exact sequence
$$1 \longrightarrow SG \longrightarrow G \xrightarrow{\det} F^\times \longrightarrow 1$$
We have a split extension of groups (split by the map j) which is $F^\times$-stable:
$$0 \longrightarrow M \longrightarrow SG \xrightarrow{\pi} S(G_1 \times G_2) \longrightarrow 1.$$
The resulting Hochschild-Serre spectral sequence has the form
$$E^2_{p,q} = \mathrm{H}_p(S(G_1 \times G_2), \mathrm{H}_q(M,\mathbb{Z})) \Longrightarrow \mathrm{H}_{p+q}(SG,\mathbb{Z}).$$
This spectral sequence exists in the category of $\mathbb{Z}[F^\times]$-modules and all differentials and edge homomorphisms are $\mathbb{Z}[F^\times]$-maps.
Since the map π is split by j it induces a split surjection on integral homology groups. Thus
$$\mathrm{H}_n(S(G_1 \times G_2),\mathbb{Z}) = E^2_{n,0} = E^\infty_{n,0} \quad \text{for all } n \geq 0.$$
Observe furthermore that the $\mathbb{Z}[F^\times]$-module $\mathrm{H}_n(S(G_1 \times G_2),\mathbb{Z})$ is multiplicative : Given $a \in F^\times$, the element
$$\rho_a := \begin{pmatrix} \mathrm{Id}_W & 0 \\ 0 & a \cdot \mathrm{Id}_V \end{pmatrix} \in G$$

has determinant a^m ($m = \dim_F(V)$) and centralizes $S(G_1 \times G_2)$. It follows that $\langle a^m \rangle$ acts trivially on $\mathrm{H}_n(S(G_1 \times G_2), \mathbb{Z})$ for all n; i.e. $\langle\langle a^m \rangle\rangle$ annihilates $\mathrm{H}_n(S(G_1 \times G_2), \mathbb{Z})$.

Recall (Example 4.7 above) that for $q \geq 1$, the modules $\mathrm{H}_q(M, \mathbb{Z}) = \bigwedge^q_{\mathbb{Z}}(M)$, with the $\mathbb{Z}[F^\times]$-action derived from the action of F by scalars on M, are additive modules.

Now if $a \in F^\times$, then conjugation by ρ_a is trivial on $S(G_1 \times G_2)$ but acts on M as scalar multiplication by a. It follows that for $q > 0$, $\langle\langle a^m \rangle\rangle$ acts as an automorphism on $\mathrm{H}_p(S(G_1 \times G_2), \mathrm{H}_q(M, \mathbb{Z}))$ for all $a \in \mathbb{Q} \setminus \{\pm 1\}$. Thus, for $q > 0$, the groups $\mathrm{H}_p(S(G_1 \times G_2), \mathrm{H}_q(M, \mathbb{Z}))$ are additive $\mathbb{Z}[F^\times]$-modules; i.e., all $E^2_{p,q}$ are additive for $q > 0$. It follows at once that the groups $E^\infty_{p,q}$ are additive for all $q > 0$. Thus, from the convergence of the spectral sequence, we have a short exact sequence

$$0 \to H \to \mathrm{H}_n(SG, \mathbb{Z}) \to E^\infty_{n,0} = j\left(\mathrm{H}_n(S(G_1 \times G_2), \mathbb{Z})\right) \to 0$$

and H has a filtration whose graded quotients are all additive .

So $\mathrm{H}_n(SG, \mathbb{Z})$ is $\mathcal{AM}$ as claimed, and $\mathrm{H}_n(SG, \mathbb{Z})_{\mathcal{M}} \cong \mathrm{H}_n(S(G_1 \times G_2), \mathbb{Z})$.

□

COROLLARY 4.24. *Suppose that $W' \subset W$. Then there is a corresponding inclusion $\mathrm{SA}(W', V) \to \mathrm{SA}(W, V)$. This inclusion induces an isomorphism*

$$\mathrm{H}_n(\mathrm{SA}(W', V), \mathbb{Z})_{\mathcal{M}} \xrightarrow[\cong]{} \mathrm{H}_n(\mathrm{SA}(W, V), \mathbb{Z})_{\mathcal{M}} \cong \mathrm{H}_n(\mathrm{SL}(V), \mathbb{Z})$$

for all $n \geq 1$.

5. The spectral sequences

Recall that F is a field of characteristic 0 throughout this section.

In this section we use the complexes $C_\bullet(W, V)$ to construct spectral sequences converging to 0 in dimensions less than $n = \dim_F(V)$, and to $\tilde{S}(W, V)$ in dimension n. By projecting onto the multiplicative part, we obtain spectral sequences with good properties: the terms in the E^1-page are just the kernels and cokernels of the stabilization maps $f_{t,n} : \mathrm{H}_n(\mathrm{SL}_t(F), \mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_{t+1}(F), \mathbb{Z})$. We then prove that the higher differentials are all zero. Since the spectral sequences converge to 0 in low degrees, this already implies the main stability result (Corollary 5.11); the maps $f_{t,n}$ are isomorphisms for $t \geq n+1$ and are surjective for $t = n$. The remainder of the paper is devoted to an analysis of the case $t = n - 1$, which requires some more delicate calculations.

Let $\mathcal{C}^\tau_\bullet(W, V)$ denote the truncated complex.

$$\mathcal{C}^\tau_p(W, V) = \begin{cases} C_p(W, V), & p \leq \dim_F(V) \\ 0, & p > \dim_F(V) \end{cases}$$

Thus

$$\mathrm{H}_p(\mathcal{C}^\tau_\bullet(W, V)) = \begin{cases} 0, & p \neq n \\ H(W, V), & p = n \end{cases}$$

where $n = \dim_F(V)$.

Thus the natural action of $\mathrm{SA}(W,V)$ on $\mathcal{C}^\tau_\bullet(W,V)$ gives rise to a spectral sequence $\mathcal{E}(W,V)$ which has the form

$$E^1_{p,q} = \mathrm{H}_p(\mathrm{SA}(W,V), \mathcal{C}^\tau_q(W,V)) \Longrightarrow \mathrm{H}_{p+q-n}(\mathrm{SA}(W,V), H(W,V)).$$

The groups $\mathcal{C}^\tau_q(W,V)$ are permutation modules for $\mathrm{SA}(W,V)$ and thus the E^1-terms (and the differentials d^1) can be computed in terms of the homology of stabilizers.

Fix a basis $\{e_1,\ldots,e_n\}$ of V. Let V_r be the span of $\{e_1,\ldots,e_r\}$ and let V'_s be the span of $\{e_{n-s},\ldots,e_n\}$, so that $V = V_r \oplus V'_{n-r}$ if $0 \le r \le n$.

For any $0 \le q \le n-1$, the group $\mathrm{SA}(W,V)$ acts transitively on the basis of $\mathcal{C}^\tau_q(W,V)$ and the stabilizer of

$$\big((0,e_1),\ldots,(0,e_q)\big)$$

is $\mathrm{SA}(W \oplus V_q, V'_{n-q})$.

Thus, for $q \le n-1$,

$$E^1_{p,q} = \mathrm{H}_p(\mathrm{SA}(W,V), \mathcal{C}^\tau_q(W,V)) \cong \mathrm{H}_p(\mathrm{SA}(W \oplus V_q, V'_{n-q}), \mathbb{Z})$$

by Shapiro's Lemma.

By the results in section 4 we have:

Lemma 5.1. *The terms $E^1_{p,q}$ in the spectral sequence $\mathcal{E}(W,V)$ are $\mathcal{AM}$ for $q > 0$, and*

$$\left(E^1_{p,q}\right)_{\mathcal{M}} = \mathrm{H}_p(\mathrm{SL}(V'_{n-q}), \mathbb{Z}) \cong \mathrm{H}_p(\mathrm{SL}_{n-q}(F), \mathbb{Z}).$$

For $q = n$, the orbits of $\mathrm{SA}(W,V)$ on the basis of $\mathcal{C}^\tau_n(W,V)$ are in bijective correspondence with $F^\times$ via

$$\big((w_1,v_1),\ldots,(w_n,v_n)\big) \mapsto \det\left([v_1|\cdots|v_n]_{\mathcal{E}}\right).$$

The stabilizer of any basis element of $\mathcal{C}^\tau_n(W,V)$ is trivial. Thus

$$E^1_{p,n} = \begin{cases} \mathbb{Z}[F^\times], & p = 0 \\ 0, & p > 0 \end{cases}$$

Of course, $E^1_{p,q} = 0$ for $q > n$.

The first column of the E^1-page of the spectral sequence $\mathcal{E}(W,V)$ has the form

$$E^1_{0,q} = \begin{cases} \mathbb{Z}, & q < n \\ \mathbb{Z}[F^\times], & q = n \\ 0, & q > n \end{cases}$$

and the differentials are easily computed: For $q < n$

$$d^1_{0,q} : E^1_{0,q} \to E^1_{0,q} = \begin{cases} \mathrm{Id}_{\mathbb{Z}}, & q \text{ is odd} \\ 0, & q \text{ is even} \end{cases}$$

and

$$d^1_{0,n} : \mathbb{Z}[F^\times] \to \mathbb{Z} = \begin{cases} \text{augmentation}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

It follows that $E^2_{0,q} = 0$ for $q \neq n$ and

$$E^2_{0,n} = \begin{cases} \mathcal{I}_{F^\times}, & n \text{ odd} \\ \mathbb{Z}[F^\times], & n \text{ even} \end{cases}$$

Note that the composite

$$\tilde{\mathrm{S}}(W,V) \xrightarrow{\text{edge}} E^\infty_{0,n} \subset E^2_{0,n} = \mathcal{A}_n$$

is just the map $D_{W,V}$ of section 3 above.

Lemma 5.2. *The map $D_{W,V}$ is a split surjective homomorphism of $\mathbb{Z}[F^\times]$-modules.*

Proof. If $W = 0$, this is Lemma 3.7 (1) and (3), since $V \cong F^n$.
In general the natural map of complexes $\mathcal{C}^\tau_\bullet(V) \to \mathcal{C}^\tau_\bullet(W,V)$ gives rise to a commutative diagram of $\mathbb{Z}[F^\times]$-modules

$$\begin{tikzcd} \tilde{\mathrm{S}}(V) \arrow[rr] \arrow[dr, "D_V"'] & & \tilde{\mathrm{S}}(W,V) \arrow[dl, "D_{W,V}"] \\ & \mathcal{A}_n & \end{tikzcd}$$

□

We let $\tilde{S}(W,V)^+ := \mathrm{Ker}(D_{W,V} : \tilde{\mathrm{S}}(W,V) \to \mathcal{A}_n)$, so that $\tilde{\mathrm{S}}(W,V) \cong \tilde{S}(W,V)^+ \oplus \mathcal{A}_n$ for all W,V.

Corollary 5.3. *In the spectral sequence $\mathcal{E}(W,V)$, we have $E^2_{0,q} = E^\infty_{0,q}$ for all $q \geq 0$.*
All higher differentials $d^r_{0,q} : E^r_{0,q} \to E^r_{r-1,q+r}$ are zero.

It follows that the spectral sequences $\mathcal{E}(W,V)$ decompose as a direct sum of two spectral sequences

$$\mathcal{E}(W,V) = \mathcal{E}^0(W,V) \oplus \mathcal{E}^+(W,V)$$

where $\mathcal{E}^0(W,V)$ is the first column of $\mathcal{E}(W,V)$ and $\mathcal{E}^+(W,V)$ involves only the terms $E^r_{p,q}$ with $q > 0$.
The spectral sequence $\mathcal{E}^0(W,V)$ converges in degree d to

$$\begin{cases} 0, & d \neq n \\ \mathcal{A}_n, & d = n \end{cases}$$

The spectral sequence $\mathcal{E}^+(W,V)$ converges in degree d to

$$\begin{cases} 0, & d < n \\ \tilde{S}(W,V)^+, & d = n \\ \mathrm{H}_{d-n}(\mathrm{SA}(W,V), H(W,V)), & d > n \end{cases}$$

By Lemma 5.1 above, all the terms of the spectral sequence $\mathcal{E}^+(W,V)$ are $\mathcal{AM}$.
We thus have

Corollary 5.4.

(1) *The* $\mathbb{Z}[F^\times]$*-modules* $\tilde{S}(W,V)^+$ *are* $\mathcal{AM}$.
(2) *The graded submodule* $\tilde{S}(F^\bullet)^+{}_{\mathcal{A}} \subset \tilde{\mathrm{S}}(F^\bullet)$ *is an ideal.*

Proof.
(1) This follows from Theorem 4.22.
(2) This follows from Lemma 4.18, since $\tilde{S}(F^\bullet)^+$ is an ideal in $\tilde{\mathrm{S}}(F^\bullet)$ by Lemma 3.7 (2).

□

COROLLARY 5.5. *The natural embedding* $H(V) \to H(W,V)$ *induces an isomorphism*

$$\tilde{S}(V)^+{}_{\mathcal{M}} \xrightarrow{\cong} \tilde{S}(W,V)^+{}_{\mathcal{M}}.$$

Proof. The map of complexes of $\mathrm{SL}(V)$-modules $\mathcal{C}^\tau_\bullet(V) \to \mathcal{C}^\tau_\bullet(W,V)$ gives rise to a map of spectral sequences $\mathcal{E}^+(V) \to \mathcal{E}^+(W,V)$ and hence a map $\mathcal{E}^+(V)_{\mathcal{M}} \to \mathcal{E}^+(W,V)_{\mathcal{M}}$. The induced map on the E^1-terms is

$$\begin{tikzcd} \mathrm{H}_p(\mathrm{SL}_{n-q}(F),\mathbb{Z}) \arrow[r, "\mathrm{Id}"] \arrow[d, "\cong"] & \mathrm{H}_p(\mathrm{SL}_{n-q}(F),\mathbb{Z}) \arrow[d, "\cong"] \\ \mathrm{H}_p(\mathrm{SL}(V),\mathcal{C}^\tau_q(V))_{\mathcal{M}} \arrow[r] & \mathrm{H}_p(\mathrm{SA}(W,V),\mathcal{C}^\tau_q(W,V))_{\mathcal{M}} \end{tikzcd}$$

and thus is an isomorphism.
It follows that there is an induced isomorphism of abutments

$$\tilde{S}(V)^+{}_{\mathcal{M}} \cong \tilde{S}(W,V)^+{}_{\mathcal{M}}$$

and

$$\mathrm{H}_k(\mathrm{SL}(V),H(V))_{\mathcal{M}} \cong \mathrm{H}_k(\mathrm{SA}(W,V),H(W,V))_{\mathcal{M}}.$$

□

For convenience, we now *define*

$$\tilde{\mathrm{S}}(W,V)_{\mathcal{M}} := \frac{\tilde{\mathrm{S}}(W,V)}{\tilde{S}(W,V)^+{}_{\mathcal{A}}}$$

(even though $\tilde{\mathrm{S}}(W,V)$ is not an $\mathcal{AM}$ module).
This gives:

COROLLARY 5.6.

$$\tilde{\mathrm{S}}(W,V)_{\mathcal{M}} \cong \tilde{S}(W,V)^+{}_{\mathcal{M}} \oplus \mathcal{A}_n \cong \tilde{S}(V)^+{}_{\mathcal{M}} \oplus \mathcal{A}_n \cong \tilde{\mathrm{S}}(V)_{\mathcal{M}}$$

as $\mathbb{Z}[F^\times]$*-modules, and* $\tilde{\mathrm{S}}(F^\bullet)_{\mathcal{M}}$ *is a graded* $\mathbb{Z}[F^\times]$*-algebra.*

LEMMA 5.7. *For any* $k \geq 1$*, the corestriction map*

$$\mathrm{cor} : \mathrm{H}_i(\mathrm{SL}_k(F),\mathbb{Z}) \to \mathrm{H}_i(\mathrm{SL}_{k+1}(F),\mathbb{Z})$$

is $F^\times$*-invariant;i.e. if* $a \in F^\times$ *and* $z \in \mathrm{H}_i(\mathrm{SL}_k(F),\mathbb{Z})$*, then*

$$\mathrm{cor}(\langle a\rangle\, z) = \langle a\rangle \,\mathrm{cor}(z) = \mathrm{cor}(z).$$

Proof. Of course, cor is a homomorphism of $\mathbb{Z}[F^\times]$-modules. However, for $a \in F^\times$, $\langle a^k \rangle$ acts trivially on $\mathrm{H}_i(\mathrm{SL}_k(F), \mathbb{Z})$ while $\langle a^{k+1} \rangle$ acts trivially on $\mathrm{H}_i(\mathrm{SL}_{k+1}(F), \mathbb{Z})$ so that

$$\mathrm{cor}(\langle a \rangle z) = \mathrm{cor}(\langle a^{k+1} \rangle z) = \langle a^{k+1} \rangle \mathrm{cor}(z) = \mathrm{cor}(z).$$

$\square$

LEMMA 5.8. *For $0 \le q < n$, the differentials of the spectral sequence $\mathcal{E}^+(W,V)_{\mathcal{M}}$*

$$d^1_{p,q} : (E^1_{p,q})_{\mathcal{M}} \cong \mathrm{H}_p(\mathrm{SL}_{n-q}(F), \mathbb{Z}) \to (E^1_{p,q-1})_{\mathcal{M}} \cong \mathrm{H}_p(\mathrm{SL}_{n-q+1}(F), \mathbb{Z})$$

are zero when q is even and are equal to the corestriction map when q is odd.

Proof. d^1 is derived from the map $d_q : \mathcal{C}^\tau_q(W,V) \to \mathcal{C}^\tau_{q-1}(W,V)$ of permutation modules. Here

$$\begin{aligned} d_q\big((0,e_1),\ldots,(0,e_q)\big) &= \sum_{i=1}^{q} (-1)^{i+1}\big((0,e_1),\ldots,\widehat{(0,e_i)},\ldots,(0,e_q)\big) \\ &= \sum_{i=1}^{q} (-1)^{i+1}\phi_i\big((0,e_1),\ldots,(0,e_{q-1})\big) \end{aligned}$$

where $\phi_i \in \mathrm{SA}(W,V)$ can be chosen to be of the form

$$\phi_i = \begin{pmatrix} \mathrm{Id}_W & 0 \\ 0 & \psi_i \end{pmatrix}, \quad \psi_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \tau_i \end{pmatrix} \in \mathrm{GL}(V)$$

with $\sigma_i \in \mathrm{GL}(V_q)$ a permutation matrix of determinant ϵ_i and $\tau_i \in \mathrm{GL}(V'_{n-q})$ also of determinant ϵ_i.
ϕ_i normalises $\mathrm{SA}(W \oplus V_q, V'_{n-q},)$ and $\mathrm{SL}(V'_{n-q})$. Thus for $z \in \mathrm{H}_p(\mathrm{SL}(V'_{n-q}), \mathbb{Z})$,

$$\begin{aligned} d^1(z) &= \sum_{i=1}^{q} (-1)^{i+1}\mathrm{cor}(\tau_i z) \\ &= \sum_{i=1}^{q} (-1)^{i+1}\mathrm{cor}(\langle \epsilon_i \rangle z) \\ &= \sum_{i=1}^{q} (-1)^{i+1}\mathrm{cor}(z) = \begin{cases} \mathrm{cor}(z), & q \text{ odd} \\ 0, & q \text{ even} \end{cases} \end{aligned}$$

$\square$

Let $E := \lfloor -1, 1 \rceil \in \tilde{\mathrm{S}}(F^2)_{\mathcal{M}}$. E is represented by the element

$\tilde{E} := d_3(e_1, e_2, e_2 - e_1) = (e_2, e_2 - e_1) - (e_1, e_2 - e_1) + (e_1, e_2) \in H(F^2) \subset \mathcal{C}^\tau_2(F^2)$.

Multiplication by $\tilde{E}$ induces a map of complexes of $\mathrm{GL}_{n-2}(F)$-modules

$$\mathcal{C}^\tau_\bullet(F^{n-2})[2] \to \mathcal{C}^\tau_\bullet(F^n)$$

There is an induced map of spectral sequences $\mathcal{E}(F^{n-2})[2] \to \mathcal{E}(F^n)$, which in turn induces a map $\mathcal{E}^+(F^{n-2})[2] \to \mathcal{E}^+(F^n)$, and hence a map $\mathcal{E}^+(F^{n-2})_{\mathcal{M}}[2] \to \mathcal{E}^+(F^n)_{\mathcal{M}}$.

By the work above, the E^1-page of $\mathcal{E}^+(F^n)_{\mathcal{M}}$ has the form

$$E^1_{p,q} = \mathrm{H}_p(\mathrm{SL}_{n-q}(F), \mathbb{Z}) \quad (p > 0)$$

while the E^1-page of $\mathcal{E}^+(F^{n-2})_{\mathcal{M}}[2]$ has the form

$$E'^1_{p,q} = \begin{cases} \mathrm{H}_p(\mathrm{SL}_{(n-2)-(q-2)}(F), \mathbb{Z}) = \mathrm{H}_p(\mathrm{SL}_{n-q}(F), \mathbb{Z}), & q \geq 2, p > 0 \\ 0, & q \leq 1 \text{ or } p = 0 \end{cases}$$

LEMMA 5.9. *For $q \geq 2$ (and $p > 0$), the map*

$$E'^1_{p,q} \cong \mathrm{H}_p(\mathrm{SL}_{n-q}(F), \mathbb{Z}) \to E^1_{p,q} = \mathrm{H}_p(\mathrm{SL}_{n-q}(F), \mathbb{Z})$$

induced by $\tilde{E} * -$ is the identity map.

Proof. There is a commutative diagram

$$\begin{tikzcd} E'^1_{p,q} = \mathrm{H}_p(\mathrm{SL}_{n-q}(F), \mathbb{Z}) \arrow[r] \arrow[d, "(\tilde{E}*-)_{\mathcal{M}}"] & \mathrm{H}_p(\mathrm{SA}(F^{q-2}, F^{n-q}), \mathbb{Z}) \arrow[r, "\cong"] \arrow[d, "\tilde{E}*-"] & \mathrm{H}_p(\mathrm{SL}_{n-2}(F), \mathcal{C}^\tau_{q-2}(F^{n-2})) \arrow[d, "\tilde{E}*-"] \\ E^1_{p,q} = \mathrm{H}_p(\mathrm{SL}_{n-q}(F), \mathbb{Z}) \arrow[r] & \mathrm{H}_p(\mathrm{SA}(F^q, F^{n-q}), \mathbb{Z}) \arrow[r, "\cong"] & \mathrm{H}_p(\mathrm{SL}_n(F), \mathcal{C}^\tau_q(F^n)) \end{tikzcd}$$

We number the standard basis of F^{n-2} $e_3, \ldots, e_n$ so that the inclusion $\mathrm{SL}_{n-2}(F) \to \mathrm{SL}_n(F)$ has the form

$$A \mapsto \begin{pmatrix} I_2 & 0 \\ 0 & A \end{pmatrix}.$$

So we have a commutative diagram of inclusions of groups

$$\begin{tikzcd} \mathrm{SL}_{n-q}(F) \arrow[r] \arrow[d, "="] & \mathrm{SA}(F^{q-2}, F^{n-q}) \arrow[r] \arrow[d] & \mathrm{SL}_{n-2}(F) \arrow[d] \\ \mathrm{SL}_{n-q}(F) \arrow[r] & \mathrm{SA}(F^q, F^{n-q}) \arrow[r] & \mathrm{SL}_n(F). \end{tikzcd}$$

Let $B_\bullet = B_\bullet(\mathrm{SL}_n(F))$ be the right bar resolution of $\mathrm{SL}_n(F)$. We can use it to compute the homology of any of the groups occurring in this diagram.
Suppose now that $q \geq 2$ and we have a class, w, in $E'^1_{p,q} = \mathrm{H}_p(\mathrm{SL}_{n-q}(F), \mathbb{Z})$ represented by a cycle

$$z \otimes 1 \in B_p \otimes_{\mathbb{Z}[\mathrm{SL}_{n-q}(F)]} \mathbb{Z}.$$

Its image in $\mathrm{H}_p(\mathrm{SL}_{n-2}(F), \mathcal{C}^\tau_{q-2}(F^{n-2}))$ is represented by $z \otimes (e_3, \ldots, e_q)$. The image of this in $\mathrm{H}_p(\mathrm{SL}_n(F), \mathcal{C}^\tau_q(F^n))$ is

$$\begin{aligned} & z \otimes \left[\tilde{E} * (e_3, \ldots, e_q)\right] \\ = \quad & z \otimes [(e_2, e_2 - e_1, e_3, \ldots) - (e_1, e_2 - e_1, e_3, \ldots) + (e_1, e_2, e_3, \ldots)] \\ = \quad & z \otimes [(g_1 - g_2 + 1)(e_1, e_2, e_3, \ldots)] \in B_p \otimes_{\mathbb{Z}[\mathrm{SL}_n(F)]} \mathcal{C}^\tau_q(F^n) \end{aligned}$$

where

$$g_1 = \begin{pmatrix} 0 & -1 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \in \mathrm{SL}_n(F).$$

This corresponds to the element in $\mathrm{H}_p(\mathrm{SL}_{n-q}(F), \mathbb{Z})$ represented by

$$z(g_1 - g_2 + 1) \otimes 1 \in B_p \otimes_{\mathbb{Z}[\mathrm{SL}_{n-q}(F)]} \mathbb{Z}$$

Since the elements g_i centralize $\mathrm{SL}_{n-q}(F)$ it follows that this is $(g_1 - g_2 + 1) \cdot w = w$. $\square$

Recall that the spectral sequence $\mathcal{E}^+(F^n)_{\mathcal{M}}$ converges in degree n to $\tilde{S}(F^n)^+{}_{\mathcal{M}}$. Thus there is a filtration

$$0 = \mathcal{F}_{n,-1} \subset \mathcal{F}_{n,0} \subset \mathcal{F}_{n,1} \subset \cdots \mathcal{F}_{n,n} = \tilde{S}(F^n)^+{}_{\mathcal{M}}$$

with

$$\frac{\mathcal{F}_{n,i}}{\mathcal{F}_{n,i-1}} \cong E^{\infty}_{n-i,i}.$$

The E^1-page of $\mathcal{E}^+(F^n)_{\mathcal{M}}$ has the form

$$\begin{tikzcd}
0 & 0 & 0 & \cdots & 0 \\
0 & \mathrm{H}_1(\mathrm{SL}_2(F), \mathbb{Z}) \arrow[d] & \mathrm{H}_2(\mathrm{SL}_2(F), \mathbb{Z}) \arrow[d] & \cdots & \mathrm{H}_n(\mathrm{SL}_2(F), \mathbb{Z}) \arrow[d] \\
\vdots & \vdots \arrow[d, "\mathrm{cor}"] & \vdots \arrow[d, "\mathrm{cor}"] & \cdots & \vdots \arrow[d, "\mathrm{cor}"] \\
0 & \mathrm{H}_1(\mathrm{SL}_{n-2}(F), \mathbb{Z}) \arrow[d, "0"] & \mathrm{H}_2(\mathrm{SL}_{n-2}(F), \mathbb{Z}) \arrow[d, "0"] & \cdots & \mathrm{H}_n(\mathrm{SL}_{n-2}(F), \mathbb{Z}) \arrow[d, "0"] \\
0 & \mathrm{H}_1(\mathrm{SL}_{n-1}(F), \mathbb{Z}) \arrow[d, "\mathrm{cor}"] & \mathrm{H}_2(\mathrm{SL}_{n-1}(F), \mathbb{Z}) \arrow[d, "\mathrm{cor}"] & \cdots & \mathrm{H}_n(\mathrm{SL}_{n-1}(F), \mathbb{Z}) \arrow[d, "\mathrm{cor}"] \\
0 & \mathrm{H}_1(\mathrm{SL}_n(F), \mathbb{Z}) & \mathrm{H}_2(\mathrm{SL}_n(F), \mathbb{Z}) & \cdots & \mathrm{H}_n(\mathrm{SL}_n(F), \mathbb{Z})
\end{tikzcd}$$

THEOREM 5.10.

(1) *The higher differentials $d^2, d^3, \dots,$ in the spectral sequence $\mathcal{E}^+(F^n)_{\mathcal{M}}$ are all* 0.

(2) $\tilde{S}(F^{n-2})_{\mathcal{M}} \cong E * \tilde{S}(F^{n-2})_{\mathcal{M}}$ *and this latter is a direct summand of* $\tilde{S}(F^n)_{\mathcal{M}}$.

Proof.

(1) We will use induction on n. For $n \leq 2$ the statement is true for trivial reasons.

On the other hand, if $n > 2$, by Lemma 5.9, the map
$$\tilde{E} * - : \mathcal{E}^+(F^{n-2})_{\mathcal{M}}[2] \to \mathcal{E}^+(F^n)_{\mathcal{M}}$$
induces an isomorphism on E^1-terms for $q \geq 2$. By induction (and the fact that ${E'}^1_{p,q} = 0$ for $q \leq 1$), the result follows for n.

(2) The map of spectral sequences $\mathcal{E}^+(F^{n-2})_{\mathcal{M}}[2] \to \mathcal{E}^+(F^n)_{\mathcal{M}}$ induces a homomorphism on abutments
$$\tilde{S}(F^{n-2})^+{}_{\mathcal{M}} \xrightarrow{E*-} \tilde{S}(F^n)^+{}_{\mathcal{M}}$$
By Lemma 5.9 again, it follows that the composite
$$\tilde{S}(F^{n-2})^+{}_{\mathcal{M}} \xrightarrow{E*-} \tilde{S}(F^n)^+{}_{\mathcal{M}} \longrightarrow \left(\tilde{S}(F^n)^+{}_{\mathcal{M}}\right)/\mathcal{F}_{n,1}$$
is an isomorphism.

Thus $\tilde{S}(F^{n-2})^+{}_{\mathcal{M}} \cong E * \tilde{S}(F^{n-2})^+{}_{\mathcal{M}}$ and
$$\tilde{S}(F^n)^+{}_{\mathcal{M}} \cong \left(E * \tilde{S}(F^{n-2})^+{}_{\mathcal{M}}\right) \oplus \mathcal{F}_{n,1}.$$

□

As a corollary we obtain the following general homology stability result for the homology of special linear groups:

COROLLARY 5.11.
The corestriction maps $\mathrm{H}_p(\mathrm{SL}_{n-1}(F), \mathbb{Z}) \to \mathrm{H}_p(\mathrm{SL}_n(F), \mathbb{Z})$ *are isomorphisms for* $p < n-1$ *and are surjective when* $p = n-1$.

Proof. Using (1) of Theorem 5.10 and Lemma 5.8, we have (for the spectral sequence $\mathcal{E}^+(F^n)_{\mathcal{M}}$) that $E^\infty_{p,q} = E^2_{p,q} =$
$$\frac{\mathrm{Ker}(d^1)}{\mathrm{Im}(d^1)} = \begin{cases} \mathrm{Ker}(\mathrm{H}_p(\mathrm{SL}_{n-q}(F), \mathbb{Z}) \to \mathrm{H}_p(\mathrm{SL}_{n-q+1}(F), \mathbb{Z})) & q \text{ odd} \\ \mathrm{Coker}(\mathrm{H}_p(\mathrm{SL}_{n-q-1}(F), \mathbb{Z}) \to \mathrm{H}_p(\mathrm{SL}_{n-q}(F), \mathbb{Z})) & q \text{ even} \end{cases}$$
But the abutment of the spectral sequence is 0 in dimensions less than n. It follows that $E^\infty_{p,q} = 0$ whenever $p + q \leq n-1$. □

REMARK 5.12. Note that in the spectral sequence $\mathcal{E}^+(F^n)_{\mathcal{M}}$,
$$E^\infty_{n,0} = \mathrm{Coker}(\mathrm{H}_n(\mathrm{SL}_{n-1}(F), \mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_n(F), \mathbb{Z})) = \mathrm{SH}_n(F).$$

Clearly, the edge homomorphism $\mathrm{H}_n(\mathrm{SL}_n(F), \mathbb{Z}) \to E^\infty_{n,0} \to \tilde{S}(F^n)_{\mathcal{M}}$ is just the iterated connecting homomorphism ϵ_n of section 3 above. Thus we have:

COROLLARY 5.13. *The maps*
$$\epsilon_\bullet : \mathrm{SH}_\bullet(F) \to \tilde{S}(F^\bullet)_{\mathcal{M}}$$
define an injective homomorphism of graded $\mathbb{Z}[F^\times]$*-algebras.*

Corollary 5.14. *$\tilde{S}(F^2)_{\mathcal{M}} = \mathcal{F}_{2,1} \oplus \mathbb{Z}[F^\times]E$ and for all $n \geq 3$,*

$$\tilde{S}(F^n)_{\mathcal{M}} = (E * \tilde{S}(F^{n-2})_{\mathcal{M}}) \oplus \mathcal{F}_{n,1} \cong \tilde{S}(F^{n-2})_{\mathcal{M}} \oplus \mathcal{F}_{n,1}.$$

Proof. Clearly $\tilde{S}(F^2)^+{}_{\mathcal{M}} = \mathcal{F}_{1,2}$, while for $n \geq 3$ we have

$$\tilde{S}(F^n)_{\mathcal{M}} = \begin{cases} \tilde{S}(F^n)^+{}_{\mathcal{M}} \oplus \mathbb{Z}[F^\times]E^{*\frac{n}{2}} & n \text{ even} \\ \tilde{S}(F^n)^+{}_{\mathcal{M}} \oplus \left(\tilde{S}(F) * E^{*\frac{n-1}{2}}\right) & n \text{ odd} \end{cases}$$

□

Corollary 5.15. *For all $n \geq 3$,*

$$\tilde{S}(F^n)_{\mathcal{M}} \cong \begin{cases} \mathcal{F}_{n,1} \oplus \mathcal{F}_{n-2,1} \oplus \cdots \oplus \mathcal{F}_{2,1} \oplus \mathbb{Z}[F^\times] & n \text{ even} \\ \mathcal{F}_{n,1} \oplus \mathcal{F}_{n-2,1} \oplus \cdots \oplus \mathcal{F}_{3,1} \oplus \mathcal{I}_{F^\times} & n \text{ odd} \end{cases}$$

as a $\mathbb{Z}[F^\times]$-module.

Note that $\mathcal{F}_{1,1} = \tilde{S}(F) = \mathcal{I}_{F^\times}$, and for all $n \geq 2$, $\mathcal{F}_{n,1}$ fits into an exact sequence associated to the spectral sequence $\mathcal{E}^+(F^n)_{\mathcal{M}}$:

$$0 \to E^\infty_{n,0} = \mathcal{F}_{n,0} \to \mathcal{F}_{n,1} \to E^\infty_{n-1,1} \to 0.$$

Corollary 5.16. *For all $n \geq 2$ we have an exact sequence*

$$\mathrm{H}_n(\mathrm{SL}_{n-1}(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z}) \to \mathcal{F}_{n,1} \to \\ \mathrm{H}_{n-1}(\mathrm{SL}_{n-1}(F),\mathbb{Z}) \to \mathrm{H}_{n-1}(\mathrm{SL}_n(F),\mathbb{Z}) \to 0.$$

Lemma 5.17. *For all $n \geq 2$, the map T_n induces a surjective map $\mathcal{F}_{n,1} \to K^{\mathrm{MW}}_n(F)$.*

Proof. First observe that since $K^{\mathrm{MW}}_n(F)$ is generated by the elements of the form $[a_1]\cdots[a_n]$ it follows from the definition of T_n that $T_n : \tilde{S}(F^n) \to K^{\mathrm{MW}}_n(F)$ is surjective for all $n \geq 1$.
Next, since $K^{\mathrm{MW}}_\bullet(F)$ is multiplicative, $T_\bullet$ factors through an algebra homomorphism $\tilde{S}(F^\bullet)_{\mathcal{M}} \to K^{\mathrm{MW}}_\bullet(F)$. The lemma thus follows from Corollary 5.14 and the fact that $T_2(E) = 0$. □

Lemma 5.18. *$\mathcal{F}_{2,1} = \mathcal{F}_{2,0}$ and $T_2 : \mathcal{F}_{2,1} \to K^{\mathrm{MW}}_2(F)$ is an isomorphism.*

Proof. Since $\mathrm{H}_1(\mathrm{SL}_1(F),\mathbb{Z}) = 0$, $\mathcal{F}_{2,1} = \mathcal{F}_{2,0} = E^\infty_{2,0} = \epsilon_2(\mathrm{H}_2(\mathrm{SL}_2(F),\mathbb{Z}))$. Now apply Theorem 3.10. □

It is natural to define elements $[a,b] \in \mathcal{F}_{2,0} \subset \tilde{S}(F^2)_{\mathcal{M}}$ by $[a,b] := T_2^{-1}([a][b])$.

Lemma 5.19. *In $\tilde{S}(F^2)_{\mathcal{M}}$ we have the formula*

$$[a,b] = \lfloor a \rceil * \lfloor b \rceil - \langle\langle a\rangle\rangle\langle\langle b\rangle\rangle E.$$

Proof. The results above show that the maps T_2 and D_2 induce an isomorphism

$$(T_2, D_2) : \tilde{S}(F^2)_{\mathcal{M}} \cong K^{\mathrm{MW}}_2(F) \oplus \mathbb{Z}[F^\times].$$

Since $D_2(\lfloor a \rceil * \lfloor b \rceil) = \langle\langle a\rangle\rangle\langle\langle b\rangle\rangle$, while $D_2(E) = 1$, the result follows. □

Theorem 5.20.

(1) *The product $*$ respects the filtrations on $\tilde{S}(F^n)$; i.e. for all $n, m \geq 1$ and $i, j \geq 0$*
$$\mathcal{F}_{n,i} * \mathcal{F}_{m,j} \subset \mathcal{F}_{n+m,i+j}.$$
(2) *For $n \geq 1$, let $\epsilon_{n+1,1}$ denote the composite $\mathcal{F}_{n+1,1} \to E^\infty_{n,1} = E^2_{n,1} \to \mathrm{H}_n(\mathrm{SL}_n(F), \mathbb{Z})$. For all $a \in F^\times$ and for all $n \geq 1$ the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{F}_{n,0} \arrow[r, "\lfloor a \rceil *"] & \mathcal{F}_{n+1,1} \arrow[d, "\epsilon_{n+1,1}"] \\
\mathrm{H}_n(\mathrm{SL}_n(F), \mathbb{Z}) \arrow[u, "\epsilon_n"] \arrow[r, "\langle\langle a \rangle\rangle \cdot"] & \mathrm{H}_n(\mathrm{SL}_n(F), \mathbb{Z})
\end{tikzcd}$$

Proof.

(1) The filtration on $\tilde{S}(F^n)_{\mathcal{M}}$ is derived from the spectral sequence $\mathcal{E}(F^n)$. This is the spectral sequence of the double complex $B_\bullet \otimes_{\mathrm{SL}_n(F)} \mathcal{C}^\tau_\bullet(F^n)$, regarded as a filtered complex by truncating $\mathcal{C}^\tau_\bullet(F^n)$ at i for $i = 0, 1, \ldots$. Since the product $*$ is derived from a graded bilinear pairing on the complexes $\mathcal{C}^\tau_\bullet(F^n)$, the result easily follows.

(2) The spectral sequence $\mathcal{E}(F^{n+1})$ calculates
$$\mathrm{H}_\bullet(\mathrm{SL}_{n+1}(F), \mathcal{C}^\tau(F^{n+1})) \cong \mathrm{H}_\bullet(\mathrm{SL}_{n+1}(F), H(F^{n+1})[n+1]$$
(where $[n+1]$ denotes a degree shift by $n+1$).

Let $C[1, n]$ denote the truncated complex
$$\mathcal{C}^\tau_1(F^{n+1}) \xrightarrow{d_1} \mathcal{C}^\tau_0(F^{n+1})$$
and let Z_1 denote the kernel of d_1. Then
$$\mathrm{H}_\bullet(\mathrm{SL}_{n+1}(F), C[1, n]) \cong \mathrm{H}_\bullet(\mathrm{SL}_{n+1}(F), Z_1)[1].$$
If $\mathcal{F}_i$ denotes the filtration on $\mathrm{H}_\bullet(\mathrm{SL}_{n+1}(F), \mathcal{C}^\tau(F^{n+1}))$ associated to the spectral sequence $\mathcal{E}(F^{n+1})$, then from the definition of this filtration, $\mathcal{F}_1\mathrm{H}_k(\mathrm{SL}_{n+1}(F), \mathcal{C}^\tau(F^{n+1})) =$
$$\mathrm{Im}(\mathrm{H}_k(\mathrm{SL}_{n+1}(F), C[1, n]) \to \mathrm{H}_k(\mathrm{SL}_{n+1}(F), \mathcal{C}^\tau(F^{n+1}))).$$
In particular,
$$\mathcal{F}_{n+1,1} \cong \mathrm{Im}(\mathrm{H}_{n+1}(\mathrm{SL}_{n+1}(F), C[1, n]) \to \mathrm{H}_{n+1}(\mathrm{SL}_{n+1}(F), \mathcal{C}^\tau(F^{n+1})))$$
and with this identification the diagram

$$\begin{tikzcd}
\mathrm{H}_{n+1}(\mathrm{SL}_{n+1}(F), C[1, n]) \arrow[r] & \mathcal{F}_{n+1,1} \arrow[d, "\epsilon_{n+1,1}"] \\
\mathrm{H}_n(\mathrm{SL}_{n+1}(F), Z_1) \arrow[u, "\cong"] \arrow[r] & \mathrm{H}_n(\mathrm{SL}_{n+1}(F), \mathcal{C}^\tau_1(F^{n+1}))
\end{tikzcd}$$

commutes (and $\mathrm{H}_n(\mathrm{SL}_{n+1}(F), \mathcal{C}^\tau_1(F^{n+1})) \cong \mathrm{H}_n(\mathrm{SA}(F, F^n), \mathbb{Z})$ by Shapiro's Lemma, of course).

We consider $\mathrm{SL}_n(F) \subset \mathrm{SA}(F, F^n) \subset \mathrm{SL}_{n+1}(F) \subset \mathrm{GL}_{n+1}(F)$ where the first inclusion is obtained by inserting a 1 in the $(1,1)$ position. Let $B_\bullet$ denote a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathrm{GL}_{n+1}(F)]$. Let $z \in \mathrm{H}_n(\mathrm{SL}_n(F), \mathbb{Z})$ be represented by $x \otimes 1 \in B_n \otimes_{\mathbb{Z}[\mathrm{SL}_n(F)]} \mathbb{Z} = B_n \otimes_{\mathbb{Z}[\mathrm{SL}_n(F)]} \mathcal{C}_0^\tau(F^n)$. Then $\lfloor a \rceil * \epsilon_n(z)$ is represented by $z \otimes [(ae_1) - (e_1)] \in B_n \otimes_{\mathrm{SL}_{n+1}(F)} Z_1$ which maps to the element of $\mathrm{H}_n(\mathrm{SL}_{n+1}(F), \mathcal{C}_1^\tau(F^{n+1}))$ represented by $z(g-1) \otimes (e_1)$ where $g = \mathrm{diag}(a, 1, \ldots, 1, a^{-1})$. But this is just the image of $\langle\langle a \rangle\rangle z$ under the map $\mathrm{H}_n(\mathrm{SL}_n(F), \mathbb{Z}) \to \mathrm{H}_n(\mathrm{SA}(F, F^n), \mathbb{Z}) \cong \mathrm{H}_n(\mathrm{SL}_{n+1}(F), \mathcal{C}_1^\tau(F^{n+1}))$. $\square$

Lemma 5.21. *The map $T_3 : \mathcal{F}_{3,1} \to K_3^{\mathrm{MW}}(F)$ is an isomorphism.*

Proof. Consider the short exact sequence

$$0 \to E^\infty_{3,0} \to \mathcal{F}_{3,1} \to E^\infty_{2,1} \to 0.$$

Here ϵ_3 induces an isomorphism

$$E^\infty_{3,0} \cong \mathrm{Coker}(\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}) \to \mathrm{H}_3(\mathrm{SL}_3(F), \mathbb{Z})).$$

By the main result of [8] (Theorem 4.7 - see also section 2.4 of this article), T_3 thus induces an isomorphism $E^\infty_{3,0} \cong 2K_3^{\mathrm{M}}(F) \subset K_3^{\mathrm{MW}}(F)$.
On the other hand,

$$E^\infty_{2,1} \cong \mathrm{Ker}(\mathrm{H}_2(\mathrm{SL}_2(F), \mathbb{Z}) \to \mathrm{H}_2(\mathrm{SL}_3(F), \mathbb{Z})) \cong I^3(F)$$

Thus we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E^\infty_{3,0} \arrow[r] \arrow[d, "\cong"', "T_3"] & \mathcal{F}_{3,1} \arrow[r, "\rho"] \arrow[d, "T_3"] & I^3(F) \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & 2K_3^{\mathrm{M}}(F) \arrow[r] & K_3^{\mathrm{MW}}(F) \arrow[r, "p_3"] & I^3(F) \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrows are surjections.
Now the inclusion $I^3(F) \to K_2^{\mathrm{MW}}(F)$ is given by $\langle\langle a, b, c \rangle\rangle \mapsto \langle\langle a \rangle\rangle [b][c]$. Thus the inclusion $j : I^3(F) \to \mathrm{H}_2(\mathrm{SL}_2(F), \mathbb{Z})$ is given by $\langle\langle a, b, c \rangle\rangle \mapsto \langle\langle a \rangle\rangle \langle b, c \rangle$ where $\langle b, c \rangle = \epsilon_2^{-1}([b, c])$. Thus for all $a, b, c \in F^\times$ we have

$$j \circ \rho(\lfloor a \rceil * [b, c]) = \epsilon_{3,1}(\lfloor a \rceil * [b, c]) = \langle\langle a \rangle\rangle \langle b, c \rangle$$

using Theorem 5.20 (2), and thus $\rho(\lfloor a \rceil * [b, c]) = \langle\langle a, b, c \rangle\rangle \in I^3(F)$. It follows from the diagram that

$$\alpha(\langle\langle a, b, c \rangle\rangle) = \alpha \circ \rho(\lfloor a \rceil * [b, c]) = p_3 \circ T_3(\lfloor a \rceil * [b, c]) = \langle\langle a, b, c \rangle\rangle$$

so that α is the identity map, and the result follows. $\square$

Lemma 5.22. *For all $a \in F^\times$, $\lfloor a \rceil * E = E * \lfloor a \rceil$ in $\tilde{S}(F^3)_{\mathcal{M}}$.*

Proof. By the calculations above, $\mathcal{F}_{3,1} = \tilde{S}(F^3)^+{}_{\mathcal{M}} = \mathrm{Ker}(D_3)$. Thus $R_a := \lfloor a \rceil * E - E * \lfloor a \rceil \in \mathcal{F}_{3,1}$. But then $T_3(R_a) = 0$ since $T_2(E) = 0$ and thus $R_a = 0$ by the previous lemma. $\square$

Lemma 5.23.

(1) *For all* $a, b, c \in F^\times$
$$\lfloor a \rceil * [b, c] = [a, b] * \lfloor c \rceil \ \text{in } \tilde{S}(F^3)_{\mathcal{M}}.$$
(2) *For all* $a, b, c \in F^\times$
$$\lfloor a \rceil * \lfloor b \rceil * \lfloor c \rceil = \lfloor c \rceil * \lfloor a \rceil * \lfloor b \rceil \ \text{in } \tilde{S}(F^3)_{\mathcal{M}}.$$
(3) *For all* $a, b, c, d \in F^\times$
$$[a, b] * [c, d] = [a, c^{-1}] * [b, d] \ \text{in } \tilde{S}(F^4)_{\mathcal{M}}.$$

Proof. The calculations above have established that the map
$$(T_3, D_3) : \tilde{S}(F^3)_{\mathcal{M}} \to K_3^{\mathrm{MW}}(F) \oplus \mathcal{I}_{F^\times}$$
is an isomorphism.

(1) This follows from the identities
$$T_3(\lfloor a \rceil * [b, c]) = [a][b][c] = T_3([a, b] * \lfloor c \rceil)$$
and
$$D_3(\lfloor a \rceil * [b, c]) = \langle\langle a, b, c\rangle\rangle = D_3([a, b] * \lfloor c \rceil)$$
(2) This follows from the fact that $[a][b][c] = [c][a][b]$ in $K_3^{\mathrm{MW}}(F)$.

(3) We begin by observing that, since $\tilde{S}(F) \cong \mathcal{I}_{F^\times}$ as a $\mathbb{Z}[F^\times]$-module we have $\langle\langle a\rangle\rangle\lfloor b \rceil = \lfloor ab \rceil - \lfloor a \rceil - \lfloor b \rceil = \langle\langle b\rangle\rangle\lfloor a \rceil$ for all $a, b \in F^\times$.

For $x_1, \ldots, x_n \in F^\times$ and $i, j \geq 1$ with $i + j = n$ we set
$$L_{i,j}(x_1, \ldots, x_n) := \langle\langle x_1\rangle\rangle \cdots \langle\langle x_i\rangle\rangle \left(\lfloor x_{i+1} \rceil * \cdots * \lfloor x_n \rceil\right) \in \tilde{S}(F^j)_{\mathcal{M}}.$$
By the observation just made, we have
$$L_{i,j}(x_1, \ldots, x_n) = L_{i,j}(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$
for any permutation σ of $1, \ldots, n$.

So
$$\begin{aligned}[a, b] * [c, d] &= (\lfloor a \rceil * \lfloor b \rceil - \langle\langle a\rangle\rangle\langle\langle b\rangle\rangle E) * (\lfloor c \rceil * \lfloor d \rceil - \langle\langle c\rangle\rangle\langle\langle d\rangle\rangle E) \\ &= \lfloor a \rceil * \lfloor b \rceil * \lfloor c \rceil * \lfloor d \rceil - 2L_{2,2}(a, b, c, d) * E + \langle\langle a\rangle\rangle\langle\langle b\rangle\rangle\langle\langle c\rangle\rangle\langle\langle d\rangle\rangle E^{*2}\end{aligned}$$
Let $R = [a, b] * [c, d] - [a, c^{-1}] * [b, d]$.

So $R =$
$$\begin{aligned}&\lfloor a \rceil * \lfloor b \rceil * \lfloor c \rceil * \lfloor d \rceil - \lfloor a \rceil * \lfloor c^{-1} \rceil * \lfloor b \rceil * \lfloor d \rceil - 2(L_{2,2}(a, b, c, d) - L_{2,2}(a, c^{-1}, b, d)) * E \\ &\qquad + \langle\langle a\rangle\rangle\langle\langle d\rangle\rangle \left[(\langle\langle b\rangle\rangle\langle\langle c\rangle\rangle - \langle\langle c^{-1}\rangle\rangle\langle\langle b\rangle\rangle)E\right] * E.\end{aligned}$$
However, since $[b, c] = [c^{-1}, b]$ in $\tilde{S}(F^2)_{\mathcal{M}}$ we have (by Lemma 5.19)
$$(\langle\langle b\rangle\rangle\langle\langle c\rangle\rangle - \langle\langle c^{-1}\rangle\rangle\langle\langle b\rangle\rangle)E = \lfloor b \rceil * \lfloor c \rceil - \lfloor c^{-1} \rceil * \lfloor b \rceil.$$
Thus $\langle\langle a\rangle\rangle\langle\langle d\rangle\rangle \left[(\langle\langle b\rangle\rangle\langle\langle c\rangle\rangle - \langle\langle c^{-1}\rangle\rangle\langle\langle b\rangle\rangle)E\right] * E =$
$$(L_{2,2}(a, b, c, d) - L_{2,2}(a, c^{-1}, b, d)) * E$$
and hence $R =$
$$\lfloor a \rceil * \lfloor b \rceil * \lfloor c \rceil * \lfloor d \rceil - \lfloor a \rceil * \lfloor c^{-1} \rceil * \lfloor b \rceil * \lfloor d \rceil - (L_{2,2}(a, b, c, d) - L_{2,2}(a, c^{-1}, b, d)) * E.$$

Now

$$\begin{aligned}
&(L_{2,2}(a,b,c,d) - L_{2,2}(a,c^{-1},b,d)) * E \\
&\quad = \lfloor a\rceil * \lfloor d\rceil * \left[(\langle\langle b\rangle\rangle\langle\langle c\rangle\rangle - \langle\langle c^{-1}\rangle\rangle\langle\langle b\rangle\rangle)E\right] \\
&\quad = \lfloor a\rceil * \lfloor d\rceil * \left[\lfloor b\rceil * \lfloor c\rceil - \lfloor c^{-1}\rceil * \lfloor b\rceil\right] \\
&\quad = \lfloor a\rceil * (\lfloor d\rceil * \lfloor b\rceil * \lfloor c\rceil) - \lfloor a\rceil * (\lfloor d\rceil * \lfloor c^{-1}\rceil * \lfloor b\rceil) \\
&\quad = \lfloor a\rceil * \lfloor b\rceil * \lfloor c\rceil * \lfloor d\rceil - \lfloor a\rceil * \lfloor c^{-1}\rceil * \lfloor b\rceil * \lfloor d\rceil
\end{aligned}$$

using (2) in the last step.

□

THEOREM 5.24. *For all $n \geq 2$ there is a homomorphism $\mu_n : K_n^{\mathrm{MW}}(F) \to \mathcal{F}_{n,1}$ such that the composite $T_n \circ \mu_n$ is the identity map.*

Proof. For $n \geq 2$ and $a_1, \ldots, a_n \in F^\times$, let $\{\{a_1, \ldots, a_n\}\} :=$

$$\left\{ \begin{array}{ll} [a_1,a_2] * \cdots * [a_{n-1},a_n], & n \text{ even} \\ \lfloor a_1\rceil * [a_2,a_3] * \cdots * [a_{n-1},a_n], & n \text{ odd} \end{array} \right\} \in \mathcal{F}_{n,1} \subset \tilde{S}(F^n)_{\mathcal{M}}.$$

By Lemma 5.23 (1) and (3), as well as the definition of $[x,y]$, the elements $\{\{a_1, \ldots, a_n\}\}$ satisfy the 'Matsumoto-Moore' relations (see Section 2.4 above), and thus there is a well-defined homomorphism of groups

$$\mu_n : K_n^{\mathrm{MW}}(F) \to \mathcal{F}_{n,1}, \qquad [a_1]\cdots[a_n] \mapsto \{\{a_1, \ldots, a_n\}\}.$$

Since $T_n(\{\{a_1, \ldots, a_n\}\}) = [a_1]\cdots[a_n]$, the result follows. □

COROLLARY 5.25. *The subalgebra of $\mathrm{SH}_{2\bullet}(F)$ generated by $\mathrm{SH}_2(F) = \mathrm{H}_2(\mathrm{SL}_2(F), \mathbb{Z})$ is isomorphic to $K_{2\bullet}^{\mathrm{MW}}(F)$ and is a direct summand of $\mathrm{SH}_{2\bullet}(F)$.*

Proof. This is immediate from Theorems 3.10 and 5.24. □

6. DECOMPOSABILTY

Recall that F is a field of characteristic 0 throughout this section.
In [24], Suslin proved that $\mathrm{H}_n(\mathrm{GL}_n(F), \mathbb{Z})/\mathrm{H}_n(\mathrm{GL}_{n-1}(F), \mathbb{Z}) \cong K_n^{\mathrm{M}}(F)$. This is, in particular, a decomposability result. It says that $\mathrm{H}_n(\mathrm{GL}_n(F), \mathbb{Z})$ is generated, modulo the image of $\mathrm{H}_n(\mathrm{GL}_{n-1}(F), \mathbb{Z})$ by products of 1-dimensional cycles. In this section we will prove analogous results for the special linear group, with Milnor-Witt K-theory replacing Milnor K-theory. To do this, we prove the decomposability of the algebra $\tilde{S}(F^\bullet)_{\mathcal{M}}$ (for $n \geq 3$). Theorem 6.2 is an analogue of Suslin's Proposition 3.3.1. The proof is essentially identical, and we reproduce it here for the convenience of the reader. From this we deduce our decomposability result (Theorem 6.8), which requires still a little more work than in the case of the general linear group.

LEMMA 6.1. *For any finite-dimensional vector spaces W and V, the image of the pairing*

$$\tilde{\mathrm{S}}(W,V) \otimes H(W) \to \tilde{\mathrm{S}}(W \oplus V)_{\mathcal{M}} \tag{2}$$

coincides with the image of the pairing

$$\tilde{\mathrm{S}}(V) \otimes \tilde{\mathrm{S}}(W) \to \tilde{\mathrm{S}}(W \oplus V)_{\mathcal{M}} \tag{3}$$

Proof. The image of the pairing (2) is equal to the image of

$$\tilde{\mathrm{S}}(W,V)_{\mathcal{M}} \otimes H(W) \to \tilde{\mathrm{S}}(W \oplus V)_{\mathcal{M}}$$

which coincides with the image of

$$\tilde{\mathrm{S}}(V)_{\mathcal{M}} \otimes \tilde{\mathrm{S}}(W)_{\mathcal{M}} \to \tilde{\mathrm{S}}(W \oplus V)_{\mathcal{M}}$$

by the isomorphism of Corollary 5.6. □

Let $\tilde{S}(F^n)^{\mathrm{dec}} \subset \tilde{\mathrm{S}}(F^n)_{\mathcal{M}}$ be *the* $\mathbb{Z}[F^\times]$*-submodule of decomposable elements*; i.e. $\tilde{S}(F^n)^{\mathrm{dec}}$ is the image of

$$\bigoplus_{p+q=n,p,q>0} \left(\tilde{\mathrm{S}}(F^p)_{\mathcal{M}} \otimes \tilde{\mathrm{S}}(F^q)_{\mathcal{M}}\right) \xrightarrow{\ *\ } \tilde{\mathrm{S}}(F^n)_{\mathcal{M}}.$$

More generally, note that if $V = V_1 \oplus V_2 = V_1' \oplus V_2'$ and if $\dim_F(V_i) = \dim_F(V_i')$ for $i = 1,2$, then the image of $\tilde{\mathrm{S}}(V_1) \otimes \tilde{\mathrm{S}}(V_2) \to \tilde{\mathrm{S}}(V)$ coincides with $\tilde{\mathrm{S}}(V_1') \otimes \tilde{\mathrm{S}}(V_2') \to \tilde{\mathrm{S}}(V)$. This follows from the fact that there exists $\phi \in \mathrm{SL}(V)$ with $\phi(V_i) = V_i'$ for $i = 1,2$.
Therefore $\tilde{S}(F^n)^{\mathrm{dec}}$ is the image of

$$\bigoplus_{F^n=V_1\oplus V_2, V_i\neq 0} \left(\tilde{\mathrm{S}}(V_1)_{\mathcal{M}} \otimes \tilde{\mathrm{S}}(V_2)_{\mathcal{M}}\right) \xrightarrow{\ *\ } \tilde{\mathrm{S}}(F^n)_{\mathcal{M}}.$$

If $x = \sum_i n_i(x_1^i,\ldots,x_p^i) \in C_p(V)$ and $y = \sum_j m_j(y_1^j,\ldots,y_q^j) \in C_q(V)$ and if $(x_1^i,\ldots,x_p^i,y_1^j,\ldots,y_q^j) \in X_{p+q}(V)$ for all i,j, then we let

$$x \circledast y := \sum_{i,j} n_i m_j (x_1^i,\ldots,x_p^i,y_1^j,\ldots,y_q^j) \in C_{p+q}(V).$$

Of course, if $x \in C_p(V_1)$ and $y \in C_q(V_2)$ with $V = V_1 \oplus V_2$, then $x \circledast y = x * y$. Furthermore, when $x \circledast y$ is defined, we have

$$d(x \circledast y) = d(x) \circledast y + (-1)^p x \circledast d(y).$$

THEOREM 6.2. *Let $n \geq 1$. For any $a_1,\ldots,a_n,b \in F^\times$ and for any $1 \leq i \leq n$*

$$\lfloor a_1,\ldots,ba_i,\ldots,a_n\rceil \cong \langle b\rangle \lfloor a_1,\ldots,a_n\rceil \pmod{\tilde{S}(F^n)^{dec}}.$$

Proof. Let $a = a_1e_1 + \cdots + ba_ie_i + \cdots a_ne_n$.
We have

$$\begin{aligned}
&\lfloor a_1,\ldots,ba_i,\ldots,a_n\rceil - \langle b\rangle \lfloor a_1,\ldots,a_n\rceil \\
&\quad= \quad d(e_1,\ldots,e_i,\ldots,e_n,a) - d(e_1,\ldots,b_ie_i,\ldots,e_n,a) \\
&\quad= \quad d\Big((e_1,\ldots,e_{i-1}) \circledast ((e_i) - (be_i)) \circledast (e_{i+1},\ldots,e_n,a)\Big) \\
&\quad= \quad d(e_1,\ldots,e_{i-1}) \circledast ((e_i) - (be_i)) \circledast (e_{i+1},\ldots,e_n,a) \\
&\quad+ \quad (-1)^i(e_1,\ldots,e_{i-1}) \circledast ((e_i) - (be_i)) \circledast d(e_{i+1},\ldots,e_n,a)
\end{aligned}$$

Let $u = a_1e_1 + \cdots + a_{i-1}e_{i-1} + ba_ie_i = a - \sum_{j=i+1}^{n} a_je_j$.Then

$$(-1)^{i-1}(e_1,\ldots,e_{i-1}) = d\big((e_1,\ldots,e_{i-1}) \circledast (u)\big) - d(e_1,\ldots,e_{i-1}) \circledast (u)$$

and

$$(e_{i+1},\ldots,e_n,a) = d\big((u) \circledast (e_{i+1},\ldots,e_n,a)\big) + (u) \circledast d(e_{i+1},\ldots,e_n,a).$$

Thus $\lfloor a_1,\ldots,ba_i,\ldots,a_n\rceil - \langle b\rangle \lfloor a_1,\ldots,a_n\rceil = X_1 - X_2 + X_3$ where

$$\begin{aligned}
X_1 &= d(e_1,\ldots,e_{i-1}) \circledast \big((e_i) - (be_i)\big) \circledast d(u, e_{i+1},\ldots,e_n,a),\\
X_2 &= d(e_1,\ldots,e_{i-1},u) \circledast \big((e_i) - (be_i)\big) \circledast d(e_{i+1},\ldots,e_n,a), \text{ and}\\
X_3 &= d(e_1,\ldots,e_{i-1}) \circledast \Big[\big((e_i) - (be_i)\big) \circledast (u) + (u) \circledast \big((e_i) - (be_i)\big)\Big] \circledast\\
&\quad \circledast d(e_{i+1},\ldots,e_n,a)
\end{aligned}$$

We show that each X_i is decomposable: Let $V \subset F^n$ be the span of $u, e_{i+1},\ldots,e_n$ (which is also equal to the span of $a, e_{i+1},\ldots,e_n$), and let V' be the span of $e_1,\ldots,e_{i-1}$. Then $F^n = V' \oplus V$ and $d(u, e_{i+1},\ldots,e_n,a) \in H(V)$ while
$d(e_1,\ldots,e_{i-1}) \circledast \big((e_i) - (be_i)\big) \in H(V,V')$.
Thus X_1 lies in the image of

$$H(V,V') \otimes H(V) \xrightarrow{\ *\ } \tilde{S}(F^n)_{\mathcal{M}}$$

and so is decomposable.
Similarly, if we let W be the span of $e_1,\ldots,e_i$ and W' the span of $e_{i+1},\ldots,e_n$, then

$$d(e_1,\ldots,e_{i-1},u) \circledast \big((e_i) - (be_i)\big),\ d(e_1,\ldots,e_{i-1}) \circledast \Big[\big((e_i) - (be_i)\big) \circledast (u) + (u) \circledast \big((e_i) - (be_i)\big)\Big]$$

belongs to $H(W)$ and $d(e_{i+1},\ldots,e_n,a) \in H(W,W')$.Thus X_2, X_3 lie in the image of

$$H(W) \otimes H(W,W') \xrightarrow{\ *\ } \tilde{S}(F^n)_{\mathcal{M}}$$

and are also decomposable. □

Let $\tilde{S}(F^n)^{\mathrm{ind}} := \tilde{S}(F^n)_{\mathcal{M}}/\tilde{S}(F^n)^{\mathrm{dec}}$.
The main goal of this section is to show that $\tilde{S}(F^n)^{\mathrm{ind}} = 0$ for all $n \geq 3$ (Theorem 6.8 below).

Lemma 6.3. *For all $n \geq 3$, $\tilde{S}(F^n)^{ind}$ is a multiplicative $\mathbb{Z}[F^\times]$-module.*

Proof. We have

$$\mathcal{A}_n \cong \begin{cases} \mathbb{Z}[F^\times]E^{*n/2}, & n \text{ even}\\ \tilde{S}(F) * E^{*(n-1)/2}, & n \text{ odd}\end{cases}$$

and these modules are decomposable for all $n \geq 3$. It follows that the map

$$\tilde{S}(F^n)^+{}_{\mathcal{M}} \to \tilde{S}(F^n)^{\mathrm{ind}}$$

is surjective for all $n \geq 3$. □

REMARK 6.4. Since $E * \tilde{S}(F^{n-2})_{\mathcal{M}} \subset \tilde{S}(F^n)^{\mathrm{dec}}$, in fact we have that $\mathcal{F}_{n,1} \to \tilde{S}(F^n)^{\mathrm{ind}}$ is surjective.

Theorem 6.2 shows that for all $a_1, \ldots, a_n \in F^\times$

$$\lfloor a_1, \ldots, a_n \rceil \cong \left\langle \prod_i a_i \right\rangle \lfloor 1, \ldots, 1 \rceil \pmod{\tilde{S}(F^n)^{\mathrm{dec}}}.$$

In other words the map

$$\mathbb{Z}[F^\times] \to \tilde{S}(F^n)^{\mathrm{ind}}, \quad \alpha \mapsto \alpha \lfloor 1, \ldots, 1 \rceil$$

is a surjective homomorphism of $\mathbb{Z}[F^\times]$-modules. Thus, we are required to establish that $\lfloor 1, \ldots, 1 \rceil \in \tilde{S}(F^n)^{\mathrm{dec}}$ for all $n \geq 3$.

For convenience below, we will let $\tilde{\Sigma}_n(F)$ denote the free $\mathbb{Z}[F^\times]$-module on the symbols $\left[a_1, \ldots, a_n\right]$, $a_1, \ldots, a_n \in F^\times$. Let $p_n : \tilde{\Sigma}_n(F) \to \tilde{S}(F^n)$ be the $\mathbb{Z}[F^\times]$-module homomorphism sending $\left[a_1, \ldots, a_n\right]$ to $\lfloor a_1, \ldots, a_n \rceil$. We will say that $\sigma \in \tilde{S}(F^n)$ is represented by $\tilde{\sigma} \in \tilde{\Sigma}_n(F)$ if $p_n(\tilde{\sigma}) = \sigma$.

Note that $\tilde{\Sigma}_\bullet(F)$ can be given the structure of a graded $\mathbb{Z}[F^\times]$-algebra by setting

$$\left[a_1, \ldots, a_n\right] \cdot \left[a_{n+1}, \ldots, a_{n+m}\right] := \left[a_1, \ldots, a_{n+m}\right];$$

i.e., we can identify $\tilde{\Sigma}_\bullet(F)$ with the tensor algebra over $\mathbb{Z}[F^\times]$ on the free module with basis $\left[a\right]$, $a \in F^\times$.

Let $\Pi_\bullet : \tilde{\Sigma}_\bullet(F) \to \mathbb{Z}[F^\times][x]$ be the homomorphism of graded $\mathbb{Z}[F^\times]$-algebras sending $\left[a\right]$ to $\langle a \rangle\, x$.

For all $n \geq 1$ we have a commutative square of surjective homomorphisms of $\mathbb{Z}[F^\times]$-modules

$$\begin{array}{ccc} \tilde{\Sigma}_n(F) & \xrightarrow{\Pi_n} & \mathbb{Z}[F^\times] \cdot x^n \\ \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle \gamma_n} \\ \tilde{S}(F^n) & \longrightarrow & \tilde{S}(F^n)^{\mathrm{ind}} \end{array}$$

where $\gamma_n(x^n) = \lfloor 1, \ldots, 1 \rceil$.

LEMMA 6.5. *If n is odd and $n \geq 3$ then $\tilde{S}(F^n)^{ind} = 0$; i.e.,*

$$\tilde{S}(F^n)_{\mathcal{M}} = \tilde{S}(F^n)^{dec}.$$

Proof. From the fundamental relation in $\tilde{S}(F^n)$ (Theorem 3.3), if $b_1, \ldots, b_n$ are distinct elements of $F^\times$, then $0 \in \tilde{S}(F^n)$ is represented by $R_b :=$

$$[b_1, \ldots, b_n] - [1, \ldots, 1] - \sum_{j=1}^{n} (-1)^{n+j} \left\langle (-1)^{n+j} \right\rangle [b_1 - b_j, \ldots, \widehat{b_j - b_j}, \ldots, b_n - b_j, b_j]$$

in $\in \tilde{\Sigma}_n(F)$.

Now $\Pi_n(R_b) =$

$$\left[\left\langle \prod_i b_i \right\rangle - \langle 1 \rangle - \sum_{j=1}^{n} (-1)^{n+j} \langle (b_j - b_1) \cdots (b_j - b_{j-1}) \cdot (b_{j+1} - b_j) \cdots (b_n - b_j) \cdot b_j \rangle \right] x^n.$$

We choose $b_i = i$, $i = 1, \ldots, n$. Then

$$\Pi_n(R_b) = \left[\langle n! \rangle - \langle 1 \rangle - \sum_{j=1}^{n} (-1)^{n+j} \langle j!(n-j)! \rangle \right] x^n = - \langle 1 \rangle x^n \text{ since } n \text{ is odd.}$$

It follows that $-\lfloor 1, \ldots, 1 \rceil = 0$ in $\tilde{S}(F^n)^{\mathrm{ind}}$ as required. □

The case n even requires a little more work.
The maps $\{p_n\}_n$ do not define a map of graded algebras. However, we do have the following:

Lemma 6.6. *For $1 \neq a \in F^\times$, let*

$$L(x) := \langle -1 \rangle \left[1 - x, 1\right] - \langle x \rangle \left[1 - \frac{1}{x}, \frac{1}{x}\right] + \left[1, 1\right] \in \tilde{\Sigma}_2(F).$$

Then for all $a_1, \ldots, a_n \in F^\times \setminus \{1\}$, the product

$$\prod_{i=1}^{n} \lfloor 1, a_i \rceil = \lfloor 1, a_1 \rceil * \cdots * \lfloor 1, a_n \rceil \in \tilde{S}(F^{2n})$$

is represented by $\prod_i L(a_i) \in \tilde{\Sigma}_{2n}(F)$.

Proof. For convenience of notation, we will represent standard basis elements of $C_q(F^n)$ as $n \times q$ matrices $[v_1 | \cdots | v_q]$.
Let $e = (1, \ldots, 1)$ and let $\sigma_i(C)$ denote the sum of the entries in the ith row of the $n \times n$ matrix C. By Remark 3.2, if $A \in \mathrm{GL}_n(F)$ and $[A|e] \in X_{n+1}(F^n)$ then $d_{n+1}([A|e])$ represents $\langle \det A \rangle \lfloor \sigma_1(A^{-1}), \ldots, \sigma_n(A^{-1}) \rceil \in \tilde{S}(F^n)$.
Now, for $a \neq 1$, $\lfloor 1, a \rceil$ is represented in $\tilde{S}(F^2)$ by

$$d_3 \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & a \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 \\ 1 & a \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = T_1(a) - T_2(a) + T_3(a) \in C_2(F^2).$$

From the definition of the product $*$, it follows that $\lfloor 1, a_1 \rceil * \cdots * \lfloor 1, a_n \rceil$ is represented by

$$Z := \sum_{j=(j_1,\ldots,j_n) \in (1,2,3)^n} (-1)^{k(j)} \begin{bmatrix} T_{j_1}(a_1) & & \\ & \ddots & \\ & & T_{j_n}(a_n) \end{bmatrix} = \sum_j (-1)^{k(j)} T(j, a).$$

where $k(j) := |\{i \leq n | j_i = 2\}|$
Since $a_i \neq 1$ for all i, the vector $e = (1, \ldots, 1)$ is in general position with respect to the columns of all these matrices. Thus we can use the partial homotopy operator s_e to write this cycle as a boundary:

$$Z = \sum_j (-1)^{k(j)} d_{2n+1} \left([T(j, a) | e] \right).$$

By the remarks above

$$d_{2n+1}\left([T(j,a)|e]\right) = \left\langle \prod_i \det T_{j_i}(a_i)\right\rangle \times$$
$$\times \lfloor \sigma_1(T_{j_1}(a_1)), \sigma_2(T_{j_1}(a_1)), \sigma_1(T_{j_2}(a_2)), \ldots, \sigma_1(T_{j_n}(a_n)), \sigma_2(T_{j_n}(a_n))\rceil.$$

This is represented by

$$\left\langle \prod_i \det T_{j_i}(a_i)\right\rangle \times$$
$$\times\left[\sigma_1(T_{j_1}(a_1)), \sigma_2(T_{j_1}(a_1)), \sigma_1(T_{j_2}(a_2)), \ldots, \sigma_1(T_{j_n}(a_n)), \sigma_2(T_{j_n}(a_n))\right]$$
$$= \prod_{i=1}^{n}\left(\langle \det T_{j_i}(a_i)\rangle \left[\sigma_1(T_{j_i}(a_i)), \sigma_2(T_{j_i}(a_i))\right]\right) \in \tilde{\Sigma}_{2n}(F).$$

Thus Z is represented by

$$\sum_j (-1)^{k(j)} \prod_{i=1}^{n}\left(\langle \det T_{j_i}(a_i)\rangle \left[\sigma_1(T_{j_i}(a_i)), \sigma_2(T_{j_i}(a_i))\right]\right) =$$
$$= \prod_{i=1}^{n}\left(\sum_{j=1}^{3} (-1)^{j+1} \langle \det T_j(a_i)\rangle \left[\sigma_1(T_j(a_i)), \sigma_2(T_j(a_i))\right]\right) = \prod_{i=1}^{n} L(a_i) \in \tilde{\Sigma}_{2n}(F).$$

□

Observe that all of our multiplicative modules (and in particular $\tilde{S}(F^n)_{\mathcal{M}}$) have the following property: they admit a finite filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ such that each of the associated quotients M_r/M_{r-1} is annihilated by $\mathcal{I}_{(F^\times)^{k_r}}$ for some $k_r \geq 1$. From this observation it easily follows that

Lemma 6.7.

$$\tilde{S}(F^n)^{ind} = 0 \iff \tilde{S}(F^n)^{ind}/(\mathcal{I}_{(F^\times)^r} \cdot \tilde{S}(F^n)^{ind}) = 0 \textit{ for all } r \geq 1.$$

Theorem 6.8. *$\tilde{S}(F^n)^{ind} = 0$ for all $n \geq 3$.*

Proof. The case n odd has already been dealt with in Lemma 6.5
For the even case, by Lemma 6.7 it will be enough to prove that for all $r \geq 1$

$$\mathbb{Z}[F^\times/(F^\times)^r] \otimes_{\mathbb{Z}[F^\times]} \tilde{S}(F^n)^{\mathrm{ind}} = 0.$$

Fix $r \geq 1$. If $a \in (F^\times)^r \setminus \{1\}$, then

$$\Pi_2(L(a)) = \left(\langle a-1\rangle - \left\langle 1 - \frac{1}{a}\right\rangle + \langle 1\rangle\right) x^2 = \langle 1\rangle\, x^2 \in \mathbb{Z}[F^\times/(F^\times)^r]x^2$$

since

$$1 - \frac{1}{a} = \frac{a-1}{a} \equiv a - 1 \pmod{(F^\times)^r}.$$

Now let $n > 1$ and choose $a_1, \ldots, a_n \in (F^\times)^r \setminus \{1\}$. Let $\sigma = \lfloor 1, a_1 \rceil * \cdots * \lfloor 1, a_n \rceil \in \tilde{S}(F^{2n})$, so that $\sigma \mapsto 0$ in $\tilde{S}(F^{2n})^{\mathrm{ind}}$. By Lemma 6.6, σ is represented by $\tilde{\sigma} = \prod_{i=1}^{n} L(a_i)$ in $\tilde{\Sigma}_{2n}(F)$ and thus

$$\Pi_{2n}(\tilde{\sigma}) = \prod_{i=1}^{n} \left(\Pi_2(L(a_i))\right) = \langle 1 \rangle \in \mathbb{Z}[F^\times/(F^\times)^r]x^{2n}$$

so that the image of σ in $\mathbb{Z}[F^\times/(F^\times)^r] \otimes_{\mathbb{Z}[F^\times]} \tilde{S}(F^{2n})^{\mathrm{ind}}$ is $1 \otimes \lfloor 1, \ldots, 1 \rceil$. This proves the theorem. □

Corollary 6.9. *For all $n \geq 2$, the map T_n induces an isomorphism $\mathcal{F}_{n,1} \cong K_n^{\mathrm{MW}}(F)$.*

Proof. Since, by the computations above, $\tilde{S}(F^2)_{\mathcal{M}} = \tilde{S}(F)^{*2} + \mathbb{Z}[F^\times]E$ it follows, using Theorem 6.8 and induction on n, that $\tilde{S}(F^\bullet)_{\mathcal{M}}$ is generated as a $\mathbb{Z}[F^\times]$-algebra by $\{\lfloor a \rceil \in \tilde{S}(F) | 1 \neq a \in F^\times\}$ and E.
Thus E is central in the algebra $\tilde{S}(F^\bullet)_{\mathcal{M}}$ and for all $n \geq 2$,

$$\frac{\tilde{S}(F^n)_{\mathcal{M}}}{E * \tilde{S}(F^{n-2})_{\mathcal{M}}}$$

is generated by the elements of the form $\lfloor a_1 \rceil * \cdots * \lfloor a_n \rceil$, and hence also by the elements $\{\{a_1, \ldots, a_n\}\}$ since $[a, b] \equiv \lfloor a \rceil * \lfloor b \rceil \pmod{\langle E \rangle}$ for all $a, b \in F^\times$.
Since

$$\mathcal{F}_{n,1} \cong \frac{\tilde{S}(F^n)_{\mathcal{M}}}{E * \tilde{S}(F^{n-2})_{\mathcal{M}}}$$

by Corollary 5.14, it follows that $\mathcal{F}_{n,1}$ is generated by the elements $\{\{a_1, \ldots, a_n\}\}$, and thus that the homomorphisms μ_n of Theorem 5.24 are surjective. □

Corollary 6.10. *For all $n \geq 3$,*

$$\tilde{S}(F^n)_{\mathcal{M}} \cong \begin{cases} K_n^{\mathrm{MW}}(F) \oplus K_{n-2}^{\mathrm{MW}}(F) \oplus \cdots \oplus K_2^{\mathrm{MW}}(F) \oplus \mathbb{Z}[F^\times] & n \text{ even} \\ K_n^{\mathrm{MW}}(F) \oplus K_{n-2}^{\mathrm{MW}}(F) \oplus \cdots \oplus K_3^{\mathrm{MW}}(F) \oplus \mathcal{I}_{F^\times} & n \text{ odd} \end{cases}$$

as a $\mathbb{Z}[F^\times]$-module.

Corollary 6.11. *For all even $n \geq 2$ the cokernel of the map*

$$\mathrm{H}_n(\mathrm{SL}_{n-1}(F), \mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_n(F), \mathbb{Z})$$

is isomorphic to $K_n^{\mathrm{MW}}(F)$.

Proof. Recall that ϵ_2 induces an isomorphism $\mathrm{H}_2(\mathrm{SL}_2(F), \mathbb{Z}) \cong \mathcal{F}_{2,1} = \mathcal{F}_{2,0}$. Let $\langle a, b \rangle$ denote the generator $\epsilon_2^{-1}([a, b])$ of $\mathrm{H}_2(\mathrm{SL}_2(F), \mathbb{Z})$. Then for even n

$$\begin{aligned} \{\{a_1, \ldots, a_n\}\} &= [a_1, a_2] * \cdots * [a_{n-1}, a_n] \\ &= \epsilon_2(\langle a_1, a_2 \rangle) * \cdots * \epsilon_2(\langle a_{n-1}, a_n \rangle) \\ &= \epsilon_n(\langle a_1, a_2 \rangle \times \cdots \times \langle a_{n-1}, a_n \rangle) \end{aligned}$$

by Lemma 3.5 (2).

Since $\mathcal{F}_{n,1}$ is generated by the elements $\{\{a_1,\ldots,a_n\}\}$, it follows that $\mathcal{F}_{n,1} = \epsilon_n(\mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z})) = E^\infty_{n,0} = \mathcal{F}_{n,0}$, proving the result. □

COROLLARY 6.12. *For all odd $n \geq 1$ the maps*

$$\mathrm{H}_n(\mathrm{SL}_k(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_{k+1}(F),\mathbb{Z})$$

are isomorphisms for $k \geq n$.

Proof. In view of Corollary 5.11, the only point at issue is the injectivity of

$$\mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_{n+1}(F),\mathbb{Z}).$$

But the proof of Corollary 6.11 shows that the term

$$\mathcal{F}_{n+1,1}/E^\infty_{n+1,0} \cong E^\infty_{n,1} = \mathrm{Ker}(\mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_{n+1}(F),\mathbb{Z}))$$

in the spectral sequence $\mathcal{E}^+(F^{n+1})_{\mathcal{M}}$ is zero. □

COROLLARY 6.13. *If $n \geq 3$ is odd, then*

$$\mathrm{Coker}(\mathrm{H}_n(\mathrm{SL}_{n-1}(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z})) \cong 2K^{\mathrm{M}}_n(F)$$
$$\mathrm{Ker}(\mathrm{H}_{n-1}(\mathrm{SL}_{n-1}(F),\mathbb{Z}) \to \mathrm{H}_{n-1}(\mathrm{SL}_n(F),\mathbb{Z})) \cong I^n(F).$$

Proof. Since we have already proved this result for $n = 3$ above, we will assume that $n \geq 5$ (n odd).

Let $a_1,\ldots,a_n \in F^\times$ and let $z \in \mathrm{H}_{n-1}(\mathrm{SL}_{n-1}(F),\mathbb{Z})$ satisfy $\epsilon_{n-1}(z) = \{\{a_2,\ldots,a_n\}\} \in \mathcal{F}_{n-1,0} \cong K^{\mathrm{MW}}_{n-1}(F)$. Thus $\{\{a_1,\ldots,a_n\}\} = \lfloor a_1\rceil * \epsilon_{n-1}(z)$ and hence $\epsilon_{n,1}(\{\{a_1,\ldots,a_n\}\}) = \langle\langle a_1\rangle\rangle z$ by Theorem 5.20 (2). It follows that the diagram

$$\begin{tikzcd}
\mathcal{F}_{n,1} \arrow[r, "\epsilon_{n,1}"] \arrow[d, "T_n", "\cong"'] & \mathrm{H}_{n-1}(\mathrm{SL}_{n-1}(F),\mathbb{Z}) \arrow[d, "T_{n-1}\circ\epsilon_{n-1}"] \\
K^{\mathrm{MW}}_n(F) \arrow[r, "\eta"] & K^{\mathrm{MW}}_{n-1}(F)
\end{tikzcd}$$

commutes.

Now $\mathrm{Ker}(\epsilon_{n,1}) = \mathrm{Im}(\epsilon_n : \mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z}) \to \mathcal{F}_{n,1})$. Since $\mathrm{Im}(\epsilon_3) = T_3^{-1}(2K^{\mathrm{M}}_3(F))$ and $\mathrm{Im}(\epsilon_{n-3}) = \mathcal{F}_{n-3,1} = T^{-1}_{n-3}(K^{\mathrm{MW}}_{n-3}(F))$ we have

$$T_n(\mathrm{Im}(\epsilon_n)) = \mathrm{Im}(T_n \circ \epsilon_n) \supset 2K^{\mathrm{M}}_3(F)\cdot K^{\mathrm{MW}}_{n-3}(F) = 2K^{\mathrm{M}}_n(F) \subset K^{\mathrm{MW}}_n(F)$$

(using the fact that $T_\bullet$ and $\epsilon_\bullet$ are algebra homomorphisms).

Thus we get a commutative diagram

$$\begin{tikzcd}
\frac{K^{\mathrm{MW}}_n(F)}{2K^{\mathrm{M}}_n(F)} \arrow[r, "T_n^{-1}"] \arrow[d, "\eta", "\cong"'] & \frac{\mathcal{F}_{n,1}}{\mathrm{Ker}(\epsilon_{n,1})} \arrow[dl, "T_{n-1}\circ\epsilon_{n-1}\circ\epsilon_{n,1}"] \\
I^n(F) &
\end{tikzcd}$$

from which it follows that the map T_n^{-1} in this diagram is an isomorphism, and hence $\mathrm{Im}(\epsilon_n) = \mathrm{Ker}(\epsilon_{n,1}) \cong 2K^{\mathrm{M}}_n(F)$ and $\mathrm{Im}(\epsilon_{n,1}) \cong I^n(F)$. □

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